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AXIALLY-SYMMETRIC TOPOLOGICAL CONFIGURATIONS IN THE SKYRME AND FADDEEV CHIRAL MODELS

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Abstract. By definition, in chiral model the field takes values in some homogeneous space G/H. For example, in the Skyrme model (SM) the field is given by the unitary matrix $U \in SU(2)$, and in the Faddeev model (FM) — by the unit 3-vector $\mathbf{n} \in S^2$. Physically interesting configurations in chiral models are endowed with nontrivial topological invariants (charges) Q taking integer values and serving as generators of corresponding homotopic groups. For SM $Q = \deg(S^3 \to S^3)$ and is interpreted as the baryon charge B. For FM it coincides with the Hopf invariant Q_H of the map $S^3 \to S^2$ and is interpreted as the lepton charge. The energy E in SM and FM is estimated from below by some powers of charges: $E_S > \mathrm{const}|Q|$, $E_F > \mathrm{const}|Q_H|^{3/4}$.

We consider static axially-symmetric topological configurations in these models realizing the minimal values of energy in some homotopic classes. As is well-known, for $Q = 1$ in SM the absolute minimum of energy is attained by the so-called hedgehog ansatz (Skyrmion): $U = \exp[\imath\Theta(r)\,\sigma]$, $\sigma = (\sigma\mathbf{r})/r$, $r = |\mathbf{r}|$, where σ stands for Pauli matrices. We prove via the variational method the existence of axially-symmetric configurations (torons) in SM with $|Q| > 1$ and in FM with $|Q_H| \geq 1$, the corresponding minimizing sequences being constructed, with the property of * weak convergence in W^1_∞.

1 Introduction. G-invariant functionals and principle of symmetric criticality

In various physical problems arising in nuclear physics, nonlinear optics, condensed matter physics etc. there appears a necessity of searching for localized structures (solitons) realizing minimums of some G-invariant functionals (Hamiltonians H). Let us consider the field $\varphi(x) : \mathbb{R}^3 \to M$ taking values in some compact manifold M, e. g. sphere, group or homogeneous space, with the natural boundary condition at space infinity:

$$\lim_{x\to\infty} \varphi(x) = \varphi_\infty \tag{1.1}$$

motivated by boundedness of $H[\varphi]$. As is well-known [4], due to (1.1) the fields $\varphi(x)$ can be classified according to homotopy group $\pi_3(M)$, i. e. $\varphi(x)$ is endowed with the topological charge Q serving as the generator of $\pi_3(M)$. If the energy E of the field is bounded from below by some monotonically increasing function of the charge Q:

$$E \geq f(|Q|), \tag{1.2}$$

then one can search for $\varphi(x)$ as the minimum of the Hamiltonian

$$H[\varphi] = E[\varphi] - f(|Q|), \tag{1.3}$$

with Q being fixed. Thus, in some homotopic class the configuration $\varphi(x)$ should be realized as the minimum critical point of the Hamiltonian $H[\varphi]$.

Let us consider the class of G-invariant functionals $H[\varphi]$, which are invariant under the action of some group G, that is $H[\varphi_g] = H[\varphi]$, with φ_g denoting the field $\varphi(x)$ transformed by some element $g \in G$. Let us now introduce the notion of the equivariant field $\varphi_0 = \varphi_{0g}$ defined by the condition

$$\varphi_0(x) = T_g \varphi_0(g^{-1}x), \tag{1.4}$$

where T_g stands for the representation operator.

In physics the so-called reduction problem is very popular, when the G-invariant functional $H[\varphi]$ is restricted to the equivariant class $\Phi_0 = \{\varphi_0(x)\}$, i.e. $H[\varphi] \Longrightarrow H[\varphi_0]$. Then the question arises, whether the critical points of $H[\varphi]$ and $H[\varphi_0]$ coincide (the principle of symmetric criticality). The answer was given by R. Palais [6], who found a sufficient condition for such a coincidence. To sketch his idea, let us denote by X the Frechet derivative of $H[\varphi]$ at the point φ_0:

$$X = (\delta H / \delta\varphi)[\varphi_0] \tag{1.5}$$

and write down the extremum condition for $H[\varphi_0]$ in the set Φ_0:

$$(X, \delta\varphi_0) = 0 \quad \forall\, \delta\varphi_0 \in \Phi_0, \tag{1.6}$$

signifying that $X \in \Phi_0^{\perp}$, where $\Phi_0^{\perp}$ is the annulator of Φ_0. On the other hand, the G-invariance of $H[\varphi]$ yields

$$\delta H = (X, \delta\varphi) = (X_g, \delta\varphi_g) = (X, \delta\varphi_g), \tag{1.7}$$

the property $\varphi_0 = \varphi_{0g}$ being taken into account. Due to arbitrariness of $\delta\varphi_g$ one concludes from (1.7) that

$$X_g = X. \tag{1.8}$$

Let us denote by $\tilde{\Phi_0}$ the class of equivariant Frechet fields (1.8): $X \in \tilde{\Phi_0}$. Then the Palais condition

$$\tilde{\Phi_0} \cap \Phi_0^{\perp} = \emptyset, \tag{1.9}$$

in view of (1.6) and (1.8), amounts to $X = 0$. Condition (1.9) is known as a sufficient condition for the validity of the principle of symmetric criticality. Let us apply this principle to the Skyrme and Faddeev chiral models, for which condition (1.9) is satisfied due to the compactness of the invariance group G [6].

2 Topological solitons in the Skyrme model

In the Skyrme model the energy E reads

$$E = \int dx \left(-\frac{1}{4}\mathrm{tr}(\ell_i)^2 - \frac{1}{16}\mathrm{tr}[\ell_i, \ell_k]^2\right), \tag{2.1}$$

where $\ell_i = U^+\partial_i U$, $U \in SU(2)$, the integration is performed over $\mathbb{R}^3$ and the Einstein summation rule is used ($i, k = 1, 2, 3$). The topological charge Q in SM reads:

$$Q = -\frac{1}{24\pi^2}\epsilon^{ijk}\int dx\,\mathrm{tr}(\ell_i\ell_j\ell_k). \tag{2.2}$$

Using the inequality

$$-\ell_i^2 - \left(\epsilon^{ikj}[\ell_k, \ell_j]\right)^2 \geq 2|\epsilon^{ikj}\ell_i[\ell_k, \ell_j]|,$$

one easily finds the estimate

$$E > 6\pi^2\sqrt{2}|Q|$$

that allows to introduce the Hamiltonian

$$H = E - 6\pi^2\sqrt{2}|Q|, \tag{2.3}$$

which is invariant under the group

$$G = SO(3)_S \otimes SO(3)_I \tag{2.4}$$

including coordinate space rotations $SO(3)_S$ and isotopic rotations $SO(3)_I$ realized as transformations $U \to VUV^{-1}$, $V \in SU(2)$.

As can be easily verified, G-equivariant fields are trivial, and therefore we consider the two evident subgroups:

$$G_1 = \mathrm{diag}\left[SO(3)_S \otimes SO(3)_I\right], \tag{2.5}$$

$$G_2 = \mathrm{diag}\left[SO(2)_S \otimes SO(2)_I\right], \tag{2.6}$$

which include combined rotations in coordinate and isotopic spaces. As for G_1-equivariance, condition (1.4) reads

$$-\imath[\mathbf{r}\bigtriangledown]U + \frac{1}{2}[\sigma, U] = 0 \tag{2.7}$$

and leads to the well-known hedgehog configuration

$$U = \cos\Theta(r) + \imath\sin\Theta(r)(\sigma\mathbf{n}), \quad \mathbf{n} = \mathbf{r}/r, \tag{2.8}$$

first proposed by Skyrme [9]. G_1-equivariant hedgehog configurations are known as spherically-symmetric ones, for which the chiral angle $\Theta(r)$ is given by the monotonically decreasing function satisfying the following boundary conditions:

$$\Theta(0) = N\pi, \quad \Theta(\infty) = 0, \quad N \in \mathbb{Z}. \tag{2.9}$$

Using (2.2) and (2.9), one can easily find that $Q = N$. The existence of such structures can be proved by the variational method [3, 8], $N = 1$-configuration realizing the absolute minimum of the Hamiltonian (2.3). The latter property can be seen as follows.

First of all, for unitary matrix we use the representation $U = \exp[\imath(\sigma\mathbf{n})\Theta]$ and also polar coordinates β, γ for the unit vector $\mathbf{n}$. Then we introduce the three vector fields:

$$\mathbf{X} = \bigtriangledown\Theta, \quad \mathbf{Y} = \sin\Theta\bigtriangledown\beta, \quad \mathbf{Z} = \sin\Theta\sin\beta\bigtriangledown\gamma \tag{2.10}$$

and represent Hamiltonian (2.3) in the form:

$$H = \int dx\left\{\left(\mathbf{X}/\sqrt{2} + [\mathbf{YZ}]\right)^2 + \left(\mathbf{Y}/\sqrt{2} + [\mathbf{ZX}]\right)^2 + \left(\mathbf{Z}/\sqrt{2} + [\mathbf{XY}]\right)^2\right\}. \tag{2.11}$$

As follows from (2.11), the minimum of H corresponds to the anticollinearity of the correspondent pairs of vectors: $\mathbf{X}$, $[\mathbf{YZ}]$, etc. that results in the orthogonality of vectors (2.10) and the dependence of H only on modules $|\mathbf{X}|$, $|\mathbf{Y}|$, $|\mathbf{Z}|$. This fact together with the spherical rearrangement procedure [7] yields the spherical symmetry of $|\mathbf{X}|$, $|\mathbf{Y}|$ and $|\mathbf{Z}|$ and finally one gets the hedgehog ansatz:

$$\Theta = \Theta(r), \quad \beta = \vartheta, \quad \gamma = \alpha, \tag{2.12}$$

where the spherical coordinates r, ϑ, α are used.

3 Axially-symmetric configurations in the Skyrme model

Let us now consider G_2-equivariant configurations in SM with $|Q| \geq 2$. As the group G_2 includes the combined rotations around the third axes in coordinate and isotopic spaces, condition (1.4) reads

$$-\imath\partial_\alpha U + \frac{k}{2}[\sigma_3, U] = 0, \quad k \in \mathbb{Z}, \tag{3.1}$$

with the solution

$$\Theta = \Theta(r,\vartheta), \quad \beta = \beta(r,\vartheta), \quad \gamma = k\alpha + v(r,\vartheta). \tag{3.2}$$

As follows from (2.11), the absolute minimum of H in this class coresponds to the conditions

$$(\mathbf{XZ}) = (\mathbf{YZ}) = 0,$$

which are fulfilled if $v = 0$ in (3.2). For this choice we can exclude the α-coordinate and perform the minimization of the two-dimensional functional (2.11):

$$I[\Theta,\beta] = \frac{1}{2}\int\limits_0^\infty dr\, r^2 \int\limits_0^\pi d\vartheta\, \sin\vartheta\Big[\frac{1}{2}Z^2 + 3\sqrt{2}ZJ + J^2 + \left(\frac{1}{2} + Z^2\right)(\mathbf{X}^2 + \mathbf{Y}^2)\Big], \tag{3.3}$$

where the following denotations for the components of two-dimensional vectors $\mathbf{X} = \{X_1, X_2\}$, $\mathbf{Y} = \{Y_1, Y_2\}$ are used:

$$X_1 = \partial_r\Theta; \quad X_2 = \frac{1}{r}\partial_\vartheta\Theta; \quad Y_1 = \partial_r\beta; \quad Y_2 = \frac{1}{r}\partial_\vartheta\beta;$$
$$Z = \frac{k}{r\sin\vartheta}\sin\Theta\sin\beta; \quad J = X_1Y_2 - X_2Y_1.$$

Let us now obtain some apriori estimates taking into account the Legendre—Hadamard necessary condition for the minimum of functional (3.3) with respect to variations $\delta\mathbf{X} = \mathbf{x}$, $\delta\mathbf{Y} = \mathbf{y}$:

$$A(\mathbf{x}^2 + \mathbf{y}^2) + B[\mathbf{x}\mathbf{y}]_3 + ([\mathbf{x}\mathbf{Y}] + [\mathbf{X}\mathbf{y}])^2 \geq 0, \tag{3.4}$$

where $A = Z^2 + 1/2$, $B = J + 3Z/\sqrt{2}$. As follows from (3.4), one gets $|B| \leq A$, i.e. the boundedness of $|\bigtriangledown\Theta|$ and $|\bigtriangledown\beta|$ in the domain $\Omega \in \mathbb{R}^3$:

$$\Omega = \{|\sin\Theta| \geq \delta, \quad r\sin\vartheta \geq \delta_1, \quad |\sin\beta| \geq \delta_2, \quad r \leq R\} \tag{3.5}$$

for some constants $\delta > 0$, $\delta_1 > 0$, $\delta_2 > 0$, $R < \infty$. Therefore, after restricting the functional I to the domain Ω: $I \searrow I_\Omega$, one can construct the minimizing sequence $\{\Theta_n, \beta_n\} \in W^1_\infty(\Omega)$ with * weak convergence. Moreover, one finds that sequences $\{\sin\Theta_n, \sin\beta_n\}$ strongly converge in $C(\Omega)$ and sequences $\{\mathbf{X}_n, \mathbf{Y}_n\}$ * weakly converge in $L_\infty(\Omega)$, with $\{\mathbf{Y}_n\}$ being weakly converging in $L_2(\Omega)$.

Now one can use Mazur's lemma [1], which states that due to the convexity of I_Ω with respect to $\mathbf{Y}$ one can construct a minimizing convex combination of sequences:

$$\mathrm{con}\mathbf{Y}_n = \sum_{k=n}^{N} \lambda_k \mathbf{Y}_k, \quad \sum_{k=n}^{N} \lambda_k = 1,$$

strongly converging in $L_2(\Omega)$. As a result one gets the product $\{J_n\} = \{[\mathbf{X}_n \mathrm{con}\mathbf{Y}_n]_3\}$ of two sequences. The first one $\{\mathbf{X}_n\}$ * weakly converges in $L_\infty(\Omega)$ and the second one $\{\mathrm{con}\mathbf{Y}_n\}$ strongly converges in $L_2(\Omega)$. Therefore, the product $\{J_n\}$ weakly converges in $L_2(\Omega)$.

Functional (3.3) having the structure of the squared norm in $L_2(\Omega)$, one concludes that I_Ω is weakly semicontinuous from below and $\inf I_\Omega$ is attainable.

4 Topological solitons in the Faddeev model

In FM the field is given by the unit 3-vector:

$$n^a(x) : \mathbb{R}^3 \to S^2, \quad a = 1,2,3, \quad |\mathbf{n}| = 1, \tag{4.1}$$

with the boundary condition

$$n^a(\infty) = \delta^a_3. \tag{4.2}$$

In view of (4.2) there are no spherically-symmetric configurations in FM, and we consider only axially-symmetric ones. To describe the topological properties of Faddeev solitons, we introduce the auxilliary 3-vector a_i, $i = 1,2,3$, defined as follows:

$$\partial_i a_k - \partial_k a_i = 2\epsilon_{abc}\partial_i n^a \partial_k n^b n^c. \tag{4.3}$$

The topological invariant classifying configurations (4.1), (4.2) and serving as the generator of $\pi_3(S^2)$ is the so-called Hopf invariant (Hopf index) Q_H defined by the Whitehead integral

$$Q_H = -(8\pi)^{-2} \int dx\,(\mathbf{a}\mathbf{b}), \quad \mathbf{b} = \mathrm{rot}\mathbf{a}, \tag{4.4}$$

which represents the degree of knottedness of tangled **b**-lines [5] having the analogy with the hydrodynamics. In FM the energy reads

$$E = \int dx \left[\frac{1}{2}\mathbf{b}^2 + (\partial_i n^a)^2\right] \tag{4.5}$$

and can be estimated from below through the Hopf index [10]:

$$E > (4\pi)^2\sqrt{2}\, 3^{3/8}|Q_H|^{3/4} \equiv \mu|Q_H|^{3/4}. \tag{4.6}$$

Therefore, in view of (4.6), the Hamiltonian H in FM can be defined as follows:

$$H[\mathbf{n}] = E - \mu|Q_H|^{3/4}. \tag{4.7}$$

However, for the G_2-equivariant or axially-symmetric configurations we find structure (3.2), i. e. for the polar angles β, γ of $\mathbf{n}$ one has

$$\beta = \beta(\rho, z), \quad \gamma = k\alpha + v(\rho, z), \quad k \in \mathbb{Z}, \tag{4.8}$$

with ρ, α, z being cylindrical coordinates. In this case [8] the Hopf invariant can be transformed into:

$$Q_H = \frac{k}{4\pi}\int\limits_0^\infty d\rho \int\limits_{-\infty}^\infty dz \, \sin\beta \, (\partial_\rho\beta\partial_z v - \partial_z\beta\partial_\rho v) \, , \tag{4.9}$$

that allows to improve estimate (4.6). As can be shown [8], $Q_H \neq 0$ if and only if the function v has the step structure, with the jump $[v] = 2n\pi$, $n \in \mathbb{Z}$, on some line in ρ, z-plane. In this case $Q_H = kn$. Since $\mathbf{b} = 2\sin\beta[\bigtriangledown\beta \bigtriangledown \gamma]$, the surface $\beta =$ const is homeomorphic to torus T^2, with **b**-line being tangent to it and making k windings along T^2 and n transverse windings. This fact allows one to improve estimate (4.6), if we consider the energy functional in the form $E = 4\pi\sqrt{2}I$, $I = I_0 + I_1$, where

$$I_0[\beta] = \frac{1}{2}\int\limits_0^\infty \rho \, d\rho \int\limits_{-\infty}^\infty dz \Big[(\bigtriangledown\beta)^2 + \frac{k^2}{\rho^2}\sin^2\beta\,\big(1 + (\bigtriangledown\beta)^2\big)\Big],$$

$$I_1[\beta, v] = \frac{1}{2}\int\limits_0^\infty \rho \, d\rho \int\limits_{-\infty}^\infty dz \sin^2\beta\Big[(\bigtriangledown v)^2 + [\bigtriangledown\beta \bigtriangledown v]^2\Big].$$

Using simple inequalities:

$$(\bigtriangledown\beta)^2\Big[1 + \frac{k^2}{\rho^2}\sin^2\beta\Big] + \sin^2\beta(\bigtriangledown v)^2 \geq 2\sin\beta|[\bigtriangledown\beta \bigtriangledown v]|\Big[1 + \frac{k^2}{\rho^2}\sin^2\beta\Big]^{1/2},$$

$$\frac{k^2}{\rho^2} + [\bigtriangledown\beta \bigtriangledown v]^2 \geq 2\frac{|k|}{\rho}|[\bigtriangledown\beta \bigtriangledown v]|,$$

one immediately gets that

$$I > 2|k| \, |\int\limits_0^\infty d\rho \int\limits_{-\infty}^\infty dz \, \sin^2\beta[\bigtriangledown\beta \bigtriangledown v]| = 2\pi^2|kn|. \tag{4.10}$$

Now we intend to prove the existence of such configurations via the variational method in plain analogy with SM. Denoting $\mathbf{X} = \bigtriangledown\beta$, $\mathbf{Y} = \sin\beta \bigtriangledown v$, we rewrite H for $k > 0$ in the form:

$$H = \int_0^\infty \rho\, d\rho \int_{-\infty}^\infty dz \Big[\mathbf{X}^2\Big[1+\frac{k^2}{\rho^2}\sin^2\beta\Big]+\mathbf{Y}^2+[\mathbf{XY}]^2+\frac{k^2}{\rho^2}\sin^2\beta-\frac{4k}{\rho}\sin\beta[\mathbf{XY}]_3\Big]. \quad (4.11)$$

Then the necessary condition of minimum for (4.11) implies the inequality

$$\mathbf{x}^2\Big[1+\frac{k^2}{\rho^2}\sin^2\beta\Big] + \mathbf{y}^2 + ([\mathbf{X}\mathbf{y}] + [\mathbf{x}\mathbf{Y}])^2 + 2[\mathbf{XY}]\cdot[\mathbf{xy}] - \frac{4k}{\rho}\sin\beta[\mathbf{xy}]_3 > 0,$$

where $\mathbf{x} = \delta\mathbf{x}$, $\mathbf{y} = \delta\mathbf{Y}$, or equivalently

$$1 + \frac{k^2}{\rho^2}\sin^2\beta > \Big(\frac{2k}{\rho}\sin\beta - [\mathbf{XY}]_3\Big)^2. \quad (4.12)$$

Condition (4.12) is equivalent to the boundedness of derivatives $\bigtriangledown\beta$ and $\bigtriangledown v$ in the domain Ω, where $\sin\beta > \delta > 0$, $\rho > \delta_1 > 0$. Restricting the functional H to Ω, one finds the similarity of the functionals I_Ω in SM and H_Ω. Therefore, one can construct the minimizing sequence β_n, $v_n \in W^1_\infty(\Omega)$ with the * weak convergence and prove the weak semicontinuity of H_Ω from below.

The final step in studying axially-symmetric configurations in SM and FM includes the extension of the domain $\Omega \Longrightarrow \mathbb{R}^3$. To this end it should be remarked that in the complementary domain $\Omega' = \mathbb{R}^3 \backslash \Omega$ the corresponding Lagrange equations become linear and can be solved through separating variables. In view of regular behavior of these solutions in Ω' one can prove the continuous dependence of H_Ω and I_Ω on the domain Ω, the latter fact implying the attainability of $\inf H$ and $\inf I$ for axially-symmetric topological configurations in SM and FM.

5 Conclusion

Using the variational method and lower estimates of the energy functionals in axially-symmetric case through the topological charges in SM and FM, one can prove the existence of localized topological configurations in these models. Corresponding minimizing sequences have the property of * weak convergence in W^1_∞, that is in the topology $\sigma(W^1_\infty, W^1_1)$. The similarity of the Skyrme and Faddeev models based on the inverse Hopf mapping $S^3 \to S^2$ was often used in physics for estimating masses of topological solitons [2].

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