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# REFINEMENTS OF DETERMINANTAL INEQUALITIES OF JENSEN'S TYPE

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**Abstract.** In this paper some new refinements are given for Jensen's type inequalities involving the determinants of positive definite matrices. The so-called Bellman-Bergstrom-Fan functionals are considered. These functionals are not only concave, but superlinear which is a stronger condition. The results take advantage of this property. In seek of applications, results are furnished with examples.

## 1 Introduction

We start with the following notation introduced in [6] (see also [7]).

$\mathcal{M}_m$ denotes the set of positive definite matrices of order $m$. It is evident that $\mathcal{M}_m$ is closed under addition and multiplication by a positive number, i.e. if $M_1, M_2 \in \mathcal{M}_m$, $a > 0$, then $M_1 + M_2$, $aM_1 \in \mathcal{M}_m$ ($\mathcal{M}_m$ is a convex cone).

If $M \in \mathcal{M}_m$, let

$|M| :=$ the determinant of $M$,

$|M|_k = \Pi_{j=1}^{k} \lambda_j$, $k = 1, ..., m$, where $\lambda_1, ..., \lambda_m$ are the eigenvalues of $M$ arranged in non-decreasing order: $\lambda_1 \leq ... \leq \lambda_m$ (here $|M|_m = |M|$),

$M(j) :=$ the submatrix of $M$ obtained by deleting the $j^{th}$ row and column of $M$,

$M[k] :=$ the principal submatrix of $M$ formed by taking the first $k$ rows and columns of $M$; then $M[m] = M$, $M[m-1] = M(m)$ and $M[0]$ is the identity matrix.

$BBF$ means the class of Bellman-Bergstrom-Fan functionals $\sigma_i$, $\delta_j$ and $\nu_k$ defined on $\mathcal{M}_m$ by

$$\sigma_i(M) = |M|_i^{\frac{1}{i}}, \quad i = 1, ..., m,$$

$$\delta_j(M) = \frac{|M|}{|M(j)|}, \quad j = 1, ..., m,$$

and

$$\nu_k(M) = \left( \frac{|M|}{|M[k]|} \right)^{\frac{1}{(m-k)}}, \quad k = 1, ..., m,$$

respectively.

The $BBF$ functionals are superlinear (see [6]), i.e. $f \in BBF$ is both superadditive

$$f(M_1 + M_2) \geq f(M_1) + f(M_2), \quad M_1, M_2 \in \mathcal{M}_m$$

and positive homogeneous

$$f(pM) = pf(M), \quad M_1, M_2 \in \mathcal{M}_m, \quad p > 0.$$

More generally, for $f \in BBF$, $M_i \in \mathcal{M}_m$, $p_i > 0$ $(i = 1, ..., n)$, and $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, ..., n)$, we have (see also [6]):

$$f\left(\sum_{i=1}^{n} p_i M_i\right) \geq \sum_{i=1}^{n} p_i f(M_i) \geq P_n \prod_{i=1}^{n} f(M_i)^{\frac{p_i}{P_n}}, \tag{1.1}$$

which is an interpolating inequality for

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i M_i\right) \geq \prod_{i=1}^{n} f(M_i)^{\frac{p_i}{P_n}}. \tag{1.2}$$

**Remark 1.** (a) Since a functional $f \in BBF$ is superlinear, it is also concave. Inequality (1.2) comes from the second inequality in (1.1), which is just an arithmetic-geometric mean inequality, by using only the concavity of $f$ .

For $P_n = 1$, interpolations corresponding to the second inequality in (1.1) can be found in [3] and [4]. In [2] parameter dependent interpolations are given.

(b) A concave functional on $\mathcal{M}_m$ is not superlinear in general, hence the interpolations of the first inequality in (1.1) are most interesting in the case $P_n \neq 1$.

Unweighted versions of (1.1) and (1.2) are given by

$$f\left(\frac{1}{n}\sum_{i=1}^{n} M_i\right) \geq \frac{1}{n}\sum_{i=1}^{n} f(M_i) \geq \prod_{i=1}^{n} f(M_i)^{1/n}, \tag{1.3}$$

and

$$f\left(\frac{1}{n}\sum_{i=1}^{n} M_i\right) \geq \prod_{i=1}^{n} f(M_i)^{1/n}, \tag{1.4}$$

respectively.

The following interpolations of the first inequality in (1.3) are given in [6]:

$$f\left(\frac{1}{n}\sum_{i=1}^{n} M_i\right) = f_{n,n} \geq ... \geq f_{k+1,n} \geq f_{k,n} \geq ... \geq f_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(M_i), \tag{1.5}$$

where

$$f_{k,n} = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < ... < i_k \leq n} f\left(\frac{1}{k}\left(M_{i_1} + ... + M_{i_k}\right)\right).$$

[6] contains interpolations for the second inequality in (1.3) too:

$$\frac{1}{n}\sum_{i=1}^{n} f(M_i) = g_{n,n} \geq ... \geq g_{k+1,n} \geq g_{k,n} \geq ... \geq g_{1,n} = \prod_{i=1}^{n} f(M_i)^{1/n}, \tag{1.6}$$

where

$$g_{k,n}=\prod_{1\leq i_1<\ldots<i_k\leq n}\left(\frac{1}{k}\left(f(M_{i_1})+\ldots+f(M_{i_k})\right)\right)^{\frac{1}{\binom{n}{k}}}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}f(M_i)=h_{1,n}\geq\ldots\geq h_{k,n}\geq h_{k+1,n}\geq\ldots\geq h_{n,n}=\prod_{i=1}^{n}f(M_i)^{1/n}, \tag{1.7}$$

where

$$h_{k,n}=\frac{1}{\binom{n}{k}}\sum_{1\leq i_1<\ldots<i_k\leq n}\left(f(M_{i_1})\ldots f(M_{i_k})\right)^{\frac{1}{k}}.$$

There are similar interpolations for (1.4) in [6]:

$$f\left(\frac{1}{n}\sum_{i=1}^{n}M_i\right)=r_{n,n}\geq\ldots\geq r_{k+1,n}\geq r_{k,n}\geq\ldots\geq r_{1,n}=\prod_{i=1}^{n}f(M_i)^{1/n}, \tag{1.8}$$

where

$$r_{k,n}=\prod_{1\leq i_1<\ldots<i_k\leq n}f\left(\frac{1}{k}\left(M_{i_1}+\ldots+M_{i_k}\right)\right)^{\frac{1}{\binom{n}{k}}}$$

The above interpolations from [6] are based on the concavity of $f$. In this paper we give interpolations of the first inequality in (1.1) (see Remark 1 (b)), which ensure generalizations of (1.5). By using the results in the papers [3], [4] and [2], we can also generalize the second inequality in (1.3) and the inequality (1.4), and thus inequalities (1.6-1.8), but these interpolations are just concrete examples of the inequalities in the papers [3], [4] and [2] (see Remark 1 (a)).

We start with notation introduced in [5].

Let $X$ be a set. The power set of $X$ is denoted by $P(X)$. $|X|$ means the number of elements in $X$.

The usual symbol $\mathbb{N}$ is used for the set of natural numbers (including 0).

Let $u\geq 1$ and $v\geq 2$ be fixed integers. Define the functions

$$S_{v,w}:\{1,\ldots,u\}^{v}\to\{1,\ldots,u\}^{v-1},\quad 1\leq w\leq v,$$
$$S_{v}:\{1,\ldots,u\}^{v}\to P\left(\{1,\ldots,u\}^{v-1}\right),$$

and

$$T_{v}:P\left(\{1,\ldots,u\}^{v}\right)\to P\left(\{1,\ldots,u\}^{v-1}\right)$$

by

$$S_{v,w}\left(i_1,\ldots,i_v\right):=\left(i_1,i_2,\ldots,i_{w-1},i_{w+1},\ldots,i_v\right),\quad 1\leq w\leq v,$$

$$S_{v}\left(i_1,\ldots,i_v\right):=\bigcup_{w=1}^{v}\left\{S_{v,w}\left(i_1,\ldots,i_v\right)\right\},$$

and

$$T_v(I) := \begin{cases} \varnothing, & \text{if} \quad I = \varnothing \\ \bigcup\limits_{(i_1,\ldots,i_v)\in I} S_v\left(i_1,\ldots,i_v\right), & \text{if} \quad I \neq \varnothing \end{cases}.$$

Next, let the function

$$\alpha_{v,i} : \{1,\ldots,u\}^v \to \mathbb{N}, \quad 1 \le i \le u,$$

be given by: $\alpha_{v,i}\left(i_1,\ldots,i_v\right)$ means the number of occurrences of $i$ in the sequence $(i_1,\ldots,i_v)$.

For each $I \in P\left(\{1,\ldots,u\}^v\right)$ let

$$\alpha_{I,i} := \sum_{(i_1,\ldots,i_v)\in I} \alpha_{v,i}\left(i_1,\ldots,i_v\right), \quad 1 \le i \le u.$$

It is easy to see that the dependence of the functions $S_{v,w}$, $S_v$, $T_v$ and $\alpha_{v,i}$ on $u$ does not play an important role, so we can use simplified notations.

The following hypotheses will give the basic context of our results.

(H$_1$) Let $n \ge 1$ and $k \ge 2$ be fixed integers, and let $I_k$ be a subset of $\{1,\ldots,n\}^k$ such that

$$\alpha_{I_k,i} \ge 1, \quad 1 \le i \le n. \tag{1.9}$$

(H$_2$) Let $M_1, ..., M_n \in \mathcal{M}_m$.

(H$_3$) Let $p_1,\ldots,p_n$ be positive real numbers. Let $P_n := \sum\limits_{i=1}^{n} p_i$.

(H$_4$) Let the function $f : \mathcal{M}_m \to \mathbb{R}$ be a Bellman-Bergström-Fan ($BBF$) functional.

We need some further preparations.

Starting from $I_k$, we introduce the sets $I_l \subset \{1,\ldots,n\}^l$ $(k-1 \ge l \ge 1)$ inductively by

$$I_{l-1} := T_l(I_l), \quad k \ge l \ge 2.$$

Obviously, $I_1 = \{1,\ldots,n\}$, and this insures that $\alpha_{I_1,i} = 1$ $(1 \le i \le n)$. From (1.9), we have that $\alpha_{I_l,i} \ge 1$ $(k-1 \ge l \ge 1, 1 \le i \le n)$. It is evident that

$$\alpha_{1,i}(j) = \begin{cases} 1, & \text{if} \quad j = i \\ 0, & \text{if} \quad j \neq i \end{cases}, \quad 1 \le i \le n.$$

For any $k \ge l \ge 2$ and for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ let

$$\begin{aligned} &H_{I_l}\left(j_1,\ldots,j_{l-1}\right) \\ &:= \left\{((i_1,\ldots,i_l),m) \in I_l \times \{1,\ldots,l\} \mid S_{l,m}(i_1,\ldots,i_l) = (j_1,\ldots,j_{l-1})\right\}. \end{aligned}$$

Using these sets we define the functions $t_{I_k,l} : I_l \to \mathbb{N}$ $(k \ge l \ge 1)$ inductively by

$$t_{I_k,k}\left(i_1,\ldots,i_k\right) := 1, \quad (i_1,\ldots,i_k) \in I_k;$$

$$t_{I_k,l-1}(j_1,\ldots,j_{l-1}) := \sum_{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}\left(i_1,\ldots,i_l\right).$$

In the sequel, we also make use of the following hypothesis:

(H$_5$) Let $\left|H_{I_l}\left(j_1,\ldots,j_{l-1}\right)\right| = \beta_{l-1}$ for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \ge l \ge 2)$.

## 2 Refinement results

The refinement results of this section involve some special expressions, which we now describe. Assume ($H_1$)-($H_4$). We shall use the fact that $f \in BBF$ is positive homogeneous. For any $k \geq l \geq 1$ let

$$A_{l,l} = A_{l,l}\left(I_k, M_1, \ldots, M_n, p_1, \ldots, p_n\right)$$

$$:= \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} M_{i_s}}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}}\right) \tag{2.1}$$

$$= \sum_{(i_1,\ldots,i_l)\in I_l} f\left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} M_{i_s}\right),$$

and associate to each $k-1 \geq l \geq 1$ the number

$$A_{k,l} = A_{k,l}\left(I_k, M_1, \ldots, M_n, p_1, \ldots, p_n\right)$$

$$:= \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}\left(i_1, \ldots, i_l\right) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} M_{i_s}}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)$$

$$= \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}\left(i_1, \ldots, i_l\right) f\left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} M_{i_s}\right).$$

Under the above constructions we come to

**Theorem 2.1.** *Assume that ($H_1$)-($H_4$) are satisfied. Then*

*(a)*

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq A_{k,k} \geq A_{k,k-1} \geq \ldots \geq A_{k,2} \geq A_{k,1} = \sum_{r=1}^{n} p_r f(M_r). \tag{2.2}$$

*(b) Assume ($H_5$) is also satisfied. Then*

$$A_{k,l} = A_{l,l} = \frac{n}{l\,|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} f\left(\sum_{s=1}^{l} p_{i_s} M_{i_s}\right), \quad (k \geq l \geq 1),$$

*and thus*

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq A_{k,k} \geq A_{k-1,k-1} \geq \ldots \geq A_{2,2} \geq A_{1,1} = \sum_{r=1}^{n} p_r f(M_r).$$

*Proof.* We prove (a), (b) can be proved similarly. Since $f$ is a Bellman-Bergström-Fan functional, it is concave. Therefore Theorem 1 in [5] implies that

$$f\left(\frac{1}{P_n}\sum_{r=1}^{n} p_r M_r\right) \geq \bar{A}_{k,k} \geq \bar{A}_{k,k-1} \geq \ldots \geq \bar{A}_{k,2} \geq \bar{A}_{k,1} = \frac{1}{P_n}\sum_{r=1}^{n} p_r f(M_r), \tag{2.3}$$

where

$$\bar{A}_{l,l} := A_{l,l}\left(I_k, M_1, \ldots, M_n, \frac{p_1}{P_n}, \ldots, \frac{p_n}{P_n}\right), \quad k \geq l \geq 1$$

and

$$\bar{A}_{k,l} := A_{k,l}\left(I_k, M_1, \ldots, M_n, \frac{p_1}{P_n}, \ldots, \frac{p_n}{P_n}\right)$$

for $k-1 \geq l \geq 1$. The result now follows from (2.3), since $f$ is positive homogeneous. □

## 3 Applications

Throughout Examples (1-6) conditions ($H_2$)-($H_4$) will be assumed. These examples based on examples in [5].

First, we generalize (1.5).

**Example 1.** Let

$$I_k := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \leq k \leq n.$$

Then $\alpha_{I_n,i} = 1$ ($i = 1, \ldots, n$) ensuring ($H_1$) with $k = n$. It is easy to check that $T_k(I_k) = I_{k-1}$ ($k = 2, \ldots, n$), $|I_k| = \binom{n}{k}$ ($k = 1, \ldots, n$), and for every $k = 2, \ldots, n$

$$|H_{I_k}(j_1, \ldots, j_{k-1})| = n - (k-1), \quad (j_1, \ldots, j_{k-1}) \in I_{k-1},$$

and therefore, by Theorem 2.1 (b),

$$A_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f\left(\sum_{s=1}^{k} p_{i_s} M_{i_s}\right), \quad k = 1, \ldots, n.$$

and

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq A_{k,k} \geq A_{k-1,k-1} \geq \ldots \geq A_{2,2} \geq A_{1,1} = \sum_{r=1}^{n} p_r f(M_r). \tag{3.1}$$

If $p_1 = \ldots = p_n = \frac{1}{n}$, then (see (2.1))

$$A_{k,k} = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f\left(\frac{M_{i_1} + \ldots + M_{i_k}}{k}\right), \quad k = 1, \ldots, n,$$

and thus (3.1) gives the generalization of (1.5).

The structure of the second example is similar to the previous one.

**Example 2.** Let

$$I_k := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 \leq \ldots \leq i_k\right\}, \quad k \geq 1.$$

Obviously, $\alpha_{I_k,i} \geq 1$ ($i = 1, \ldots, n$), and therefore ($H_1$) is satisfied. It is not hard to see that $T_k(I_k) = I_{k-1}$ ($k = 2, \ldots$), $|I_k| = \binom{n+k-1}{k}$ ($k = 1, \ldots$), and for each $l = 2, \ldots, k$

$$|H_{I_l}(j_1, \ldots, j_{l-1})| = n, \quad (j_1, \ldots, j_{l-1}) \in I_{l-1}.$$

Consequently, by applying Theorem 2.1 (b), we deduce that

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} f\left(\sum_{s=1}^{k} p_{i_s} M_{i_s}\right), \quad k \geq 1,$$

and

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq \ldots \geq A_{k,k} \geq \ldots \geq A_{k,1} = \sum_{r=1}^{n} p_r f(M_r).$$

By taking $p_1 = \ldots = p_n = \frac{1}{n}$ we obtain (see (2.1))

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} f\left(\frac{M_{i_1} + \ldots + M_{i_k}}{k}\right), \quad k \geq 1.$$

The following two examples are particular cases of Theorem 2.1 (b).

**Example 3.** Let

$$I_k := \{1, \ldots, n\}^k, \quad k \geq 1.$$

Trivially, $\alpha_{I_k,i} \geq 1$ ($i = 1, \ldots, n$), hence ($H_1$) holds. It is evident that $T_k(I_k) = I_{k-1}$ ($k = 2, \ldots$), $|I_k| = n^k$ ($k = 1, \ldots$), and for every $l = 2, \ldots, k$

$$|H_{I_l}(j_1, \ldots, j_{l-1})| = n^l, \quad (j_1, \ldots, j_{l-1}) \in I_{l-1},$$

and so Theorem 2.1 (b) leads to

$$A_{k,k} = \frac{1}{kn^{k-1}} \sum_{(i_1, \ldots, i_k) \in I_k} f\left(\sum_{s=1}^{k} p_{i_s} M_{i_s}\right), \quad k \geq 1,$$

and

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq \ldots \geq A_{k,k} \geq \ldots \geq A_{1,1} = \sum_{r=1}^{n} p_r f(M_r), \quad k \geq 1.$$

Especially, for $p_1 = \ldots = p_n = \frac{1}{n}$ we find (see (2.1)) that

$$A_{k,k} = \frac{1}{n^k} \sum_{(i_1, \ldots, i_k) \in I_k} f\left(\frac{M_{i_1} + \ldots + M_{i_k}}{k}\right), \quad k = 1, \ldots, n.$$

**Example 4.** For $1 \le k \le n$ let $I_k$ consist of all sequences $(i_1, \dots, i_k)$ of $k$ distinct numbers from $\{1, \dots, n\}$. Then $\alpha_{I_n,i} \ge 1$ $(i = 1, \dots, n)$, hence $(\mathrm{H}_1)$ is valid. It is immediate that $T_k(I_k) = I_{k-1}$ $(k = 2, \dots, n)$, $|I_k| = n(n-1)\dots(n-k+1)$ $(k = 1, \dots, n)$, and for each $k = 2, \dots, n$

$$|H_{I_k}(j_1, \dots, j_{k-1})| = (n-(k-1))\,k, \quad (j_1, \dots, j_{k-1}) \in I_{k-1},$$

and from them, on account of Theorem 2.1 (b), follows

$$A_{k,k} = \frac{n}{kn(n-1)\dots(n-k+1)} \sum_{(i_1,\dots,i_k)\in I_k} f\left(\sum_{s=1}^{k} p_{i_s} M_{i_s}\right), \quad k = 1, \dots, n$$

and

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \ge A_{n,n} \ge \dots \ge A_{k,k} \ge \dots \ge A_{1,1} = \sum_{r=1}^{n} p_r f(M_r).$$

If we set $p_1 = \dots = p_n = \frac{1}{n}$, then by (2.1)

$$A_{k,k} = \frac{1}{n(n-1)\dots(n-k+1)} \sum_{(i_1,\dots,i_k)\in I_k} f\left(\frac{M_{i_1} + \dots + M_{i_k}}{k}\right), \quad k = 1, \dots, n.$$

Next two interesting corollaries of Theorem 2.1 (a) are given.

**Example 5.** Let $c_i \ge 1$ be an integer $(i = 1, \dots, n)$, let $k := \sum_{i=1}^{n} c_i$, and let $I_k = P^{c_1,\dots,c_n}$ consist of all sequences $(i_1, \dots, i_k)$ in which the number of occurrences of $i \in \{1, \dots, n\}$ is $c_i$ $(i = 1, \dots, n)$. Evidently, $(\mathrm{H}_1)$ is satisfied. A simple calculation shows that

$$I_{k-1} = \bigcup_{i=1}^{n} P^{c_1,\dots,c_{i-1},c_i-1,c_{i+1},\dots,c_n}, \quad \alpha_{I_k,i} = \frac{k!}{c_1!\dots c_n!} c_i, \quad i = 1, \dots, n,$$

and

$$t_{I_k,k-1}(i_1, \dots, i_{k-1}) = k,$$
$$\text{if} \quad (i_1, \dots, i_{k-1}) \in P^{c_1,\dots,c_{i-1},c_i-1,c_{i+1},\dots,c_n}, \quad i = 1, \dots, n.$$

According to Theorem 2.1 (a)

$$f\left(\sum_{r=1}^{n} p_r M_r\right) = A_{k,k}$$

$$= \frac{c_1!\dots c_n!}{k!} \sum_{(i_1,\dots,i_k)\in I_k} f\left(\sum_{s=1}^{k} \frac{p_{i_s}}{c_{i_s}} M_{i_s}\right) \ge A_{k,k-1}$$

$$= \frac{1}{k-1} \sum_{i=1}^{n} c_i f\left(\sum_{r=1}^{n} p_r M_r - \frac{p_i}{c_i} M_i\right) \ge \sum_{r=1}^{n} p_r f(M_r).$$

**Example 6.** Let

$$I_2 := \left\{(i_1, i_2) \in \{1, \ldots, n\}^2 \mid i_1|i_2\right\}.$$

The notation $i_1|i_2$ means that $i_1$ divides $i_2$. Since $i|i$ $(i = 1, \ldots, n)$, $(H_1)$ holds. In this case

$$\alpha_{I_2,i} = \left[\frac{n}{i}\right] + d(i), \quad i = 1, \ldots, n,$$

where $\left[\frac{n}{i}\right]$ is the largest natural number that does not exceed $\frac{n}{i}$, and $d(i)$ denotes the number of positive divisors of $i$. By Theorem 2.1 (a), we have

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq$$

$$= \sum_{(i_1,i_2)\in I_2} f\left(\frac{p_{i_1}}{\left[\frac{n}{i_1}\right] + d(i_1)} M_{i_1} + \frac{p_{i_2}}{\left[\frac{n}{i_2}\right] + d(i_2)} M_{i_2}\right) \geq \sum_{r=1}^{n} p_r f(M_r).$$

## 4 Generalizations

In this section, we give the generalization of some refinements given in Section 2. Here we consider the notations developed in [1].
Let $X$ be a set. The power set of $X$ is denoted by $P(X)$. $|X|$ means the number of elements in $X$. For every nonnegative integer $m$, let

$$P_m(X) := \{Y \subset X \mid |Y| = m\}.$$

We need to introduce two further hypotheses:
$(H_6)$ Let $S_1, \ldots, S_n$ be finite, pairwise disjoint and nonempty sets, let

$$S := \bigcup_{j=1}^{n} S_j,$$

and let $c$ be a function from $S$ into $\mathbb{R}$ such that

$$c(s) > 0, \quad s \in S, \quad \text{and} \quad \sum_{s\in S_j} c(s) = 1, \quad j = 1, \ldots, n.$$

Let the function $\tau : S \to \{1, \ldots, n\}$ be defined by

$$\tau(s) := j, \quad \text{if} \quad s \in S_j.$$

$(H_7)$ Suppose $\mathcal{A} \subset P(S)$ is a partition of $S$ into pairwise disjoint and nonempty sets. Let

$$k := \max\{|A| \mid A \in \mathcal{A}\},$$

and let

$$\mathcal{A}_l := \{A \in \mathcal{A} \mid |A| = l\}, \quad l = 1, \ldots, k.$$

We note that $\mathcal{A}_l$ $(l = 1, \ldots, k-1)$ may be the empty set, and of course, $|S| = \sum_{l=1}^{k} l\,|\mathcal{A}_l|$. The empty sum of numbers or vectors is taken to be zero whereas empty product is taken to be one.

By virtue of the above considerations we give another refinement of the first inequality in (1.1).

**Theorem 4.1.** *If ($H_2$-$H_4$) and ($H_6$-$H_7$) are satisfied, then*

$$f\left(\sum_{j=1}^{n} p_j M_j\right) \geq N_k \geq N_{k-1} \geq \ldots \geq N_2 \geq N_1 = \sum_{j=1}^{n} p_j f(M_j),$$

*where*

$$N_k := \sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(\left(\sum_{s\in A} c(s)p_{\tau(s)}\right) f\left(\frac{\sum\limits_{s\in A} c(s)p_{\tau(s)}M_{\tau(s)}}{\sum\limits_{s\in A} c(s)p_{\tau(s)}}\right)\right)\right) \tag{4.1}$$
$$= \sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(f\left(\sum_{s\in A} c(s)p_{\tau(s)}M_{\tau(s)}\right)\right)\right),$$

*and for every $1 \leq m \leq k-1$ the number $N_{k-m}$ is given by*

$$N_{k-m} := \sum_{l=1}^{m}\left(\sum_{A\in\mathcal{A}_l}\left(\sum_{s\in A} c(s)p_{\tau(s)}f(M_{\tau(s)})\right)\right) + \sum_{l=m+1}^{k}\left(\frac{m!}{(l-1)\ldots(l-m)}\right.$$
$$\left.\cdot\sum_{A\in\mathcal{A}_l}\left(\sum_{B\in P_{l-m}(A)}\left(\left(\sum_{s\in B} c(s)p_{\tau(s)}\right) f\left(\frac{\sum\limits_{s\in B} c(s)p_{\tau(s)}M_{\tau(s)}}{\sum\limits_{s\in B} c(s)p_{\tau(s)}}\right)\right)\right)\right)$$
$$= \sum_{l=1}^{m}\left(\sum_{A\in\mathcal{A}_l}\left(\sum_{s\in A} c(s)p_{\tau(s)}f(M_{\tau(s)})\right)\right) + \sum_{l=m+1}^{k}\left(\frac{m!}{(l-1)\ldots(l-m)}\right.$$
$$\left.\cdot\sum_{A\in\mathcal{A}_l}\left(\sum_{B\in P_{l-m}(A)}\left(f\left(\sum_{s\in B} c(s)p_{\tau(s)}M_{\tau(s)}\right)\right)\right)\right).$$

*Proof.* We can prove as in Theorem 2.1, by applying Theorem 1 in [1]. □

## 5 Discussion and applications

The first application of Theorem 4.1 leads to a generalization of Theorem 2.1.

**Theorem 5.1.** *Assume ($H_1$)-($H_4$) are satisfied. For $j = 1, \ldots, n$, we introduce the sets*

$$S_j := \{((i_1, \ldots, i_k), l) \mid (i_1, \ldots, i_k) \in I_k, \quad 1 \leq l \leq k, \quad i_l = j\}.$$

*Let $c$ be a positive function on $S := \bigcup_{j=1}^{n} S_j$ such that*

$$\sum_{((i_1,\ldots,i_k),l)\in S_j} c\left(\left(i_1,\ldots,i_k\right),l\right) = 1, \quad j = 1,\ldots,n.$$

*Then we have*

$$f\left(\sum_{j=1}^{n} p_j M_j\right) \geq N_k \geq N_{k-1} \geq \ldots \geq N_2 \geq N_1 = \sum_{j=1}^{n} p_j f(M_j), \tag{5.1}$$

*where the numbers $N_{k-m}$ $(0 \leq m \leq k-1)$ can be written in the following forms:*

$$N_k = \sum_{(i_1,\ldots,i_k)\in I_k} \left(f\left(\sum_{l=1}^{k} c\left(\left(i_1,\ldots,i_k\right),l\right) p_{i_l} M_{i_l}\right)\right),$$

*and for every $1 \leq m \leq k-1$*

$$N_{k-m} := \frac{m!}{(k-1)\ldots(k-m)} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{1\leq l_1<\ldots<l_{k-m}\leq k}\right.$$

$$\left.\left(f\left(\sum_{l=1}^{k-m} c\left(\left(i_1,\ldots,i_k\right),l_j\right) p_{i_{l_j}} M_{i_{l_j}}\right)\right)\right).$$

An immediate consequence of the previous result is Theorem 2.1: by choosing

$$c\left(\left(i_1,\ldots,i_k\right),l\right) := \frac{1}{|S_j|} = \frac{1}{\alpha_{I_k,j}} \quad \text{if} \quad \left(\left(i_1,\ldots,i_k\right),l\right) \in S_j,$$

we can see that the inequality (5.1) corresponds to the inequality (2.2).

By applying Theorem 5.1 to either the set

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \leq k \leq n,$$

or the set

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 \leq \ldots \leq i_k\right\}, \quad 1 \leq k,$$

generalizations of Example 1 and Example 2 are obtained. Therefore Theorem 4.1 also provides the generalizations of the corresponding results given in [6].

Now we apply Theorem 4.1 to some special situations based on examples in [1].

**Example 7.** Let $n$, $m$, $r$ be fixed integers, where $n \geq 3$, $m \geq 2$ and $1 \leq r \leq n-2$. In this example, for every $i = 1, 2, \ldots, n$ and for every $l = 0, 1, \ldots, r$ the integer $i + l$ will be identified with the uniquely determined integer $j$ from $\{1,\ldots,n\}$ for which

$$l + i \equiv j \pmod{n}. \tag{5.2}$$

Introducing the notation

$$D := \{1, \ldots, n\} \times \{0, \ldots, r\},$$

let for every $j \in \{1, \ldots, n\}$

$$S_j := \{(i,l) \in D \mid i + l \equiv j \pmod{n}\} \bigcup \{j\},$$

and let $\mathcal{A} \subset P(S)$ ($S := \bigcup_{j=1}^{n} S_j$) contain the following sets:

$$A_i := \{(i,l) \in D \mid l = 0, \ldots, r\}, \quad i = 1, \ldots, n$$

and

$$A := \{1, \ldots, n\}.$$

Let $c$ be a positive function on $S$ such that

$$\sum_{(i,l)\in S_j} c\,(i,l) + c\,(j) = 1, \quad j = 1, \ldots, n.$$

A careful verification shows that the sets $S_1, \ldots, S_n$, the partition $\mathcal{A}$ and the function $c$ defined above satisfy the conditions ($H_6$) and ($H_7$),

$$\tau\,(i,l) = i + l, \quad (i,l) \in D,$$

(by the agreement (see (5.2)), $i + l$ is identified with $j$)

$$\tau\,(j) = j, \quad j = 1, \ldots, n,$$

$$|S_j| = r + 2, \quad j = 1, \ldots, n,$$

and

$$|A_i| = r + 1, \quad i = 1, \ldots, n, \quad |A| = n.$$

Now we suppose that ($H_2$)-($H_4$) are satisfied. Then by Theorem 4.1, we have

$$f\left(\sum_{j=1}^{n} p_j M_j\right) \geq N_n = \sum_{i=1}^{n}\left(f\left(\sum_{l=0}^{r} c\,(i,l)\,p_{i+l} M_{i+l}\right)\right) \tag{5.3}$$

$$+ f\left(\sum_{j=1}^{n} c(j) p_j M_j\right) \geq \sum_{j=1}^{n} p_j f(M_j).$$

In case

$$p_j := \frac{1}{n}, \quad j = 1, \ldots, n,$$

$$c\,(i,l) := \frac{1}{m\,(r+1)}, \quad (i,l) \in D, \quad c(j) := \frac{m-1}{m} \quad j = 1, \ldots, n,$$

it follows from (5.3) and (4.1) that

$$f\left(\frac{1}{n}\sum_{j=1}^{n} M_j\right) \geq \frac{1}{mn}\sum_{i=1}^{n} f\left(\frac{M_i + M_{i+1} + \ldots + M_{i+r}}{r+1}\right)$$

$$+\frac{m-1}{m} f\left(\frac{1}{n}\sum_{j=1}^{n} M_j\right) \geq \frac{1}{n}\sum_{j=1}^{n} f(M_j).$$

**Example 8.** Let $n$ and $k$ be fixed positive integers. Let

$$D := \left\{(i_1, \ldots, i_n) \in \{1, \ldots, k\}^n \mid i_1 + \ldots + i_n = n + k - 1\right\},$$

and for each $j = 1, \ldots, n$, denote $S_j$ the set

$$S_j := D \times \{j\}.$$

For every $(i_1, \ldots, i_n) \in D$ designate by $A_{(i_1,\ldots,i_n)}$ the set

$$A_{(i_1,\ldots,i_n)} := \left\{\left((i_1, \ldots, i_n), l\right) \mid l = 1, \ldots, n\right\}.$$

It is obvious that $S_j$ $(j = 1, \ldots, n)$ and $A_{(i_1,\ldots,i_n)}$ $((i_1, \ldots, i_n) \in D)$ are decompositions of $S := \bigcup_{j=1}^{n} S_j$ into pairwise disjoint and nonempty sets, respectively. Let $c$ be a function on $S$ such that

$$c\left((i_1, \ldots, i_n), j\right) > 0, \quad \left((i_1, \ldots, i_n), j\right) \in S$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} c\left((i_1, \ldots, i_n), j\right) = 1, \quad j = 1, \ldots, n. \tag{5.4}$$

In summary we have that the conditions (H$_6$) and (H$_7$) are valid, and

$$\tau\left((i_1, \ldots, i_n), j\right) = j, \quad \left((i_1, \ldots, i_n), j\right) \in S.$$

Now we suppose that (H$_2$)-(H$_4$) are satisfied. Then by Theorem 4.1, we have

$$f\left(\sum_{j=1}^{n} p_j M_j\right) \geq N_n = \sum_{(i_1,\ldots,i_n)\in D} f\left(\sum_{l=1}^{n} c\left((i_1, \ldots, i_n), l\right) p_l M_l\right)$$

$$\geq \sum_{j=1}^{n} p_j f(M_j). \tag{5.5}$$

If we set

$$p_j := \frac{1}{n}, \quad j = 1, \ldots, n,$$

and

$$c\left((i_1, \ldots, i_n), j\right) := \frac{i_j}{\binom{n+k-1}{k-1}},$$

then (5.4) holds, since by some combinatorial considerations

$$|D| = \binom{n+k-2}{n-1},$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} i_j = \frac{n+k-1}{n}\binom{n+k-2}{n-1} = \binom{n+k-1}{k-1}, \quad j=1,\ldots,n.$$

In this situation (5.5) can therefore be expressed thus

$$f\left(\frac{1}{n}\sum_{j=1}^{n} M_j\right) \geq \frac{1}{\binom{n+k-2}{k-1}} \sum_{(i_1,\ldots,i_n)\in D} f\left(\frac{1}{n+k-1}\sum_{l=1}^{n} i_l M_l\right) \geq \frac{1}{n}\sum_{j=1}^{n} f(M_j).$$

## 6 Parameter dependent refinements

Now, we give parameter dependent refinements for determinantal inequalities of Jensen's type. We use the constructions introduced by L. Horváth in [2].

**Theorem 6.1.** *Let $\lambda \geq 1$ be a real number. Suppose ($H_2$)-($H_4$) are satisfied, consider the sets*

$$S_k := \left\{(i_1,\ldots,i_n)\in \mathbb{N}^n \mid \sum_{j=1}^{n} i_j = k\right\}, \quad k\in\mathbb{N},$$

*and for $k\in\mathbb{N}$ define the numbers*

$$C_k(\lambda) = C_k(M_1,\ldots,M_n;p_1,\ldots,p_n;\lambda)$$

$$:= \frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\ldots,i_n)\in S_k}\frac{k!}{i_1!\ldots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}p_j\right) f\left(\frac{\sum\limits_{j=1}^{n}\lambda^{i_j}p_jM_j}{\sum\limits_{j=1}^{n}\lambda^{i_j}p_j}\right)$$

$$= \frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\ldots,i_n)\in S_k}\frac{k!}{i_1!\ldots i_n!} f\left(\sum_{j=1}^{n}\lambda^{i_j}p_jM_j\right).$$

*Then*

$$f\left(\sum_{j=1}^{n}p_jM_j\right) = C_0(\lambda)\geq C_1(\lambda)\geq\ldots\geq C_k(\lambda)\geq\ldots\geq\sum_{j=1}^{n}p_jf(M_j), \quad k\in\mathbb{N}.$$

*Proof.* It is similar to the proof of Theorem 2.1, by applying Theorem 1 in [2]. □

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