Short communications

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IN CATEGORY OF SETS AND RELATIONS, IT IS POSSIBLE TO DESCRIBE FUNCTIONS IN PURELY CATEGORY TERMS

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Abstract. We prove that in the category of sets and relations, it is possible to describe functions in purely category terms.

1 Formulation of the problem

Before we formulate our problem in precise terms, let us first explain why the category of sets and relations is a reasonable way to describe the physical world.

While this idea has been systematically used for a long time in such disciplines as optimal control theory, mathematical theory of dynamical systems, etc. (see, for example, [4]), it is not yet universally accepted. So, for the benefit of the readers who are not familiar with this idea, in this section, we provide the corresponding motivation.

Category of sets and relations is a reasonable way to describe the physical world. Real world consists of systems and objects. These systems and objects usually change – they change if some action is performed, and often they change by themselves, even without any action. In general, a change can be caused by another object – e.g., the state of a measuring instrument changes in response to the change in the environment.

In the simplest case, the resulting state of a system is uniquely determined by the original state. In this case, the change can be represented as a function – for self-change, as a function from a set of states to itself; in other cases, as a function from the set of states of one system to the set of states of another system.

In many cases, the situation is more complex. First, some actions are not always possible; in this case, we have a partially defined function. Second, often, the same action, when applied to the same state, can lead to several possible changed states. In other words, a generic change is described by a partially defined multi-valued function– i.e., in mathematical terms, by a *relation* $R : X \to Y$, i.e., by a subset $R \subseteq X \times Y$. The fact that $(x, y) \in R$ is usually denoted by xRy; this notation is well known for the standard relations such as $=$, $<$, etc.

Thus, we arrive to the following description of the world:

- we have objects – each of which is characterized by a *set* X of its states, and
- we have changes (actions, self-changes, etc.) – each of which is described as a relation $R : X \to Y$.

For each object X, there is an identity relation $\mathrm{id}_X : X \to X$ (defined as $\{(x, x) : x \in X\}$) which corresponds to no changes. Also, for every two relations $R : X \to Y$ describing the effect of X on Y and $S : Y \to Z$ describing the effect of Y on Z, we can describe the resulting indirect effect of X on Z as a composition $R \circ S : X \to Z$ which is defined in a natural way: $x \in X$ can lead to $z \in Z$ if $x \in X$ can lead to some state $y \in Y$ which, in its turn, can lead to z. This corresponds to the usual composition of relations

$$xR \circ Sz \Leftrightarrow \exists y \in Y\, (xRy \,\&\, ySz).$$

Composition is known to be associative, so sets and relations form a *category* – i.e., set of objects X and morphisms $f : X \to Y$ with the notion of a composition $f \circ g$ which is defined for all $f : X \to Y$ and $g : Y \to Z$ and which is associative $(f \circ (g \circ h) = (f \circ g) \circ h)$ for all $f : X \to Y$, $g : Y \to Z$, and $h : Z \to T$; see, e.g., [1, 3]. A category also has, for every object, a special identity morphism id_X for which $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$.

In this category **Rel**:

- objects are sets X,
- morphisms are relations $R : X \to Y$, and
- composition is a usual composition of relations.

Question: in the sets-and-relations category, can we describe functions in purely category terms? A category description is used in many areas of mathematics, because it often allows us to abstract ourselves from the specifics of a given representation. Often, once we represent some property in equivalent purely category terms, it helps us prove results about this property.

From this viewpoint, it is reasonable to ask whether in our sets-and-relations category, the original ideal relations – everywhere defined functions – can be described in purely category terms.

What we do in this paper. In this paper, we show that in the category **Rel** of sets and relations, it is indeed possible to describe functions in purely category terms.

Comment. This result seems to be implicitly assumed in [2], but, to the best of our knowledge, its precise formulation and proof have never been published before.

We therefore believe that, in view of the above-mentioned importance of the category of sets and relations, such a proof will be of interest.

2 Main result

In order to prove that in the category **Rel** of sets and relations, functions can be described in purely category terms, we will prove that several set-theoretic notions can be described in category terms.

Lemma 2.1. *In* **Rel**, *the empty set* $\emptyset$ *can be described in category terms.*

Proof. Indeed, every non-empty set X has at least two different morphisms (relations) $f : X \to X$: an empty relation $R = \emptyset$ and an identity relation id_X. When $X = \emptyset$, then $X \times X = \emptyset$ and thus, the only possible relation $R \subseteq X \times X$ is this empty set.

Thus, among all the objects X, the empty set can be described as the object for which there is exactly one morphism $f : X \to X$. □

Lemma 2.2. *In* **Rel**, *one-element sets can be described in category terms.*

Proof. In a one-element set $X = \{a\}$, we have $X \times X = \{(a, a)\}$, this there are exactly two relations $R \subseteq X \to X$: the empty relation $R = \emptyset$ and the identity relation $\mathrm{id}_X = \{(a, a)\}$. On the other hand, if a set X contains at least two different elements $a \neq b$, then it has at least three different relations $R : X \to X$: $R = \emptyset$, $R = \{(a, a)\}$ and $R = \{(b, b)\}$.

Thus, among all the objects X, one-element sets can be described as objects for which there are exactly two morphisms $f : X \to X$. □

Lemma 2.3. *In* **Rel**, *it is possible to describe subsets* $s \subseteq X$ *in category terms.*

Remark 1. *To be more precise, in this category, it is possible to describe category objects which are in 1-1 correspondence with subsets.*

Proof. Let us fix a one-element set $A = \{a\}$. Then, for every set X, relations $R : A \to X$ are sets $R \subseteq A \times X = \{(a, x) : x \in X\}$. Each such subset has the form $\{a\} \times s$, where $s \subseteq X$ is the set of all the elements $x \in X$ for which $(a, x) \in R$. Vice versa, each such subset $s \subseteq X$ corresponds to a relation $\{a\} \times s$.

Thus, subsets $s \subseteq X$ are in 1-1 correspondence with morphisms $f : A \to X$. So, subsets of the set X can be described as morphisms $f : A \to X$, where A is some fixed one-element set. □

Lemma 2.4. *In* **Rel**, *it is possible to describe empty relations in category terms.*

Proof. If a relation $R : X \to Y$ is an empty set $R = \emptyset$, then for every $S : Y \to Y$, we have $R \circ S = \emptyset$, i.e., $R \to S = R$. Vice versa, if $R \neq \emptyset$, then for $S = \emptyset$, we have $R \circ S = \emptyset \neq R$.

Thus, an empty relation $f : X \to Y$ can be described as a morphism for which $f \circ g = f$ for all morphisms $g : Y \to Y$. □

Lemma 2.5. *In* **Rel**, *it is possible to describe subset relation between subsets in category terms.*

Proof. Let us recall that we have identified each subset $s \subseteq X$ with a relation $\{a\} \times s \subseteq A \times X$. If $s \subseteq s'$, then for every relation $R : X \to Y$, we have $s \circ R \subseteq s' \circ R$. Thus, if $s' \circ R = \emptyset$, then $s \circ R = \emptyset$.

Vice versa, let $s \not\subseteq s'$. This means that there exists an element $x \in s$ for which $x \notin s'$. For $Y = X$ and $R = \{(x, x\}$, we then have $s' \circ R = \emptyset$, while $s \circ R = \{(a, x)\} \neq \emptyset$.

Thus, for subsets $s, s' \subseteq X$, we have $s \subseteq s'$ if and only if for every $R : X \to Y$, $s' \circ Y = \emptyset$ implies $s \circ Y = \emptyset$. □

Lemma 2.6. *In* **Rel***, elements of a set X can be described in category terms.*

Remark 2. *To be more precise, in this category, it is possible to describe category objects which are in 1-1 correspondence with elements.*

Proof. Indeed, elements $x \in X$ are in 1-1 correspondence with 1-element sets $\{x\}$, and 1-element sets $s \subseteq X$ can be described as subsets for which there is exactly one subset $s' \subseteq s$ which is different from s.

Indeed, for $s = \{x\}$, the only subset $s' \subseteq s$ with $s' \neq s$ is $s' = \emptyset$, while if s contains at least two different elements x and x', then, in addition to $s' = \emptyset$, we also have $s' = \{x\} \subseteq s$ and $s' = \{x\} \neq s$. □

Theorem 2.1. *In the sets-and-relations category* **Rel***, functions can be described in category terms.*

Proof. Let us recall that elements $x \in X$ are identified with 1-element sets $\{x\}$ and these sets, in their turn, and identified with relations $\{a\} \times \{x\} = \{(a, x)\} : A \to R$. Let us prove that a relation $R : X \to Y$ is a function if and only if for every element x, the composition $x \times R : A \to Y$ is also a 1-element set.

Indeed, if R is a function, i.e., $R = \{(x, f(x)) : x \in X\}$, then for every x, we have $\{(a, x)\} \circ R = \{(a, f(x))\}$, i.e., a 1-element set. In general, $\{(a, x)\} \circ R = \{(a, y) : xRy\}$. So, this composition is a 1-element set if and only for every x there exists exactly one y with xRy – i.e., if and only if f is a function. □

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