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ON BIFURCATION OF NOETHER POINTS IN DISCRETE SPECTRUM OF LINEAR OPERATORS

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Abstract. In this work in the development of our previous articles [3, 4] the perturbation of the Noether points in discrete spectrum of linear operator-functions is considered. With the help of the perturbed operator regularization an approach is suggested allowing to reduce the problem of its eigenvalues determination (which can turn out to be multiple) to simple ones together with correspondent eigenelements.

1 Introduction

Foundations of perturbation theory of discrete spectrum were created in the F. Rellich articles [5, 6, 7]. His investigations were continued in the works of T. Kato [2], M.I. Vishik and L.A. Ljusternik [11]. V.A. Trenogin has investigated these problems by bifurcation theory methods [8]. Detailed review of his investigation is contained in the monograph [10]. All these works were devoted mainly to perturbation of the Fredholm eigenvalues. At the consideration of the Noetherian points in the discrete spectrum some difficulties arise stipulated by the nonzero index of eigenvalues. However there is S.G. Krein and V.P. Trofimov's result [9] according to which all Noetherian eigenvalues form an analytic set. The dependence of perturbed eigenvalues on a small parameter and their quantity were not investigated there. In this article with the help of the regularization approach suggested in the works [3, 4] such problems are reduced to the perturbation problem for the Fredholm eigenvalues. Terminology and notation of the monograph [10] are used.

2 Reduction of the Noether case to the Fredholm one

Let E_1, E_2 be Banach spaces and $\lambda_0 \in G \subset C$ be a Noether point of a linear bounded holomorphic in $t \in G$ operator-function $A(t) : E_1 \to E_2$, $\{\varphi_{i0}\}_1^n$ and $\{\psi_{i0}\}_1^m$ be basis elements of the eigen- and defect-subspaces $N(A(\lambda_0))$ and $N^*(A(\lambda_0))$ corresponding to λ_0 and $\{\gamma_{i0}\}_1^n \in E_1^*$, $\{z_{i0}\}_1^m \in E_2$ be the appropriate biorthogonal systems. The projection operators $P = \sum\limits_{i=1}^{n} \langle \cdot, \gamma_{i0} \rangle \varphi_{i0}$, $Q = \sum\limits_{i=1}^{m} \langle \cdot, \psi_{i0} \rangle z_{i0}$

generate the expansions $E_1 = E_1^n \dot{+} E_1^{\infty-n}$, $E_2 = E_{2,m} \dot{+} E_{2,\infty-m}$. For definitiveness assume that $n > m$.

Definition 1. We say that elements $\varphi_i^{(1)} \equiv \varphi_i, \varphi_i^{(2)}, \dots, \varphi_i^{(p_i)}$ form a generalized $A(\lambda)$ - Jordan chain of length p_i, corresponding to φ_i, if the following idendities hold true:

$$A(\lambda)\varphi_i^{(s)} = \sum_{j=1}^{s-1} A_j \varphi_i^{(s-j)},\ s = \overline{2, p_i},$$

where $A_j = \frac{1}{j!} A^{(j)}(\lambda)$, $j = 1, 2, 3, \dots$. Moreover, for all defect functionals $\psi_l \in N^*(A(\lambda))$ the equality $\langle \sum_{j=1}^{s-1} A_j \varphi_i^{(s-j)}, \psi_l \rangle = 0$, $s = \overline{2, p_i}$, holds and $\langle \sum_{j=1}^{p_i} A_j \varphi_i^{(p_i+1-j)}, \psi_k \rangle \neq 0$ for at least one functional $\psi_k \in N^*(A(\lambda))$.

Considering the finite-dimensional space F_{n-m} of elements of arbitrary nature with the norm $\|\cdot\|_F$, construct the space $E_2 \dot{+} F_{n-m} = \mathcal{E}_2$ as the space of elements $\{u, a\}$, $u \in E_2, a \in F_{n-m}$ with the norm $\|\{u, a\}\|_3 = \|u\|_2 + \|a\|_F$. In the space $\mathcal{E}_2$ the subspace of the elements $\{u, 0\}$ is selected, isomorphic to E_2 and identified with E_2. Select some basis $\{e_{m+1}, e_{m+2}, ..., e_n\}$ in the space F_{n-m} , set $z_i = \{0, e_i\}$, $i = \overline{m+1, n}$ and introduce the relevant biorthogonal elements $\psi_{m+1}, \psi_{m+2}, ..., \psi_n$. A similar technique of introducing ideal elements is used in the monograph [11].

Consider the operator-function

$$\overline{A(t)} \equiv A(t) + \sum_{i=m+1}^{n} \langle \cdot, \gamma_{i0} \rangle z_{i0}. \tag{2.1}$$

Theorem 2.1. *Any eigenvalue λ_0 of the operator $A(t)$ turns out to be a Fredholm point of the discrete spectrum of operator-function (2.1).*

Proof. The normal solvability of the operator $\overline{A(\lambda_0)}$ follows by the normal solvability of $A(\lambda_0)$. It is required to prove that the zero- and defect-subspaces for $\overline{A(\lambda_0)}$ have equal dimensions. Since $N\left(\overline{A(\lambda_0)}\right) \subset N(A(\lambda_0))$, then the zero and defect-functionals of the operator $\overline{A(\lambda_0)}$ can be found in the form

$$\overline{\varphi_i} = \varphi_{i0} + \sum_{j=m+1}^{n} c_{ij} \varphi_{j0},\ i = \overline{1, m} \tag{2.2}$$

and

$$\overline{\psi_i} = \psi_{i0} + \sum_{j=m+1}^{n} d_{ij} \psi_{j0},\ i = \overline{1, m}. \tag{2.3}$$

Next $\overline{A(\lambda_0)}\overline{\varphi_i} = \sum_{j=m+1}^{n} \langle \varphi_{i0}, \gamma_{j0} \rangle z_{j0} + \sum_{j=m+1}^{n} \sum_{s=m+1}^{n} c_{is} \langle \varphi_{i0}, \gamma_{j0} \rangle z_{j0} = 0$, whence it follows that

$$\sum_{s=m+1}^{n} c_{is} \langle \varphi_{s0}, \gamma_{j0} \rangle = - \langle \varphi_{i0}, \gamma_{j0} \rangle,\ j = \overline{m+1, n}. \tag{2.4}$$

By biorthogonality of the systems $\{\varphi_{i0}\}_1^n$ and $\{\gamma_{i0}\}_1^n$ the right-hand sides of system (2.4) are zeros, therefore for all $i, s = \overline{m+1, n}$ $c_{is} = 0$, whence it follows that $N\left(\overline{A(\lambda_0)}\right) = \{\varphi_{i0}\}_1^m$. By a similar argument it follows that $N^*\left(\overline{A(\lambda_0)}\right) = \{\psi_{i0}\}_1^m$. □

For illustration of Theorem 2.1 consider the following example.

Example. Consider the operator $A : R^3 \to R^2$, representable in the canonical basis of the space R^3 in a matrix form

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Zeros of this operator are $\varphi_1 = e_1, \varphi_2 = e_1 + e_2 + e_3$. Corresponding biorthogonal vectors are $\gamma_1 = e_1 - e_2, \gamma_2 = e_2 + 2e_3$.

The conjugate operator is

$$A^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} : R^2 \to R^3,$$

The defective functional is $\psi_1 = e_1 = (1, 0)$. Hence $z_1 = e_1$.

The operator A is Noetheran. Expand the space R^2 to R^3. In this case, the ideal element is $z_2 = e_3$.

The space R^2 is isomorphic to the space $(R^2, 0)$. Hence, the operator A induces the operator

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} : R^3 \to (R^2, 0).$$

Since

$$\langle x, \gamma_2 \rangle z_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 2 \end{pmatrix} x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix} x,$$

the regularized operator has the form:

$$\overline{A} = A + \langle \cdot, \gamma_2 \rangle z_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Since $rank\ \overline{A} = rank\ \overline{A^*} = 2$, the regularized operator will be Fredholm with $N(\overline{A}) = \{\varphi_1\}$, $N(\overline{A}^*) = \{(\psi_1, 0)\}$.

3 Reduction of the case of multiple eigenvalues to the case of simple ones

Now let ε be a small complex parameter,

$$A(t; \varepsilon) = \sum_{k+l=0}^{\infty} A_{lk} \mu^l \varepsilon^k : E_1 \to E_2,\ \mu = t - \lambda_0,$$

be a perturbed operator-function, $A(t;0) = A(t)$, and

$$\overline{A(t;\varepsilon)} = A(t;\varepsilon) + \sum_{i=m+1}^{n} \langle \cdot, \gamma_{i0} \rangle z_{i0}.$$

It is required to find the eigenvalues $\lambda_0 + \mu(\varepsilon)$ of the operator $\overline{A(\lambda;\varepsilon)}$, such that $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$ and the eigenelements $\varphi_i(\varepsilon), \psi_i(\varepsilon)$ corresponding to them.

For every $i = \overline{1,m}$ we construct the operators

$$\tilde{A}_i(t;\varepsilon) \equiv \overline{A(t;\varepsilon)} + \sum_{j \neq i} \langle \cdot, \gamma_{j0} \rangle z_{j0}. \tag{3.1}$$

Theorem 3.1. *For every $i = \overline{1,m}$ and sufficiently small ε $\lambda_i(\varepsilon)$ are simple eigenvalues of operator (3.1) with the relevant eigenelements $\tilde{\varphi}_i(\varepsilon) = \varphi_i(\varepsilon) + \sum_{s \neq i} a_{is}\varphi_s(\varepsilon)$ and defect-functionals $\tilde{\psi}_i(\varepsilon) = \psi_i(\varepsilon) + \sum_{s \neq i} b_{is}\psi_s(\varepsilon)$.*

Proof. By the equality

$$0 = \tilde{A}_i(\lambda_i;\varepsilon)\tilde{\varphi}_i(\varepsilon) =$$
$$= \overline{A(\lambda_i;\varepsilon)}\varphi_i(\varepsilon) + \sum_{j \neq i} a_{ij}\overline{A(\lambda_i;\varepsilon)}\varphi_j(\varepsilon) + \sum_{j \neq i} \langle \varphi_i(\varepsilon), \gamma_{j0} \rangle z_{j0} + \sum_{j \neq i}\sum_{s \neq i} a_{is} \langle \varphi_s(\varepsilon), \gamma_{j0} \rangle z_{j0}$$

or which is the same

$$0 = \sum_{j \neq i} a_{ij}\overline{A(\lambda_i;\varepsilon)}\varphi_j(\varepsilon) + \sum_{j \neq i} \langle \varphi_i(\varepsilon), \gamma_{j0} \rangle z_{j0} + \sum_{j \neq i}\sum_{s \neq i} a_{is} \langle \varphi_s(\varepsilon), \gamma_{j0} \rangle z_{j0},$$

after applying the functionals ψ_{k0}, $k \neq i$ one has

$$\sum_{s \neq i} a_{is}\left[\langle \varphi_s(\varepsilon), \gamma_{k0} \rangle + \left\langle \overline{A(\lambda_i;\varepsilon)}\varphi_s(\varepsilon), \psi_{k0} \right\rangle\right] = -\langle \varphi_i(\varepsilon), \gamma_{k0} \rangle, \ k \neq i. \tag{3.2}$$

In view of closeness of φ_{i0} to φ_i and smallness of the parameter ε the determinant of system (3.2) is close to 1, and therefore the coefficients a_{ij} are uniquely determined. The uniqueness of $\tilde{\psi}_i(\varepsilon)$ can be proved analogously. □

4 Eigenvalues bifurcation

Theorem 4.1. *If $\langle A_1\varphi_{i0}, \psi_{i0} \rangle \neq 0$ for every $i = \overline{1,m}$, then operator (3.1) has exactly m eigenvalues $\lambda_i(\varepsilon)$ with relevant eigenelements $\tilde{\varphi}_i(\varepsilon)$, which are analytic in ε.*

Proof. Let $\tilde{\varphi}_i(\varepsilon)$ be eigenelements corresponding to the eigenvalues $\lambda_i(\varepsilon)$. Then $\tilde{A}_i(\lambda_i;\varepsilon)\tilde{\varphi}_i(\varepsilon) = 0$, that can be rewritten in the form

$$\overline{A(\lambda_0)}\tilde{\varphi}_i(\varepsilon) = \overline{A}(\mu_i(\varepsilon))\tilde{\varphi}_i(\varepsilon) + H_i(\lambda_i(\varepsilon);\varepsilon)\tilde{\varphi}_i(\varepsilon) - \sum_{j \neq i} \langle \tilde{\varphi}_i(\varepsilon), \gamma_{j0} \rangle z_{j0}, \tag{4.1}$$

where $\overline{A}\left(\mu_i(\varepsilon)\right)=\overline{A(\lambda_0)}-\overline{A\left(\lambda_i(\varepsilon)\right)}$, $H_i\left(\lambda_i(\varepsilon);\varepsilon\right)=\overline{A\left(\lambda_i(\varepsilon)\right)}-\overline{A\left(\lambda_i(\varepsilon);\varepsilon\right)}$. Using the E. Schmidt's regularization (see, [10]) $\tilde{A}(\lambda_0)=\overline{A(\lambda_0)}+\sum\limits_{i=1}^{n}\langle\cdot,\gamma_{i0}\rangle z_{i0}$, $\Gamma=\tilde{A}^{-1}(\lambda_0)$, equation (4.1) could be reduced to the equivalent system

$$\begin{cases} \tilde{A}(\lambda_0)\tilde{\varphi}_i(\varepsilon)=\overline{A}\left(\mu_i(\varepsilon)\right)\tilde{\varphi}_i(\varepsilon)+H_i\left(\lambda_i(\varepsilon);\varepsilon\right)\tilde{\varphi}_i(\varepsilon)+\xi_i z_{i0}, \\ \xi_i=\langle\tilde{\varphi}_i(\varepsilon),\gamma_{i0}\rangle. \end{cases} \tag{4.2}$$

By rewriting the first equation in (4.2) in the form

$$\tilde{\varphi}_i(\varepsilon)=\xi_i\left[I-\Gamma\overline{A}\left(\mu_i(\varepsilon)\right)-\Gamma H_i\left(\lambda_i(\varepsilon);\varepsilon\right)\right]^{-1}\varphi_{i0} \tag{4.3}$$

and substituting $\tilde{\varphi}_i(\varepsilon)$ in the second equation, one obtains the branching equation for the eigenvalue λ_0 :

$$\langle\left[\overline{A}\left(\mu_i(\varepsilon)\right)+H_i\left(\lambda_i(\varepsilon);\varepsilon\right)\right]\left[I-\Gamma\overline{A}\left(\mu_i(\varepsilon)\right)-\Gamma H_i\left(\lambda_i(\varepsilon);\varepsilon\right)\right]^{-1}\varphi_{i0},\psi_{i0}\rangle=0. \tag{4.4}$$

However, due to the analyticity of the operator $A(t;\varepsilon)$ in a neighborhood of $(\lambda_0,0)$, the left-hand side of equation (4.4) can be represented as a series in integer powers of μ_i and ε :

$$\left\langle\left[\sum_{s=1}^{\infty}A_{s0}\mu_i^s+\sum_{s=1}^{\infty}\sum_{l=1}^{\infty}A_{sl}\mu_i^s\varepsilon^l\right]\left[I-\Gamma\sum_{s=1}^{\infty}A_{s0}\mu_i^s-\Gamma\sum_{s=1}^{\infty}\sum_{l=1}^{\infty}A_{sl}\mu_i^s\varepsilon^l\right]^{-1}\varphi_{i0},\psi_{i0}\right\rangle=0$$

or

$$\sum_{s=1}^{\infty}\sum_{l=0}^{\infty}L_{sli}\mu_i^s\varepsilon^l=0, \tag{4.5}$$

where

$$L_{sli}=\sum_{(s,l)=(s_1,l_1)+\cdots+(s_k,l_k)}\langle A_{s_1l_1}\Gamma A_{s_2l_2}\ldots\Gamma A_{s_kl_k}\varphi_{i0},\psi_{i0}\rangle.$$

Under the assimptions of the theorem the coefficient $L_{10i}=\langle A_{10}\varphi_{i0},\psi_{i0}\rangle\neq 0$. Therefore $\mu_i(\varepsilon)$ for all $i=\overline{1,n}$ are determined by equation (4.5) in the form of a series in integer powers of ε. Determination of $\mu_i(\varepsilon)$ according to (4.5) allows us to find the relevant eigenelement $\tilde{\varphi}_i(\varepsilon)$ also in the form of a series in integer powers of ε. □

Next we consider the case of the presence of generalized Jordan chains (GJChs) $\left\{\varphi_{i0}^{(j)}\right\}_{i=\overline{1,n}}^{j=\overline{1,p_i}}$, such that $\langle\sum\limits_{k=1}^{p_i}A_{k0}\varphi_{i0}^{(p_i+1-k)},\psi_{i0}\rangle\neq 0$ for all $i=\overline{1,m}$. This implies that all GJChs are finite. In this case in equation (4.5)

$$L_{p_i0i}=\langle\sum_{k=1}^{p_i}A_{k0}\varphi_{i0}^{(p_i+1-k)},\psi_{i0}\rangle\neq 0.$$

The decreasing part of the Newton diagram for branching equation (4.5) consists of one segment joining the points $(1,1)$ and $(p_i,0)$ (it will be so when all L_{0ji} are nonzero and $L_{11i}\neq 0$) or of two segments: the one indicated above one and the segment joining the points $(1,1)$ and $(0,q_i)$ where q_i is the number of the first nonzero member in the sequence $\{L_{0ji}\}$. To the first segment the exponent $\frac{1}{p_i-1}$ corresponds while to the second one an integer exponent. Consequently the operator $A(\lambda,\varepsilon)$ for sufficiently small ε has exactly $N=\sum\limits_{i=1}^{m}p_i$ (N is the root number) different continuons in ε eigenvalues $\lambda_i(\varepsilon)$, $\lambda_i(0)=\lambda_0$, where m eigenvalues are represented by convergent series in integer powers of ε and $N-m$ eigenvalues by convergent series in integer powers of $\varepsilon^{\frac{1}{p_i-1}}$. To every $\lambda_i(\varepsilon)$ the eigenelement $\tilde{\varphi}_i(\varepsilon)$ corresponds, represented by a convergent series in the same powers of ε, which are in the corresponding representation of $\lambda_i(\varepsilon)$. Thus the following result is proved.

Theorem 4.2. *Let in the presence of GJChs* $L_{p_i0i}=\langle\sum\limits_{k=1}^{p_i}A_{k0}\varphi_{i0}^{(p_i+1-k)},\psi_{i0}\rangle\neq 0$ *for all* $i=\overline{1,m}$. *If* $L_{0ji}=0$, $j=\overline{1,\infty}$ *and* $L_{11i}\neq 0$, *then there exist* N *simple eigenvalues with relevant eigenelements represented by convergent series in integer powers of* $\varepsilon^{\frac{1}{p_i-1}}$. *If* $L_{0ji}=0$, $j=\overline{1,q_i-1}$, $L_{0q_ii}\neq 0$, $L_{11i}\neq 0$, *then there exist exactly* N *eigenvalues with relevant eigenelements,* m *of which are represented by convergent series in integer powers of* ε, *while the other* $N-m$ *by convergent series in integer powers of* $\varepsilon^{\frac{1}{p_i-1}}$.

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