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DEGENERATION OF STEKLOV–TYPE BOUNDARY CONDITIONS IN ONE SPECTRAL HOMOGENIZATION PROBLEM

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Abstract. We consider a singularly perturbed Steklov–type problem for the second order linear elliptic equation in a bounded two–dimensional domain. We assume that the Steklov spectral condition rapidly alternates with the homogeneous Dirichlet condition on the boundary. The alternating parts of the boundary with the Dirichlet and Steklov conditions have the same small length of order ε. It is proved that when the small parameter tends to zero the eigenvalues of this problem degenerate, i.e. they tend to infinity. Moreover, it is proved that the eigenvalues of the initial problem are of order ε^{-1} when ε tends to zero.

1 Introduction

Problems with boundary conditions of rapidly changing type of has been attracting the attention of mathematicians for approximately thirty years. There exist papers with pure mathematical problems as well as applied problems with such perturbation of boundary conditions. In such problems it is supposed that the boundary of the domain is divided into two parts with different boundary conditions. Moreover it is supposed that both parts have a microinhomogeneous structure. It means that in the two–dimensional case both sets are unions of a large number of nonintersecting small curve segments with vanishing lengths as the small parameter tends to zero. And in a multi–dimensional case one of the parts is a union of a large number of spots with the sizes depending on the small parameter. One studies an asymptotic behavior of the solutions and the eigenelements of a boundary–value problem in this domain with such boundary conditions as the small parameter tends to zero. Note that the works, where the authors studied the problems of this type, appeared in the 80–s (see for example [13], [14], [33], [34], and [43]). Then, one can find many other papers of 90-th, 2000-th and 2010-th which continue investigations of such problems (see, for instance, [1] — [3], [6] — [12], [15] — [29], [35], [37] — [40], [44], and [56]). The main idea of these papers could be formulated in the following way: a solution of the boundary–value problem with rapidly alternating type of boundary conditions converges as the small parameter characterizing the size of the microstructure, tends to zero, to a solution of the problem with so-called effective boundary conditions not depending on the small parameter. In the limit (homogenized) problems (with effective boundary conditions) the solution depends on the ratio between the sizes of the parts of the boundary with different types of boundary conditions in the initial problem. In papers [15], [22], [23] and [25] the authors consider the alternation of the Dirichlet boundary conditions and the Neumann or Fourier (Robin or third type) boundary conditions. Under the assumption of periodicity of the microstructure there were proved some estimates of the rate of convergence. In [15] the author gives a complete classification of all the cases depending on different ratios of the small parameters (the length of the parts of the boundary in the given problem). In the paper there are the estimates of the deviation of the solutions to the initial problem from the solution to the respective homogenized problem. Moreover the author studied the spectral properties of these problems. In [23] it are considered problems with rapidly changing type of boundary conditions in multi–dimensional domains. Namely, it was proved that the structure of the homogenized problem depends on the asymptotics of the first eigenvalue the respective spectral cell problem. This asymptotics was constructed by the authors and it was applied for the estimation of the rate of convergence of solutions to the initial problem to the respective solution of the homogenized problem as the small parameter tens to zero. In [24] the authors studied boundary-value problems for the Laplacian in three–dimensional domain. It is assumed that the boundary of the domain consists of two parts, one of which has a periodic micro structure. For instance, it consists of periodically situated spots or holes. In the first case there is a bounded domain with a microinhomogeneous structure of the boundary, in the second case there are two domains connected by these holes. In the second case the authors obtain two different limit problems (in two subdomains). The authors provide the complete classification of the homogenized problems by their dependance on the size of the small parameters characterizing the periodical changing of conditions rate. Moreover the respecting spectral problems are considered. The convergence theorems for their eigenvalues and eigenfunctions are proved. The asymptotic expansions of the solutions to some boundary value problems with rapidly changing type of boundary conditions was constructed in [6] — [8], [12], [16], [22], [35], and [38] — [40]. In the papers [6], [9], [38] — [40], and [56] two–dimensional problems are considered. In [16] the author constructs a complete asymptotic expansion for the solution to the Poisson equation in multi–dimensional layer with rapidly changing type of boundary conditions. In addition, the complete expansion of the eigenvalues to the Laplace operator in a cylinder with rapidly alternating boundary conditions of the Dirichlet and Neumann types was constructed in [8] and [10]. In [10] the author considers the case of alternating Neumann and Fourier boundary conditions. The special case of the Dirichlet boundary conditions on the lateral part of the boundary was considered in [8]. The author assumes that the size of the parts of the boundary with the Dirichlet conditions has the same order as the size of the parts with the Neumann conditions. In these papers the author proves that the eigenvalues of the initial problem has the multiplicity less than or equal to two (i.e. either they are simple or double). In addition in [10] the author constructed the leading terms of the asymptotic expansions of the eigenvalues and eigenfunctions in the case of other boundary conditions on the lateral part of the boundary in more general situation. Also it was proved that these eigenvalues converge to the respective simple eigenvalues of the limit (homogenized) problem. In [11] the author studies a singularly perturbed spectral problem for the Laplacian in a cylinder with rapidly changing boundary conditions on a lateral surface, divided into many stripes with the alternation of Dirichlet and Neumann boundary conditions on them. The leading terms of the asymptotic expansion of the eigenvalues was constructed in the case of slow deformation of the width of the strips. In addition for rapidly changing width of the strips the author derived the estimates of the rate of convergence of eigenelements to the initial problem. In [11] the author generalized the results of the paper [7]. Also it should be noted that there exist a number of papers with rapidly alternating boundary conditions and singularly perturbed density (see, for instance, [17] — [21], [28], and [29]). Papers [28] and [29] are devoted to aperiodic case. It is assumed that the Dirichlet boundary condition is set on the parts of the outer boundary in a neighborhood of concentrated masses and it rapidly alternates with Neumann boundary condition. In addition, these concentrated masses has aperiodic structure. The authors proved the homogenization theorem and estimated the rate of convergence of respective eigenelements. In the papers [17] — [21] the authors considered the periodic case. They assumed that concentrated masses are situated periodically along flat part of the boundary. In these papers it was also proved the homogenization theorem and constructed asymptotic expansions of the respective eigenpairs.

Next, we discuss a little bit the Steklov–type spectral problems. One can find papers devoted to different aspects of investigations of this problem starting from [54], [55] (see, for example [4], [5], [31], [36], [41], [42], [45], [46], [50], and also a close work [53]). The paper [41] is devoted to the investigations of a spectral Steklov–type problem in a thin domain with a nonsmooth boundary. The authors constructed leading terms of the asymptotic expansion of eigenvalues and eigenfunctions. In paper [5] the authors derive the formula showing the connection of the first eigenvalue to a Steklov–type problem in the domain with micro perforation and the constant in the Sobolev inequality for traces. In the paper one can find the finite elements method for the approximation of the optimal shape of the cavities. In [50] the author considers a domain $\Omega \subset \mathbb{R}^2$ and studies the asymptotic behavior of the eigenvalues and respective eigenfunctions to the Steklov–type spectral problem depending on the small parameter ε, as $\varepsilon \to 0$. It is assumed that the Steklov condition is set on the small periodically alternating parts of the boundary having the length of order ε. The Dirichlet condition is set on the leftover part of the boundary. To prove the homogenization theorem for this problem the author studies the local spectral problem in the cell of periodicity, also the author describes low frequencies of the homogenized (limit) problem. In [36] the author considers the Steklov–type problem for the p-Laplacian. The existence is proved of infinite number of eigenvalues. In [46] the authors investigate the Steklov spectral problem in a domain with a degenerate corner point on the boundary. They state that the spectrum on the real nonnegative semi-axis can be discrete as well as continuous, depending on the characteristics of sharpening. The elliptic problem with critical growth with Steklov–type conditions in a bounded domain was considered in [4]. The existence of a nontrivial nodal solution is proved. The authors used the estimates for the accumulation (concentration) of Sobolev minimizers on the boundary. In the paper [31] the author proved the homogenization result for the problem with rapidly alternating boundary conditions (the Dirichlet and the Steklov conditions) in the case of the Steklov spectral condition in the limit, i.e. the limit (homogenized) problem has the classical Steklov condition without the alternation. In the paper [52] we estimated the eigenfunctions of singularly perturbed Steklov-type problem and proved an inequality for traces of functions from the Sobolev space W_2^1.

In the present paper we study the behavior of the eigenelements to the singularly perturbed Steklov-type problem in the case of the absence of the spectral parameter at the limit, i.e. at the limit we get the degenerate Dirichlet boundary-value problem for a homogeneous elliptic equation of the second order. Hence, the limit (homogenized) problem has the classical homogeneous Dirichlet boundary condition without the alternation and only trivial solution.

In Section 2 we formulate the main Theorem. Section 3 is devoted to auxiliary propositions. In Section 4 we prove the main result.

Some of the results of this paper are formulated in [32].

2 Setting of problem and main theorem

We assume that the domain $\Omega \subset \mathbb{R}^2$ is situated in the upper halfplane (see Figure 1)

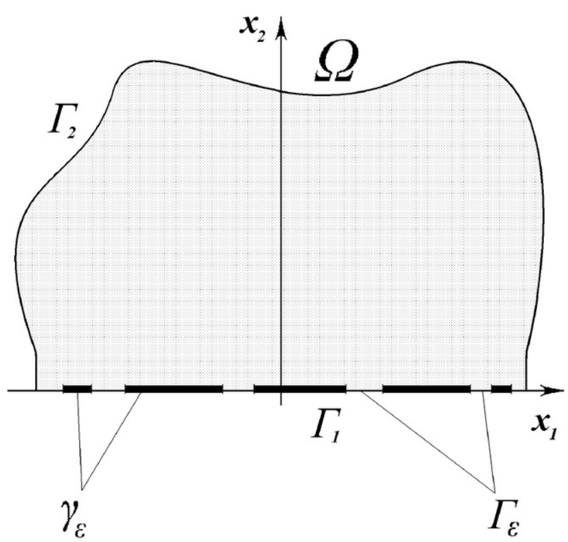

Figure 1: Domain Ω.

with piecewise smooth boundary $\partial\Omega$ which consists of three parts $\partial\Omega = \Gamma_\varepsilon \cup \gamma_\varepsilon \cup \Gamma_2$, where $\Gamma_\varepsilon \cup \gamma_\varepsilon =: \Gamma_1$ is a segment $\left[-\frac{1}{2}; \frac{1}{2}\right]$ on the abscissa axis and the part Γ_2 coincides with the straight lines $x_1 = -\frac{1}{2}$ and $x_1 = \frac{1}{2}$ in the vicinity of the points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$ respectively, Γ_2 is smooth. We suppose that

$$\gamma_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} \gamma_\varepsilon^i, \qquad \Gamma_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} \Gamma_\varepsilon^i,$$

and the parts γ_ε^i and Γ_ε^i rapidly alternate.

Suppose that for any i the following conditions:

$$C^-\varepsilon \le |\Gamma_\varepsilon^i| \le C^+\varepsilon, \quad C^-\varepsilon \le |\gamma_\varepsilon^i| \le C^+\varepsilon, \quad \text{where} \quad 0 < C^- < C^+ < +\infty,$$

hold. Here and throughout ε is a positive small parameter.

Remark 1. It is easy to calculate that N_ε is of order $\frac{1}{\varepsilon}$.

In the domain Ω we consider the following Steklov–type spectral problem:

$$\begin{cases} L[U_\varepsilon] \equiv \sum\limits_{i,j=1}^{2} \dfrac{\partial}{\partial x_j}\left(a^{ij}(x)\dfrac{\partial U_\varepsilon}{\partial x_i}\right) = 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \Gamma_2 \cup \Gamma_\varepsilon \\ \dfrac{\partial U_\varepsilon}{\partial \gamma} \equiv \sum\limits_{i,j=1}^{2} a^{ij}(x)\dfrac{\partial U_\varepsilon}{\partial x_i}\nu_j = \lambda_\varepsilon u_\varepsilon & \text{on } \gamma_\varepsilon. \end{cases} \tag{2.1}$$

Definition 1. A function $u_\varepsilon \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)\backslash\{0\}$ is called an eigenfunction of problem (2.1), corresponding to the eigenvalue λ_ε, if for any function $v \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$ the following integral identity:

$$\int\limits_{\Omega} \sum_{i,j=1}^{2} a^{ij} \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx = \lambda_\varepsilon \int\limits_{\gamma_\varepsilon} u_\varepsilon\, v\, ds \tag{2.2}$$

holds true.

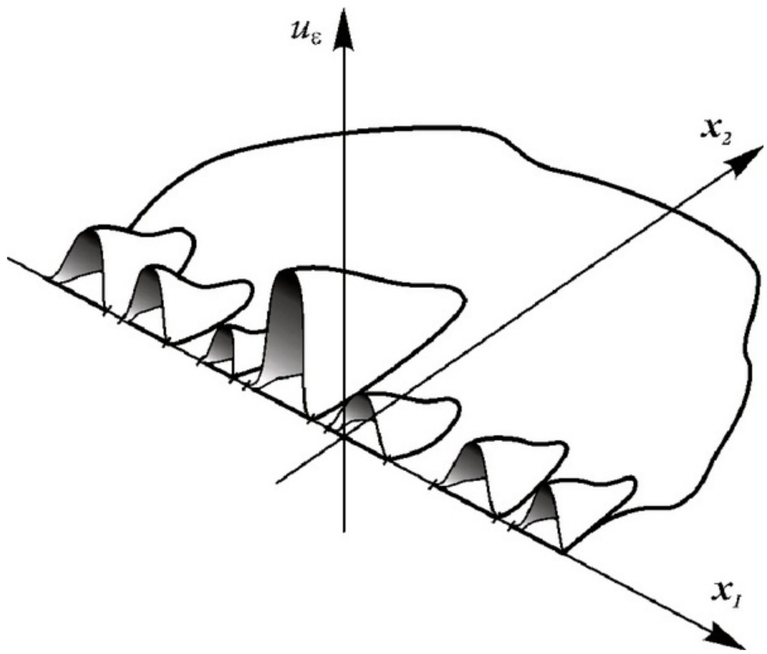

Figure 2: Normalized eigenfunction with “hoods”.

It is known (see for example [57]) that all eigenvalues of problem (2.1) are real and, moreover, positive numbers and they satisfy the following inequality:

$$0 \le \lambda_\varepsilon^1 \le \lambda_\varepsilon^2 \le \dots, \quad \lambda_\varepsilon^k \to \infty \text{ as } k \to \infty.$$

Here we enumerate the eigenvalues λ_ε^k taking into account their multiplicity.

Theorem 2.1. *The first eigenvalue to problem* (2.1) *is of order* $\frac{1}{\varepsilon}$, *i.e. it satisfies the following relation:*

$$\frac{C_1}{\varepsilon} \le \lambda_\varepsilon^1 \le \frac{C_2}{\varepsilon},$$

where C_1 *and* C_2 *are positive constants. Moreover, the first eigenfunction* u_ε^1 *normalized by the condition* $\int\limits_{\Gamma_1} (u_\varepsilon^1)^2 ds = 1$ *converges to zero strongly in* $L_2(\Omega)$ *and weakly in* $H^1(\Omega)$ *as* ε *tends to zero.*

3 Auxiliary propositions

We introduce the following notation:

$$\mu_\varepsilon = \inf_{v \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon) \setminus \{0\}} \frac{\int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx}{\int\limits_{\gamma_\varepsilon} v^2 \, ds}. \tag{3.1}$$

Now we prove the following lemma (an analogous lemma was proved in [15]).

Lemma 3.1. *The number* μ_ε *is the first eigenvalue* λ_ε^1 *of problem* (2.1).

Proof. It is sufficient to show that there exists an eigenfunction u^1 of problem (2.1) corresponding to the first eigenvalue λ_ε^1, which satisfies the relation:

$$\mu_\varepsilon = \frac{\int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial u^1}{\partial x_i} \frac{\partial u^1}{\partial x_j} \, dx}{\int\limits_{\gamma_\varepsilon} (u^1)^2 \, ds} = \lambda_\varepsilon^1 .$$

Suppose that $\{v^{(n)}\}$ is a minimizing sequence for (3.1), i.e.

$$v^{(n)} \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon), \ \|v^{(n)}\|^2{}_{L_2(\gamma_\varepsilon)} = 1,$$

and

$$\int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v^{(n)}}{\partial x_i} \frac{\partial v^{(n)}}{\partial x_j} \, dx \to \mu_\varepsilon, \qquad \text{as} \quad n \to \infty.$$

It is clear that $\{v^{(n)}\}$ is bounded in $H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$. Hence, by the Rellich theorem there exists a subsequence $\{v^{(k)}\}$ weakly convergent in $H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$ and strongly in $L_2(\gamma_\varepsilon)$. Consequently for any $\eta > 0$ there exists such $K = K(\eta)$, that

$$\|v^{(k)} - v^{(l)}\|^2{}_{L_2(\gamma_\varepsilon)} < \eta \qquad \text{as} \quad k, l > K.$$

Using the parallelogram equality written in the following form

$$\left\|\frac{v^{(k)}+v^{(l)}}{2}\right\|^2_{L_2(\gamma_\varepsilon)} = \frac{1}{2}\left\|v^{(k)}\right\|^2_{L_2(\gamma_\varepsilon)} + \frac{1}{2}\left\|v^{(l)}\right\|^2_{L_2(\gamma_\varepsilon)} - \left\|\frac{v^{(k)}-v^{(l)}}{2}\right\|^2_{L_2(\gamma_\varepsilon)},$$

we get

$$\left\|\frac{v^{(k)}+v^{(l)}}{2}\right\|^2_{L_2(\gamma_\varepsilon)} > 1-\frac{\eta}{4} \tag{3.2}$$

Due to the definition of μ_ε we conclude that for all functions $v \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$ we have

$$\int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial v^{(n)}}{\partial x_i}\frac{\partial v^{(n)}}{\partial x_j}\,dx \geq \mu_\varepsilon \|v\|^2_{L_2(\gamma_\varepsilon).} \tag{3.3}$$

Inequalities (3.2) and (3.3) lead to the following estimate:

$$\int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial\left(\frac{v^{(k)}+v^{(l)}}{2}\right)}{\partial x_i}\frac{\partial\left(\frac{v^{(k)}+v^{(l)}}{2}\right)}{\partial x_j}\,dx > \mu_\varepsilon\left(1-\frac{\eta}{4}\right).$$

Since $\{v^{(k)}\}$ is a minimizing sequence, there exists $K_1 = K_1(\eta)$ such that

$$\int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial v^{(k)}}{\partial x_i}\frac{\partial v^{(k)}}{\partial x_j}\,dx < \mu_\varepsilon+\eta, \qquad \int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial v^{(l)}}{\partial x_i}\frac{\partial v^{(l)}}{\partial x_j}\,dx < \mu_\varepsilon+\eta \qquad \text{as} \quad k,l > K_1$$

for any η. Without loss of generality we assume that $K_1 = K$ does not depend on ε. Then,

$$\int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial\left(\frac{v^{(k)}-v^{(l)}}{2}\right)}{\partial x_i}\frac{\partial\left(\frac{v^{(k)}-v^{(l)}}{2}\right)}{\partial x_j}\,dx = \frac{1}{2}\int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial v^{(k)}}{\partial x_i}\frac{\partial v^{(k)}}{\partial x_j}\,dx$$

$$+\frac{1}{2}\int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial v^{(l)}}{\partial x_i}\frac{\partial v^{(l)}}{\partial x_j}\,dx - \int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial\left(\frac{v^{(k)}+v^{(l)}}{2}\right)}{\partial x_i}\frac{\partial\left(\frac{v^{(k)}+v^{(l)}}{2}\right)}{\partial x_j}\,dx$$

$$\leq \frac{\mu_\varepsilon+\eta}{2} + \frac{\mu_\varepsilon+\eta}{2} - \mu_\varepsilon\left(1-\frac{\eta}{4}\right) = \eta\left(1+\frac{\mu_\varepsilon}{4}\right) \to 0, \qquad \eta \to 0.$$

Finally, due to the completeness of the space $H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$, we conclude that there exists a function $v^* \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$ such that the sequence $\{v^{(k)}\}$ converges to this function in the space $H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$ and

$$\int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial v^*}{\partial x_i}\frac{\partial v^*}{\partial x_j}\,dx = \mu_\varepsilon, \qquad \|v^*\|^2_{L_2(\gamma_\varepsilon)} = 1.$$

Given an arbitrary function $v \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)$ we define the following function:

$$g(t) = \frac{\int\limits_\Omega \sum_{i,j=1}^{2} a^{ij} \frac{\partial(v^*+tv)}{\partial x_i}\frac{\partial(v^*+tv)}{\partial x_j}\,dx}{\|v^*+tv\|^2_{L_2(\gamma_\varepsilon)}}.$$

The function $g(t)$ is continuously differentiable in a neighborhood of the point $t = 0$. This expression has the minimum equals to μ_ε as $t = 0$. By the Fermat theorem we have

$$0 = g'|_{t=0} = \frac{2\|v^*\|^2_{L_2(\gamma_\varepsilon))} \int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v^*}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx - 2 \int\limits_{\gamma_\varepsilon} v^* v\, ds \int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v^*}{\partial x_i} \frac{\partial v^*}{\partial x_j}\, dx}{\|v^*\|^4_{L_2(\gamma_\varepsilon)} (= 1)}$$

$$= 2 \int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v^*}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx - 2\mu_\varepsilon \int\limits_{\gamma_\varepsilon} v^* v\, ds.$$

Thus, we showed that

$$\int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v^*}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx = \mu_\varepsilon \int\limits_{\gamma_\varepsilon} v^* v\, ds$$

for any $v \in H^1(\Omega, \gamma_\varepsilon)$, that is v^* satisfies the integral identity of the problem (2.1), and in addition

$$\frac{\int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v^*}{\partial x_i} \frac{\partial v^*}{\partial x_j}\, dx}{\int\limits_{\gamma_\varepsilon} v^{*2}\, ds} = \inf_{v \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)\setminus\{0\}} \frac{\int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx}{\int\limits_{\gamma_\varepsilon} v^2\, ds} = \mu_\varepsilon.$$

Consequently, $v^* = u^1$ and $\lambda_\varepsilon^1 = \mu_\varepsilon$. Lemma is proved. □

Let us consider the boundary value problem corresponding to spectral problem (2.1), namely

$$\begin{cases} L[U_\varepsilon] = 0 & \text{in} \quad \Omega, \\ U_\varepsilon = 0 & \text{on} \quad \Gamma_2 \cup \Gamma_\varepsilon, \\ \frac{\partial U_\varepsilon}{\partial \gamma} = g(x) & \text{on} \quad \gamma_\varepsilon \end{cases} \tag{3.4}$$

The following statement can be found in [52].

Lemma 3.2. *The family of solutions u_ε to problem* (3.4) *converges in the norm $L_2(\Omega)$ and weakly in $H^1(\Omega)$ as $\varepsilon \to 0$, to the unique zero solution of the problem*

$$\begin{cases} L[U] = 0 & \textit{in} \quad \Omega, \\ U = 0 & \textit{on} \quad \partial\Omega. \end{cases} \tag{3.5}$$

4 Proof of Theorem 2.1 on convergence of eigenelements

Proof. By Lemma 3.1 and the ellipticity of the matrix a^{ij} we get that for any function $u \in H^1(\Omega, \Gamma_\varepsilon)$ satisfying $\int\limits_{\Gamma_1} u^2 ds = 1$, the following inequality holds:

$$\lambda_\varepsilon^1 = \inf_{v \in H^1(\Omega, \Gamma_2 \cup \Gamma_\varepsilon)\setminus\{0\}} \frac{\int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx}{\int\limits_{\gamma_\varepsilon} v^2\, ds} \le \int\limits_\Omega \sum_{i,j=1}^2 a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx \le \varkappa_1 \int\limits_\Omega |\nabla u|^2\, dx.$$

• In order to obtain an upper bound for the eigenvalue λ_ε^1 of the spectral problem (2.1), keeping in mind the variational definition of eigenvalues (see, for instance, [51]), we construct the following function u. Assume that on the segments γ_ε^i (of order $\mathcal{O}(\varepsilon)$) in the plane (x_1, u) we have isosceles triangle with the height equals to one. Then, instead of smooth "hoods" (see Figure 2) we construct tetrahedrons with the ε-height of the lower base (see Figure 3).

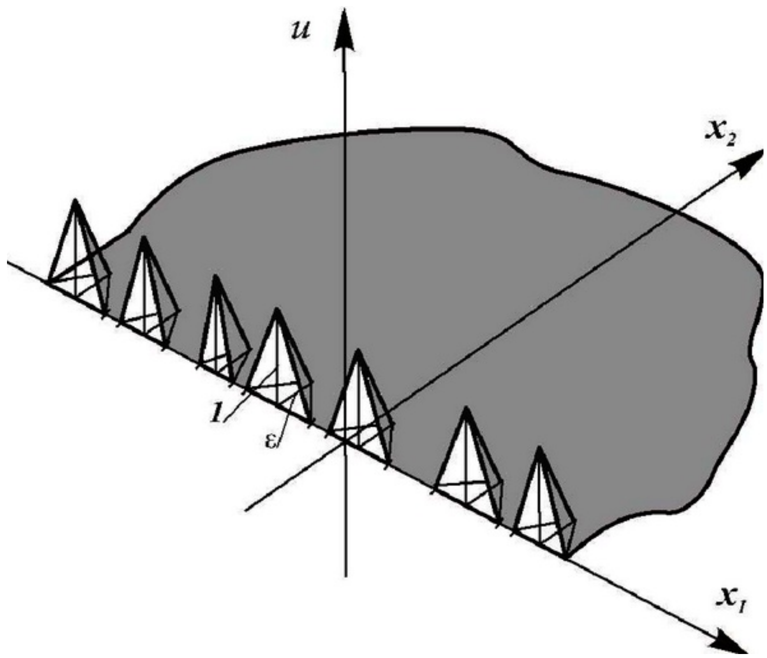

Figure 3: The function u.

Let us estimate $\int\limits_{\Omega} |\nabla u|^2 dx$.

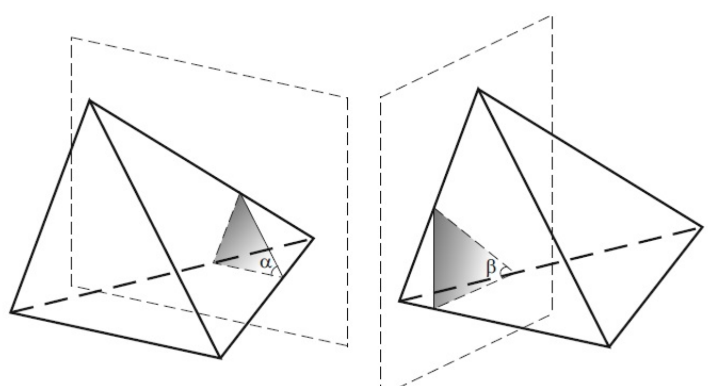

Figure 4: The section of tetrahedrons.

Note that $\tan\alpha = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$ and $\tan\beta = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$ (see Fig. 4). Also it should be noted that $\tan\alpha$ and $\tan\beta$ are independent of the position of the respective parallel cross-sections.

Consequently, we have

$$|\nabla u|^2 = \mathcal{O}\left(\frac{1}{\varepsilon^2}\right). \tag{4.1}$$

The area of the lower base of constructed tetrahedrons equals to $\frac{1}{2} \cdot c_2\varepsilon \cdot \varepsilon$. Hence, bearing in mind Remark 1 and estimate (4.1), we get

$$\int\limits_{\Omega} |\nabla u|^2 dx \le \frac{c_1}{\varepsilon^2} \cdot \frac{1}{2} c_2 \varepsilon^2 \cdot N_\varepsilon \le \frac{c_3}{\varepsilon}.$$

Thus,

$$\lambda_\varepsilon^1 \le \varkappa_1 \frac{c_3}{\varepsilon} \le \frac{C_2}{\varepsilon}.$$

• In order to obtain the lower bound for λ_ε^1 we literally follow the steps of the proof of Theorem 2 from [52]. We split the domain Ω in the strips Ω_ε^i having the width of order of ε, parallel to the axis x_2, with boundaries in points p_i (the centers of the segments Γ_ε^i) (see Figure 5). We use the notation $\Upsilon_\varepsilon^i := \Gamma_\varepsilon \cap \overline{\Omega_\varepsilon^i}$ and $H_*^1 := \bigcup\limits_{i=1}^{N_\varepsilon} H^1(\Omega_\varepsilon^i, \Upsilon_\varepsilon^i)$.

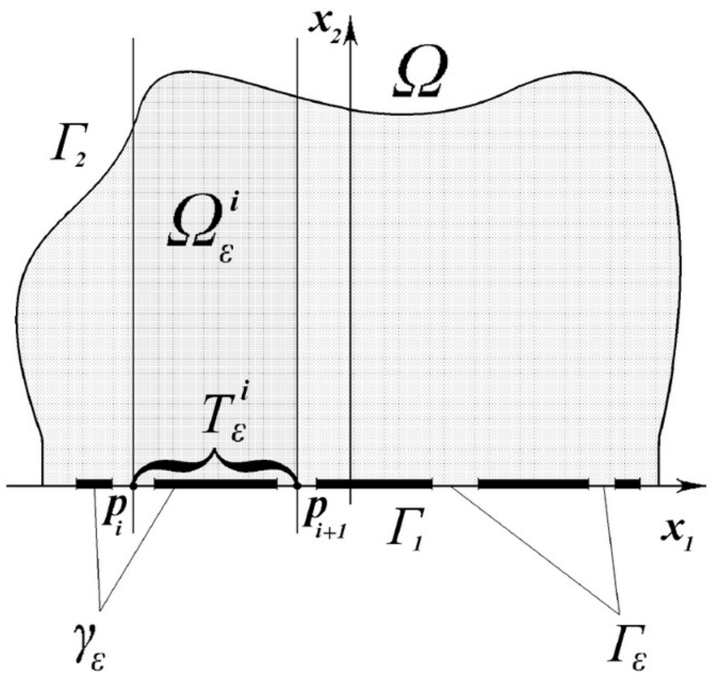

Figure 5: Strips in the domain.

Consider the strip Ω_ε^i. We show that

$$\int\limits_{\Omega_\varepsilon^i} \sum_{i,j=1}^{2} a^{ij} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j}\, dx \ge C_\varepsilon \int\limits_{\gamma_\varepsilon^i} u_\varepsilon^2\, ds \tag{4.2}$$

for normalized functions u_ε in the following sense: $\int\limits_{\gamma_\varepsilon^i} u_\varepsilon^2\, ds = \varepsilon$. Suppose also that $T_\varepsilon^i = \gamma_\varepsilon^i \cup \Upsilon_\varepsilon^i$. We transform the strip Ω_ε^i by means of the change of variables $x \mapsto \xi$, where $\xi = \frac{x}{\varepsilon}$ are the fast variables. Consider the following auxiliary boundary value problem:

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{in} & \Omega^i, \\ v_\varepsilon = 0 & \text{on} & \Upsilon^i, \\ v_\varepsilon = \widetilde{\varphi} & \text{on} & \gamma^i, \\ \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on} & \partial\Omega^i \setminus T^i, \\ \|\widetilde{\varphi}\|^2_{L_2(\gamma^i)} = 1, \end{cases} \tag{4.3}$$

where $\widetilde{\varphi}(\xi) := \varphi(\varepsilon\xi)$ and φ is the trace of the function u_ε on γ^i_ε, Ω^i is the transformed domain in fast variables, and for the transformed parts of the boundary we use the same notation but without the index ε.

Then, we use the statement from [52, Lemma 1].

Lemma 4.1. *The following formula*

$$\int\limits_{T^i} v_\varepsilon(\xi_1, \xi_2) d\xi_1 = const = \int\limits_{T^i} \widetilde{\varphi} d\xi_1 =: C^\varepsilon_\infty \tag{4.4}$$

holds true.

By Lemma 4.1 and the conditions of problem (4.3) it follows that

$$C^\varepsilon_\infty = \int\limits_{T^i} \widetilde{\varphi} d\xi_1 \leq \sqrt{|\gamma^i|} \|\widetilde{\varphi}\|^2_{L_2(\gamma^i)} = \sqrt{|\gamma^i|}. \tag{4.5}$$

By the maximum principle and using the Hopf–Oleinik Lemma (see, for instance, [48]) the maximum (minimum) of the function v_ε can be only on T^i. It is easy to verify that the function $M(\xi_2^*) = \max\limits_{\Omega^i;\xi_2=\xi_2^*} v_\varepsilon(\xi)$ monotonically decreases on ξ_2^*, and the function $m(\xi_2^*) = \min\limits_{\Omega^i;\xi_2=\xi_2^*} v_\varepsilon(\xi)$ monotonically increases.

Consider the difference $\mathbf{osc}_{N+1}(v_\varepsilon) = M(N+1) - m(N+1)$. Without loss of generality we assume that $m(N) = 0$. Further, by using the Harnack theorem on sequence of harmonic functions in the following form (see [49])

$$m(N+1) \geq \alpha M(N+1), \quad 0 < \alpha < 1$$

and the approach of [52] we get the estimate

$$\mathbf{osc}_{N+1}(v_\varepsilon) \leq e^{-\theta N} \, \mathbf{osc}_1(v_\varepsilon), \; \theta > 0.$$

Due to the elliptic estimates (see, for example, [49]) we have

$$|\mathbf{osc}_1(v_\varepsilon)| \leq K_0 \|\widetilde{\varphi}\|^2_{L_2(\gamma^i)}.$$

Thus,

$$\mathbf{osc}_{N+1}(v_\varepsilon) \leq K_0 \|\widetilde{\varphi}\|^2_{L_2(\gamma^i)} e^{-\theta N}, \quad \theta > 0. \tag{4.6}$$

From (4.6) we derive that for any $\delta > 0$ there exists such N_0 that $\mathbf{osc}_{N_0}(v_\varepsilon) = \delta$.

Let us write the following evident inequality:

$$(v_\varepsilon(\xi_1,N_0)-\widetilde{\varphi}(\xi_1))^2=\left(\int\limits_0^{N_0}\frac{\partial v_\varepsilon}{\partial\xi_2}d\xi_2\right)^2\leq N_0\int\limits_0^{N_0}\left(\frac{\partial v_\varepsilon}{\partial\xi_2}\right)^2d\xi_2. \tag{4.7}$$

Integrating (4.7) over T^i, we obtain

$$\begin{aligned}&\int\limits_{T^i}\left(\frac{C^\varepsilon_\infty}{|T^i|}+(v_\varepsilon(\xi_1,N_0)-\frac{C^\varepsilon_\infty}{|T^i|})-\widetilde{\varphi}(\xi_1)\right)^2d\xi_1\\ &=\frac{(C^\varepsilon_\infty)^2}{|T^i|}+\int\limits_{T^i}\left(v_\varepsilon(\xi_1,N_0)-\frac{C^\varepsilon_\infty}{|T^i|}\right)^2d\xi_1+\int\limits_{T^i}|\widetilde{\varphi}(\xi_1)|^2d\xi_1\\ &+2\frac{C^\varepsilon_\infty}{|T^i|}\int\limits_{T^i}\left(v_\varepsilon(\xi_1,N_0)-\frac{C^\varepsilon_\infty}{|T^i|}\right)d\xi_1-2\frac{C^\varepsilon_\infty}{|T^i|}\int\limits_{T^i}\widetilde{\varphi}(\xi_1)d\xi_1\\ &-2\int\limits_{T^i}\widetilde{\varphi}(\xi_1)\left(v_\varepsilon(\xi_1,N_0)-\frac{C^\varepsilon_\infty}{|T^i|}\right)d\xi_1\leq N_0\int\limits_{T^i}\int\limits_0^{N_0}\left(\frac{\partial v_\varepsilon}{\partial\xi_2}\right)^2d\xi.\end{aligned} \tag{4.8}$$

Using (4.3) and (4.4), we rewrite (4.8) in the form

$$\frac{(C^\varepsilon_\infty)^2}{|T^i|}+\delta^2+1+0-2\frac{(C^\varepsilon_\infty)^2}{|T^i|}+\mathcal{O}(\delta)\leq N_0\int\limits_{T^i}\int\limits_0^{N_0}|\nabla_\xi v_\varepsilon|^2d\xi. \tag{4.9}$$

Hence

$$\frac{1-\frac{(C^\varepsilon_\infty)^2}{|T^i|}+\mathcal{O}(\delta)}{N_0}\leq\int\limits_{T^i}\int\limits_0^{N_0}|\nabla_\xi v_\varepsilon|^2d\xi\leq\int\limits_{\Omega^i}|\nabla_\xi v_\varepsilon|^2d\xi.$$

Bearing in mind (4.3) и (4.5), finally we get

$$\frac{1-\frac{(\sqrt{|\gamma^i|})^2}{|T^i|}}{N_0}=\frac{1-\frac{|\gamma^i|}{|T^i|}}{N_0}\int\limits_{\gamma^i}\widetilde{\varphi}^2d\xi_1\leq\int\limits_{\Omega^i}|\nabla_\xi v_\varepsilon|^2d\xi.$$

Transforming variables in the reverse order (from the fast to the initial variables), we derive the inequality

$$\frac{\mathcal{K}}{\varepsilon}\int\limits_{\gamma^i_\varepsilon}\varphi^2dx_1\leq\frac{1}{\varepsilon^2}\varepsilon^2\int\limits_{\Omega^i_\varepsilon}|\nabla_x u_\varepsilon|^2dx.$$

Thus, we proved inequality (4.2) with the constant $C_\varepsilon=\frac{\mathcal{K}}{\varepsilon}$, and the lower bound for λ^1_ε is obtained.

The convergence of the function u^1_ε to zero weak in $H^1(\Omega)$ and strong in $L_2(\Omega)$ follows by Lemma 3.2, bearing in mind the scheme of O.A.Oleinik, G.A. Yosifian and A.S. Shamaev from [47, Ch. III, §1.2].

Now we completed the proof of Theorem 2.1. □

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