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SOME INEQUALITIES FOR SECOND ORDER DIFFERENTIAL OPERATORS WITH UNBOUNDED DRIFT

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Abstract. We study coercive estimates for some second-order degenerate and damped differential operators with unbounded coefficients. We also establish the conditions for invertibility of these operators.

1 Introduction

For the Sturm-Liouville operator $l_0 y = -y'' + q(x)y$ $(x \in \mathbb{R})$, coercive estimates and other properties associated which Sobolev spaces are well known (see [1, 3, 4, 15]). Properties of the operator $ly = -y'' + ry' + qy$ with the intermediate coefficient r subordinated to the potential q in some sense, are studied in [5, 9].

In this work, we consider the minimal closed differential operator

$$Ly = -\rho(x)(\rho(x)y')' + r(x)y' + q(x)y$$

in $L_2(\mathbb{R})$, where ρ, r are continuously differentiable functions, and q is a continuous function. We do not assume that ρ, r, q are bounded in $\mathbb{R}$. The aim of this work is to show that the operator L is continuously invertible when these coefficients satisfy some conditions and to obtain the following estimate for $y \in D(L)$

$$\|-\rho(\rho y')'\|_2 + \|ry'\|_2 + \|qy\|_2 \le C\,\|Ly\|_2\,, \tag{1.1}$$

where $D(L)$ is the domain of L, $\|\cdot\|_2$ is the norm in $L_2(\mathbb{R})$, and C independent of y.

Estimate (1.1) already implies that the domain of L coincides with the subspace generated by the norm $\|-\rho(\rho y')'\|_2 + \|ry'\|_2 + \|qy\|_2$. This fact enables us to use the methods of the embedding theory of weighted Sobolev spaces for studying many important properties (for example, regularity, spectral or approximation properties) of L (see [8, 12, 13, 16]).

The operator L has numerous applications in mathematical physics and stochastic processes. For example, in the theory of Brownian motion the Ornstein - Uhlenbek operator is used (see [10]), which is an operator of type L, and the Fokker - Plank and Kramer differential operators are generalizations of the Ornstein-Uhlenbek operator. The Ornstein-Uhlenbek operator was studied in works of M. Smoluchowski, A. Fokker,

M. Plank, H.C. Burger, R. Furth, L. Zernike, S. Goudsmitt, M.C. Wang (see [20] and the references therein). On the other hand, the operator L is used to describe the problem of the propagation of small oscillations in a viscoelastic compressible medium [17, 19]. Also, the operator L is used in the study of the vibrational motion in mediums with resistance, where the resistance depends on the velocity [18].

Recently in works of J. Pruss, R. Shnaubelt, A. Rhandi, G. Da Prato, V. Vespri, P. Clement, G. Metafune, D. Pallara, M. Hieber, L. Lorenzi and others the following Ornstein-Uhlenbek-type operator

$$A_0 u = -div\,(a\nabla u) + F\cdot\nabla u - Vu$$

was investigated with various properties (see [2] and references therein). In this works are imposed the additional conditions which are sufficient to control the drift term $F\cdot\nabla u$ by $-div\,(a\nabla u)$ and Vu.

The results of the present paper show that if the intermediate coefficient r is quickly growing, then the one dimensional operator L is invertible and has regular properties. Estimate (1.1) is useful for evolutionary partial differential equations associated with the operator L (see [7]).

The paper is organized as follows. In Section 2 we prove several auxiliary statements and the invertibility of the operator

$$ly = -\rho(\rho y')' + ry'$$

for a certain class of ρ and r. In Section 3 we prove inequality (1.1) under some additional conditions. We present some examples in Section 4.

Inequality (1.1) for operator l in the case $\rho = 1$ was obtained in [11]. The coercive estimate of L in $L_1(\mathbb{R})$ was proved in [14].

We denote by $C(\mathbb{R})$ the class of the continuous functions, and by $C^{(s)}(\mathbb{R})$ ($s =$ 1, 2, ...) the class of all s times continuously differentiable functions and by $C_0^{(s)}(\mathbb{R})$ ($s = 1, 2, ...$) the subset of all compactly supported functions in $C^{(s)}(\mathbb{R})$.

2 Auxiliary statements and existence of the resolvent for a degenerate operator

Denote by l the closure in $L_2(\mathbb{R})$ of the differential expression

$$l_0 y = -\rho(\rho y')' + ry'$$

on $C_0^{(2)}(\mathbb{R})$, where $\rho \in C^{(1)}(\mathbb{R})$, $r \in C(\mathbb{R})$. The operator l is a degenerate operator, since it does not have the lower-order term. The domain $D(l)$ is contained in the space $L_2(\mathbb{R})$ only in the case when the functions ρ and r satisfy some additional conditions.

In this section, we give some sufficient conditions for bounded invertibility of the operator l. We denote

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)}\left\|h^{-1}\right\|_{L_2(t,+\infty)}(t>0),$$

$$\beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)}\left\|h^{-1}\right\|_{L_2(-\infty,\tau)}(\tau<0),$$

$$\gamma_{g,h} = \max\left(\sup_{t>0}\alpha_{g,h}(t),\ \sup_{\tau<0}\beta_{g,h}(\tau)\right),$$

where g and h are given functions.

Lemma 2.1. [11]. *Let g and h be continuous functions on $\mathbb{R}$ and $\gamma_{g,h} < \infty$. Then for any $y \in C_0^{(1)}(\mathbb{R})$ the following inequality holds:*

$$\int_{-\infty}^{\infty} |g(x)y(x)|^2\,dx \le c_1 \int_{-\infty}^{\infty} |h(x)y'(x)|^2\,dx.$$

Moreover, the least such constant c_1 satisfies $\gamma_{g,h} \le c_1 \le 2\gamma_{g,h}$.

Lemma 2.2. *Let $\rho \in C^{(1)}(\mathbb{R})$ and $r \in C(\mathbb{R})$ satisfy the following conditions*

$$r \ge 1, \gamma_{1,\sqrt{r}} < \infty. \tag{2.1}$$

Then for $y \in D(l)$ the following estimate holds:

$$\left\|\sqrt{r}y'\right\|_2 + \|y\|_2 \le \left(1+\sqrt{2\gamma_{1,\sqrt{r}}}\,\right)\left\|\frac{1}{\sqrt{r}}ly\right\|_2. \tag{2.2}$$

Proof. Let $y \in C_0^{(2)}(\mathbb{R})$. Integrating by parts, we have

$$(ly,\, y') = \int_{\mathbb{R}} r(x)(y')^2dx. \tag{2.3}$$

By Hölder's inequality,

$$|\,(L_0y,\, y')| \le \left\|\frac{1}{\sqrt{r}}L_0y\right\|_2 \left\|\sqrt{r}y'\right\|_2. \tag{2.4}$$

Since $r \ge 1$, from (2.3) and (2.4) it follows that

$$\left\|\sqrt{r}y'\right\|_2 \le \left\|\frac{1}{\sqrt{r}}L_0y\right\|_2. \tag{2.5}$$

On the other hand, using Lemma 2.1, we get

$$\|y\|_2 \le 2\gamma_{1,\sqrt{r}}\left\|\sqrt{r}y'\right\|_2.$$

Then

$$\left\|\sqrt{r}y'\right\|_2 + \|y\|_2 \le \left(1+2\gamma_{1,\sqrt{r}}\right)\left\|\sqrt{r}y'\right\|_2.$$

So, using (2.5) we obtain that (2.2) holds for any $y \in C_0^{(2)}(\mathbb{R})$.

Let $y \in D(l)$. Then there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset C_0^{(2)}(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - ly\|_2 \to 0$ as $n \to \infty$. Since (2.2) holds for all y_n $(n \in \mathbb{N})$. Then passing to limit as $n \to \infty$ we obtain the desired estimate for $y \in D(l)$. $\square$

Theorem 2.1. *Let $r \in C(\mathbb{R})$, $\rho \in C^{(1)}(\mathbb{R})$ be such that*

$$r \geq \rho^2, \gamma_{1,\sqrt{r}} < \infty \tag{2.6}$$

and for some $N > 0$ the following inequality holds

$$1 \leq \rho(x) \leq c_2 \left(1 + x^2\right)^N . \tag{2.7}$$

Then the operator l is invertible and the inverse operator l^{-1} is defined on the whole $L_2(\mathbb{R})$.

Proof. Inequality (2.2) implies that the inverse l^{-1} exists. It suffices to show that $R(l) = L_2(\mathbb{R})$. Assume that $R(l\,) \neq L_2(\mathbb{R})$. Then there exists a non-zero element $v \in L_2(\mathbb{R})$ such that $v \perp R(l)$. It follows that

$$l^* v \equiv \; (\rho(\rho\, v)')' + (rv)' = 0,$$

where l^* is the adjoint operator of l. Put $\rho v = z$, then

$$\left(\rho z' + \frac{r}{\rho} z\right)' = 0,$$

or

$$\left(z \exp\left[\int_a^x \frac{r(t)}{\rho^2(t)} dt\right]\right)' = \frac{c}{\rho} \exp\left(\int_a^x \frac{r(t)}{\rho^2(t)} dt\right),$$

where c is a constant.

If $c \neq 0$, then we can assume that $c = -1$. Inequalities (2.6), (2.7) imply that

$$\left(z(x) \;\exp\left[\int_a^x \frac{r(t)}{\rho^2(t)} dt\right]\right)' \leq c_1 < 0,\; x \in (a,\, +\infty).$$

Hence (2.6) and (2.7) imply that $v \notin L_2(\mathbb{R})$.

If $c = 0$, then we have

$$v = \frac{c_2}{\rho(x)} \exp\left[- \int_a^x \frac{r(t)}{\rho^2(t)} dt\right].$$

By (2.7), there exists $x_0 < a$ such that $|v(x)| \geq \delta > 0$ for any $x \leq x_0$. So $v \notin L_2(\mathbb{R})$.

Hence, we obtained a contradiction. Thus $R(l\,) = L_2(\mathbb{R})$. □

Definition 1. l is called separable in $L_2(\mathbb{R})$, if there exists $c > 0$ such that

$$\|\rho(\rho y')'\|_2 + \|r y'\|_2 \leq c_3 \; \|l y\|_2 \tag{2.8}$$

for all $y \in D(l)$.

Put $\rho y' = z$. Then

$$ly = -\rho z' + \frac{r}{\rho} z.$$

Let $\lambda \geq 0$, and ρ be a bounded function. We define $K_\lambda : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ as follows:

$$K_\lambda z = -z' + \left(\frac{r}{\rho^2} + \lambda\right) z, \ z \in D(K_\lambda),$$

where $D(K_\lambda)$ is the domain of K_λ. Note that K_λ is separable in $L_2(\mathbb{R})$, if for some $c_4 > 0$,

$$\|z'\|_2 + \left\| \left(\frac{r}{\rho^2} + \lambda\right) z \right\|_2 \leq c_4 \|K_\lambda z\|_2$$

for all $z \in D(K_\lambda)$.

Lemma 2.3. *Let $\rho \in C^{(1)}(\mathbb{R})$, $1 \leq \rho \leq s$, $r \in C(\mathbb{R})$ satisfy (2.2). Then l is separable in $L_2(\mathbb{R})$ if and only if*

$$K_\lambda z = -z' + \left(\frac{r}{\rho^2} + \lambda\right) z$$

is separable in $L_2(\mathbb{R})$ for some $\lambda \geq 0$.

Proof. Assume that l is separable in $L_2(\mathbb{R})$. Put $\rho y' = z$. Then

$$\| - \rho z'\|_2 + \left\| \frac{r}{\rho} z \right\|_2 \leq c_5 \|\rho^{-1} K_0 z\|_2.$$

Hence,

$$\| - z'\|_2 + \left\| \frac{r}{\rho^2} z \right\|_2 \leq c_5 \|K_0 z\|_2. \tag{2.9}$$

It is easy to check that for any $z \in D(K_\lambda)$ the following estimate holds:

$$\left\| \sqrt{\frac{r}{\rho^2} + \lambda}\, z \right\|_2 \leq \left\| \frac{1}{\sqrt{\frac{r}{\rho^2} + \lambda}} K_\lambda z \right\|_2 . \tag{2.10}$$

Therefore,

$$\left(\frac{1}{s^2} + \lambda\right) \|z\|_2 \leq \|K_\lambda z\|_2 , \ z \in D(K_\lambda). \tag{2.11}$$

By (2.9) and (2.11), we have that

$$\| - z'\|_2 + \left\| \left(\frac{r}{\rho^2} + \lambda\right) z \right\|_2 \leq c_5 \|K_0 z\|_2 + \lambda \|z\|_2 \leq (c_5 + 2) \|K_\lambda z\|_2. \tag{2.12}$$

So, K_λ is separable in $L_2(\mathbb{R})$.

Let K_λ be separable in $L_2(\mathbb{R})$, i.e.

$$\| - z'\|_2 + \left\| \left(\frac{r}{\rho^2} + \lambda\right) z \right\|_2 \leq c_6 \|K_\lambda z\|_2, \ z \in D(K_\lambda).$$

By (2.11), we obtain that

$$\|K_\lambda z\|_2 \leq \|K_0 z\|_2 + \frac{\lambda}{\lambda + 1/s^2} \|K_\lambda z\|_2,$$

hence

$$\|K_\lambda z\|_2 \le (s^2\lambda+1)\|K_0 z\|_2.$$

So, it follows that

$$\|-\rho z'\|_2 + \left\|\frac{r}{\rho}z\right\|_2 \le s\left[\|-z'\|_2 + \left\|\frac{r}{\rho^2}z\right\|_2\right] \le c_6\|K_\lambda z\|_2 + \lambda\|z\|_2$$

$$\le (c_6+1)\|K_\lambda z\|_2 \le 2c_6(s^2\lambda+1)\|K_0 z\|_2.$$

Taking $z/\rho = y'$, we get that

$$\|-\rho(\rho y')'\|_2 + \|ry'\|_2 \le c_7\|ly\|_2.$$

□

Lemma 2.4. *Let $\rho \in C^{(1)}(\mathbb{R})$, $1 \le \rho \le s$ and $r \in C(\mathbb{R})$. Suppose that*

$$\sup_{|x-\eta|\le 2} \frac{r(x)}{r(\eta)} < \infty \tag{2.13}$$

and condition (2.2) hold. Then l is separable in $L_2(\mathbb{R})$.

Proof. By Lemma 2.3, it is enough to prove that K_λ is separable in $L_2(\mathbb{R})$ for some $\lambda \ge 0$.

Theorem 2.1 implies that K_λ is continuously invertible on $L_2(\mathbb{R})$ for all $\lambda \ge 0$. Next, we show a useful representation of K_λ^{-1}. Let $\Delta_j = (j-1, j+1)$ $(j \in \mathbb{Z})$, and $\{\varphi_j\}_{j=-\infty}^{+\infty}$ be a sequence in $C_0^\infty(\Delta_j)$ such that

$$0 \le \varphi_j \le 1, \left|\varphi_j'(x)\right| \le m \ (j \in \mathbb{Z}), \sum_{j=-\infty}^{+\infty} \varphi_j^2(x) = 1.$$

We extend the restriction of $r(x)\rho^{-2}(x)$ to the interval Δ_j to $\mathbb{R}$ as a piecewise continuous function $\psi_j(x)$ with period 2. Let $K_{\lambda,j}$ be the closure in $L_2(\Delta_j)$ of the differential operator $-z' + (\psi_j(x)+\lambda)\, z$ on $C_0^{(1)}(\Delta_j)$. Similarly to (2.10), we obtain that

$$\left\|\sqrt{\psi_j+\lambda}\; z\right\|_{2,\Delta_j} \le \left\|\frac{1}{\sqrt{\psi_j+\lambda}} K_{\lambda,j}\, z\right\|_{2,\Delta_j}, z \in C_0^{(1)}(\Delta_j), j \in \mathbb{Z}.$$

Hence,

$$\left(\frac{1}{s^2}+\lambda\right)\|z\|_{2,\Delta_j} \le \|K_{\lambda,j} z\|_{2,\Delta_j}, z \in D(K_{\lambda,j}), j \in \mathbb{Z}. \tag{2.14}$$

So, $K_{\lambda,j}^{-1}$ exists. On the other hand, by Theorem 2.1, $K_{\lambda,j}^{-1}$ is defined on the whole $L_2(\Delta_j)$.

Define B_λ and M_λ as follows:

$$B_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j'(x) K_{\lambda,j}^{-1}\varphi_j f, \quad M_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j(x) K_{\lambda,j}^{-1}\varphi_j f, f \in L_2(\mathbb{R}).$$

Since $\operatorname{supp}\varphi_j \subset \Delta_{j-1}\bigcup\Delta_j\bigcup\Delta_{j+1}$ $(j\in\mathbb{Z})$, at each point $x\in\mathbb{R}$ the sums of the right-hand side of B_λ and M_λ contain no more than two summands, so B_λ and M_λ are well-defined on the whole $L_2(\mathbb{R})$. Moreover, it is clear that

$$K_\lambda M_\lambda = E - B_\lambda. \tag{2.15}$$

Notice that in $(j, j+1)$ $(j\in\mathbb{Z})$ only the functions φ_j and φ_{j+1} are not equal to zero. So, we have that

$$\|B_\lambda f\|_2^2 = \left\|\sum_{j=-\infty}^{+\infty}\varphi_j'(x)K_{\lambda,j}^{-1}\varphi_j f\right\|_2^2 = \int_{-\infty}^{\infty}\left|\sum_{j=-\infty}^{+\infty}\varphi_j'(x)K_{\lambda,j}^{-1}\varphi_j f\right|^2 dx$$

$$= \sum_{i=-\infty}^{+\infty}\int_i^{i+1}\left(\sum_{j=-\infty}^{+\infty}\left|\varphi_j'(x)\right|\left|\left[K_{\lambda,j}^{-1}\left(\varphi_j f\right)\right](x)\right|\right)^2 dx$$

$$= \sum_{i=-\infty}^{+\infty}\int_i^{i+1}\left[\left|\varphi_i'\right|\left|\left(K_{\lambda,i}^{-1}\left(\varphi_k f\right)\right)\right| + \left|\varphi_{i+1}'\right|\left|\left(K_{\lambda,i+1}^{-1}\left(\varphi_{i+1}f\right)\right)\right|\right]^2 dx$$

$$\le 2\sum_{i=-\infty}^{+\infty}\left(\int_{\Delta_i}\left|\varphi_i'\right|^2\left|K_{\lambda,i}^{-1}\left(\varphi_i f\right)\right|^2 dx + \int_{\Delta_{i+1}}\left|\varphi_{i+1}'\right|^2\left|K_{\lambda,i+1}^{-1}\left(\varphi_{i+1}f\right)\right|^2 dx\right)$$

$$= 4\sum_{i=-\infty}^{+\infty}\int_{\Delta_i}\left|\varphi_i'(x)\right|^2\left|K_{\lambda,i}^{-1}\left(\varphi_i f\right)(x)\right|^2 dx.$$

Furthermore

$$\|B_\lambda f\|_2^2 \le 4m^2\sum_{j=-\infty}^{+\infty}\left(\left\|K_{\lambda,j}^{-1}\right\|_{L_2(\Delta_j)\to L_2(\Delta_j)}^2\left\|\varphi_j f\right\|_{2,\,\Delta_j}^2\right)$$

$$\le 8m^2\sup_{j\in Z}\left\|K_{\lambda,j}^{-1}\right\|_{L_2(\Delta_j)\to L_2(\Delta_j)}^2\int_{\mathbb{R}}\left(\sum_j\varphi_j^2\right)|f|^2 dx$$

$$= 8m^2\sup_{j\in\mathbb{Z}}\left\|K_{\lambda,j}^{-1}\right\|_{L_2(\Delta_j)\to L_2(\Delta_j)}^2\|f\|_2^2.$$

By inequality (2.14),

$$\left\|K_{\lambda,j}^{-1}\right\|_{L_2(\Delta_j)\to L_2(\Delta_j)} \le \frac{s^2}{1+s^2\lambda}.$$

Thus $\|B_\lambda f\|_2 \le \frac{2\sqrt{2}\,ms^2}{1+s^2\lambda}\|f\|_2$, $f\in L_2(\mathbb{R})$. Let $\lambda_0 = (4\sqrt{2}\,ms^2-1)s^{-2}$. Then

$$\|B_\lambda\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} \le \frac{1}{2}$$

holds for any $\lambda\ge\lambda_0$. So, $E+B_\lambda$ $(\lambda\ge\lambda_0)$ is invertible. By (2.15), we get

$$K_\lambda^{-1} = M_\lambda(E-B_\lambda)^{-1}, \lambda\ge\lambda_0. \tag{2.16}$$

Now, we can prove (2.8). Let $m_1 = \sup\limits_{|x-\eta|\leq 2} \frac{r(x)}{r(\eta)}$. By (2.16) and the properties of $\varphi_j\,(j\in\mathbb{Z})$, we obtain that

$$\left\|\left(\frac{r}{\rho^2}+\lambda\right)K_\lambda^{-1}f\right\|_2 \leq 4\sqrt{2}\,(m_1s^2+1)\,\|f\|_2\,.$$

Then, for $\lambda \geq (4\sqrt{2}ms^2-1)s^{-2}$, we have that

$$\|z'\|_2+\left\|\left(\frac{r}{\rho^2}+\lambda\right)z\right\|_2 \leq (1+4\sqrt{2}+4\sqrt{2}m_1s^2)\,\|K_\lambda z\|_2\,. \tag{2.17}$$

Put $z=\rho y'$. By (2.17), we get that

$$\left\|\rho\,(\rho y')'\right\|_2+\|ry'\|_2 \leq 8\sqrt{2}ms(1+4\sqrt{2}+4\sqrt{2}m_1s^2)\,\|ly\|_2\,, y\in D(l), \tag{2.18}$$

hence l is separable. □

3 Separability of the damped differential operator

Denote by L the closure in $L_2(\mathbb{R})$ of the differential expression

$$\tilde{L}y=-\rho(\rho y')'+ry'+qy$$

on $C_0^{(2)}(\mathbb{R})$, where ρ is a continuously differentiable function, r and q are continuous functions.

Theorem 3.1. *Let ρ be a bounded continuously differentiable function, r and q be continuous functions. Suppose that $\rho\geq 1$, r and q satisfy conditions (2.2), (2.17) and $\gamma_{q,r}<\infty$. Then L is continuously invertible, and L^{-1} is defined on the whole $L_2(\mathbb{R})$. Furthermore, there exists c_8 such that*

$$\|-\rho(\rho y')'\|_2+\|ry'\|_2+\|qy\|_2 \leq c_8\,\|Ly\|_2\,, \tag{3.1}$$

for any $y\in D(L)$.

Proof. We consider the equation

$$Ly=f. \tag{3.2}$$

A function $y\in L_2(\mathbb{R})$ is called a solution to (3.2), if there is a sequence $\{y_n\}_{n=1}^{+\infty}\subset C_0^{(2)}(\mathbb{R})$ such that $\|y_n-y\|_2\to 0$, $\|Ly_n-f\|_2\to 0$ $(n\to+\infty)$. It is clear that L is continuously invertible if and only if there exists a unique solution y to (3.2) for each $f\in L_2(\mathbb{R})$. Putting $x=at$ $(a>0)$, we rewrite (3.2) in the following form:

$$-\tilde{\rho}(t)(\tilde{\rho}(t)\tilde{y}_t')_t'+1/\alpha\tilde{r}(t)\tilde{y}_t'+1/\alpha^2\tilde{q}(t)\tilde{y}=\tilde{f}, \tag{3.3}$$

where

$$\tilde{y}(t)=y(at), \tilde{\rho}(t)=\rho(at), \tilde{r}(t)=r(at), \tilde{q}(t)=q(at), \tilde{f}(t)=f(at)/a^2.$$

Let

$$\hat{l}_0\tilde{y} = -\tilde{\rho}(t)(\tilde{\rho}(t)\tilde{y}_t')_t' + \tilde{r}/a\ \tilde{y}_t',$$

then from (3.3) we obtain

$$\hat{l}_0\tilde{y} + \tilde{q}(t)/a^2\tilde{y} = \tilde{f}(t). \tag{3.4}$$

Note that $\tilde{r}/a$ satisfies the conditions of Lemma 2.3, so the operator $\hat{l}_0$ is continuously invertible. By (2.18),

$$\left\|-\tilde{\rho}(t)(\tilde{\rho}(t)\tilde{y}_t')_t'\right\|_2 + \left\|\tilde{r}/a\ \tilde{y}_t'\right\|_2 \le T\left\|\hat{l}_0\tilde{y}\right\|_2, \forall\, \tilde{y} \in D(\hat{l}_0), \tag{3.5}$$

where $T = 8\sqrt{2}ms(1+4\sqrt{2}+4\sqrt{2}m_1s^2)$.

It is clear that $\gamma_{\tilde{q},\tilde{r}} = 1/a\ \ \gamma_{q,r}$. By Lemma 2.1 and (3.5),

$$\left\|\frac{1}{a^2}\tilde{q}\tilde{y}\right\|_2 \le \frac{2\gamma_{\tilde{q},\tilde{r}}}{a^2}\left\|\tilde{r}\tilde{y}'\right\|_2 \le 2\gamma_{q,r}a^{-2}\left\|\frac{1}{a}\tilde{r}\tilde{y}'\right\|_2 \le \frac{2T\gamma_{q,r}}{a^2}\left\|\hat{l}_0\tilde{y}\right\|_2.$$

Choose $a = 2\sqrt{T\ \gamma_{q,r}}$, then

$$\left\|\frac{1}{a^2}\tilde{q}\tilde{y}\right\|_2 \le \frac{1}{2}\left\|\hat{l}_0\tilde{y}\right\|_2. \tag{3.6}$$

By Theorem 1.16 in Chapter IV of [6], $\hat{l}_0 + \frac{1}{a^2}\tilde{q}_1(t)E$ is invertible and $R\left(\hat{l}_0 + \frac{1}{a^2}\tilde{q}_1E\right) = L_2(\mathbb{R})$. Let $\tilde{y}$ be a solution to (3.4). Then, by (3.5) and (3.6), we get that

$$\begin{aligned}\left\|-\tilde{\rho}(t)(\tilde{\rho}(t)\tilde{y}')'\right\|_2 + \left\|\frac{1}{a}\tilde{r}\tilde{y}'\right\|_2 + \left\|\frac{1}{a^2}\tilde{q}\tilde{y}\right\|_2 \\ \le \left[T\left(1+\frac{2\gamma_{q,r}}{a^2}\right)\right]\left\|\hat{l}_0\tilde{y}\right\|_2.\end{aligned} \tag{3.7}$$

On the other hand,

$$\left\|\hat{l}_0\tilde{y}\right\|_2 \le \left\|\left(\hat{l}_0 + \frac{1}{a^2}\tilde{q}E\right)\tilde{y}\right\|_2 + \left\|\frac{1}{a^2}\tilde{q}\tilde{y}\right\|_2. \tag{3.8}$$

Using (3.4) and (3.6), we obtain that

$$\left\|\frac{1}{a^2}\tilde{q}\tilde{y}\right\|_2 \le \left\|\left(\hat{l}_0 + \frac{1}{a^2}\tilde{q}E\right)\tilde{y}\right\|_2,$$

and

$$\left\|\hat{l}_0\tilde{y}\right\|_2 \le \left\|\left(\hat{l}_0 + \frac{1}{a^2}\tilde{q}E\right)\tilde{y}\right\|_2 + \left\|\frac{1}{a^2}\tilde{q}\tilde{y}\right\|_2 \le 2\left\|\left(\hat{l}_0 + \frac{1}{a^2}\tilde{q}E\right)\tilde{y}\right\|_2. \tag{3.9}$$

So, (3.7) and (3.9) imply that the inequality

$$\left\|-\tilde{\rho}(\tilde{\rho}\tilde{y}')'\right\|_2 + \left\|\frac{1}{a}\tilde{r}\tilde{y}'\right\|_2 + \left\|\frac{1}{a^2}\tilde{q}\tilde{y}\right\|_2 \le 2\left[T\left(1+\frac{2\gamma_{q,r}}{a^2}\right)\right]\left\|\tilde{f}\right\|_2$$

holds for any solution $\tilde{y}$ to (3.4). Let $t = x/a$. Rewriting the above formula, we obtain (3.1). □

4 Examples

1. Let $L_0 y = (1+x^2)\left((1+x^2)\, y'\right)' + (5+x^4)\, y'$. Then all conditions of Theorem 2.1 are satisfied. Hence, L_0 is invertible, and L_0^{-1} is continuous.

2. We consider

$$Ly = -y'' + (1+x^2)^{\omega} y' + |x|^{\sigma} y,$$

where $\omega > 0$, $\sigma \geq 0$. If $\omega \geq \sigma/2 + 3/4$, then the conditions of Theorem 3.1 are satisfied. So L has a bounded inverse L^{-1}, and there exists $c_9 > 0$ such that

$$\|y''\|_2 + \left\|\left(1+x^2\right)^{\omega} y'\right\|_2 + \||x|^{\sigma} y\|_2 \leq c_9 \|Ly\|_2$$

for all $y \in D(L)$.
3. By Theorem 3.1, $\tilde{L}y = -y'' + \exp(1+x^2)y' + \exp|x|\, y$ is continuously invertible on $L_2(\mathbb{R})$. Moreover, for all $y \in D(\tilde{L})$,

$$\|y''\|_2 + \left\|\exp(1+x^2)\, y'\right\|_2 + \|\exp|x|\, y\|_2 \leq c_{10} \left\|\tilde{L}y\right\|_2,$$

where c_{10} is independent of y.

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