EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 6, Number 2 (2015), 41 – 62

ON ESTIMATES OF THE APPROXIMATION NUMBERS OF THE HARDY OPERATOR

E.N. Lomakina

Communicated by V.D. Stepanov

Key words: Lebesgue space, Lorentz space, Hardy operator, approximation numbers, Schatten-von Neumann norm.

AMS Mathematics Subject Classification: 47B06, 47G10, 47B10.

Abstract. We obtain two–sided estimates which describe the behaviour of the approximation numbers of the Hardy operator and Schatten–Neumann norms in the new case, when the compact operator

$$Tf(x) = \int_0^x f(\tau)\, d\tau, \quad x > 0,$$

is acting from a Lebesgue space to a Lorentz space ($T : L^r_v(R^+) \to L^{pq}_\omega(R^+)$) under the condition $1 < p < r \le q < \infty$.

1 Introduction

Let $\mathcal{B}(X,Y)$ be the space of all linear bounded operators from a Banach space X to a Banach space Y. *The approximation numbers* of an operator $T \in \mathcal{B}(X,Y)$ are equal to the distances in $\mathcal{B}(X,Y)$ between the operator T and subspaces of finite–dimentional operators in $\mathcal{B}(X,Y)$:

$$a_n(T) := \inf\{\|T - L\|_{X\to Y} : \quad L : X \to Y, \ \text{rank } L \le n-1\}, \quad n \in \mathbb{N},$$

where rank $L := \dim \mathcal{R}(L)$.

In recent two decades, great attention has been paid to the study of the approximation numbers of the Hardy operator

$$Tf(x) = v(x)\int_0^x f(\tau)u(\tau)\, d\tau, \quad x > 0$$

in weighted Lebesgue spaces. First steps in this direction were made by D.E. Edmunds, W.D. Evans, D.J. Harris in the article [1], where implicit estimates for the approximation numbers of the operator $T : L^p(R^+) \to L^q(R^+)$ in the case $1 < p \le q < \infty$ were obtained in the form

$$\frac{1}{4}\varepsilon N^{\frac{1}{q}-\frac{1}{p}} \le a_N(T), \qquad a_{N+2}(T) \le \varepsilon.$$

The next work by the same authors [2] was devoted to asymptotic estimates of the a–numbers of the operator $T : L^p(R^+) \to L^p(R^+)$. The study was further developed in the article [5] by E.N. Lomakina and V.D. Stepanov, where implicit estimates for the Hardy operator in Lebesgue spaces in the case $1 < q < p < \infty$ were found and asymptotic estimates for T when $1 < p, q < \infty$ were derived. The authors also studied Schatten–Neumann (weak) norm estimates for T in all cases of integration parameters p, q and established the equivalence of the Schatten–Neumann norm of the operator to an integral generalization of the Hilbert-Schmidt formula in [5]. Later, M.A. Lifshitz and W. Linde in [4] considered the explicit asymptotic estimates for the approximation and entropy numbers of the Hardy operator in the case $1 < p, q < \infty$. The results of the monograph [4] was recently essentially complemented in paper A. A. Vasil'eva [9]. The next work by E.N. Lomakina and V.D. Stepanov [6] dealt with Lorentz spaces. In that work there was investigated the boundedness, compactness and measure of non–compactness of the Hardy operator with Oinarov's kernel, and estimates of the approximation numbers of the operator $T : L^{rs}_v(R^+) \to L^{pq}_u(R^+)$ when $1 < \max(r, s) \le \min(p, q) < \infty$. Further in [7], the authors continued to work with even more general Banach function spaces satisfying the $\ell-$ condition introduced by E.I. Berezhnoi.

In this work we obtain two–sided estimates which describe the behaviour of the approximation numbers of the Hardy operator and Schatten–Neumann norms in the new case, when the compact operator

$$Tf(x) = \int_0^x f(\tau)\, d\tau, \quad x > 0, \tag{1.1}$$

is acting from a Lebesgue space to a Lorentz space ($T : L^r_v(R^+) \to L^{pq}_\omega(R^+)$) under the condition $1 < p < r \le q < \infty$. The equivalence

$$\left(\sum_n a_n^s(T)\right)^{1/s} \approx \left(\int_0^\infty \left(\int_0^x v^{1-r'}(t)\, dt\right)^{s/r'} \left(\int_x^\infty \omega(t)\, dt\right)^{\frac{s}{p}-1} \omega(x)\, dx\right)^{1/s}$$

of the Schatten–Neumann norm of the operator $T : L^r_v(R^+) \to L^{pr}_\omega(R^+)$, when $1 < p < r < \infty$ and $1 < s < \infty$, to the integral expression depending only on the weights v and ω is established in this paper as wells. The weights v, ω are supposed to be Lebesgue measurable and non–negative function on $(0, \infty)$ such that $v(x) < \infty$, $\omega(x) < \infty$ a.e. on $(0, \infty)$.

Let $L^r_v(R^+)$ be the weighted Lebesgue space of all measurable functions f on $(0, \infty)$ satisfying the condition $\|f\|_{L^r_v} < \infty$:

$$L^r_v(R^+) = \left\{ f : \|f\|_{L^r_v} = \left(\int_0^\infty |f(x)|^r v(x)\, dx\right)^{1/r} < \infty \right\}.$$

Let a function f be defined on the measurable space $((0, \infty), \omega(x)dx)$. Given $1 < p, q < \infty$ and a weight $\omega(x)$ on $R^+ = (0, \infty)$ the Lorentz space $L^{pq}_\omega \equiv L^{pq}_\omega(R^+)$ consists of all measurable functions f such that

$$\|f\|_{L^{pq}_\omega} = \left(\int_0^\infty \frac{q}{p}\left(t^{\frac{q}{p}-1} f^*_\omega(t)^q\right) dt\right)^{1/q} < \infty,$$

where f^*_ω is the non–increasing rearrangement of the function f with respect to the measure ωdx:

$$f^*_\omega(t) = \inf\{\lambda > 0 : \quad \omega\left(\{x > 0 \ : \ |f(x)| > \lambda\}\right) \leq t\}.$$

For $1 < p, q < \infty$ and $1 < r < \infty$ the Lorentz space L^{pq}_ω and the Lebesgue space L^r_v are Banach function spaces with absolutely continuous norms. Recall that a norm in a Banach function space X is *absolutely continuous* if $\|f\chi_{E_n}\|_X \to 0$ for all $f \in X$ and any sequences of sets $\{E_n\} \subset R^+$ such that $\chi_{E_n}(x) \to 0$ a.e.

The dual space to L^{pq}_ω is the space

$$L^{p'q'}_\omega = \left\{ g : \int_0^\infty \mid fg \mid < \infty \text{ for all } f \in L^{pq}_\omega \right\}$$

with the norm

$$\|g\|_{L^{p'q'}_\omega} = \sup\left\{ \int_0^\infty \mid fg \mid : \ \|f\|_{L^{pq}_\omega} \leq 1 \right\} \tag{1.2}$$

and the related Hölder's inequality has the form

$$\left| \int_0^\infty f(x)g(x)\omega(x)\,dx \right| \leq \|f\|_{L^{pq}_\omega} \|g\|_{L^{p'q'}_\omega}.$$

Boundedness and compactness criteria for the operator (1.1) are contained in the following theorems.

Theorem 1.1. ([3],[8]) *Let* $1 < p, q < \infty$, $1 < r < \infty$ *and* $r \leq q$. *The inequality*

$$\|Tf\|_{L^{pq}_\omega} \leq C\|f\|_{L^r_v}$$

with operator (1.1) *holds for all* $f \geq 0$ *with a constant* C *independent of* f *if and only if*

$$A = \sup_{t>0} A(t) = \sup_{t>0} \left(\int_t^\infty \omega(x)\,dx \right)^{1/p} \left(\int_0^t v^{1-r'}(x)\,dx \right)^{1/r'} < \infty.$$

Theorem 1.2. ([3]) *Let* $1 < p, q < \infty$, $1 < r < \infty$ *and* $r \leq q$. *The operator* $T : L^r_v(R^+) \to L^{pq}_\omega(R^+)$ *of form* (1.1) *is compact if and only if*

$$A < \infty \quad \text{and} \quad \lim_{t\to 0+} A(t) = \lim_{t\to\infty} A(t) = 0.$$

2 Estimates of the approximation numbers of the Hardy operator

Let the operator $T : L^r_v(R^+) \to L^{pq}_\omega(R^+)$ be compact. Given a number ε, such that $0 < \varepsilon < \|T\|$, we choose a finite sequence of increasing numbers

$$0 = c_0 < c_1 < c_2 < ...c_{N-1} < c_N < c_{N+1} = \infty$$

such that

$$A[c_1] = A[c_N] = \varepsilon, \tag{2.1}$$

where $N = N(\varepsilon)$ and

$$A[c_1] = \sup_{0<t<c_1} A(t), \qquad A[c_N] = \sup_{c_N<t<\infty} A(t).$$

The rest of the numbers of the sequence $c_1 < c_2 < ...c_{N-1} < c_N$ are to be defined after proving the following theorem.

Theorem 2.1. *Let* $1 < p, q < \infty$, $1 < r < \infty$ *and* $r \le q$. *Let* $0 < a < b < \infty$, $I = (a, b)$,

$$F(x) = \int_a^x f(\tau)\, d\tau, \quad x \in I; \quad F_I = \frac{1}{\mu(I)} \int_I F(x) d\mu(x),$$

where

$$\mu(I) = \int_I d\mu(x) = \int_I g(x)\omega(x)\, dx;$$

and let a function g *on the interval* I *be defined by the condition*

$$(1-\delta) \left(\int_I \omega(x)\, dx \right)^{1/p} \|\chi_I g\|_{L_\omega^{p'q'}} \le \int_I g(x)\omega(x)\, dx$$

with $0 < \delta < 1$. *(The existence of the function* g *is guaranteed by the relation* (1.2)*).*

Then there exists a point $c \in I$ *such that*

$$\|\mathcal{T}_I\|_{L_\omega^{pq} \to L_v^r} = \sup_{f \ne 0} \frac{\|\chi_I (F - F_I)\|_{L_\omega^{pq}}}{\|\chi_I f\|_{L_v^r}} \approx \max(A(a,c), B(c,b)),$$

where

$$A(a,c) = \sup_{a<x<c} \left(\int_a^x \omega(t)\, dt \right)^{1/p} \left(\int_x^c v^{1-r'}(t)\, dt \right)^{1/r'},$$

$$B(c,b) = \sup_{c<x<b} \left(\int_c^x v^{1-r'}(t)\, dt \right)^{1/r'} \left(\int_x^b \omega(t)\, dt \right)^{1/p}.$$

Proof. Necessity. Fix $f \in L_v^r$, $c \in (a,b)$ and put

$$\Psi_c(x) = \begin{cases} -\int_x^c f(\tau)\, d\tau, & a \le x < c, \\ \int_c^x f(\tau)\, d\tau, & c \le x \le b \end{cases}$$

and $\Psi_{c,I} = \frac{1}{\mu(I)} \int_I \Psi_c \, d\mu$. Then

$$F(x) - F_I = \Psi_c(x) - \Psi_{c,I}. \tag{2.2}$$

To prove the lower estimate we choose $f \in L_v^r(I)$ so that supp$f \subseteq (a,c)$ and assume that the inequality

$$\left\|\chi_I (F - F_I)\right\|_{L_\omega^{pq}} \le C \|\chi_I f\|_{L_v^r}$$

holds for all $f \in L_v^r(R^+)$ with a constant C independent of f. Then

$$C\left\|\chi_{(a,c)} f\right\|_{L_v^r} \geq \left\|\chi_{(a,c)}(F - F_I)\right\|_{L_\omega^{pq}} = \left\|\chi_{(a,c)}\Big(\Psi_c - \Psi_{c,I}\Big)\right\|_{L_\omega^{pq}}$$

$$\geq \left\|\chi_{(a,c)}\Psi_c\right\|_{L_\omega^{pq}} - \frac{1}{\mu(I)}\left|\int_I \Psi_c(x)\, d\mu(x))\right| \left(\int_a^c \omega(t)\, dt\right)^{1/p}$$

$$\geq \left\|\chi_{(a,c)}\Psi_c\right\|_{L_\omega^{pq}} - \frac{1}{\mu(I)}\left\|\chi_{(a,c)}\Psi_c\right\|_{L_\omega^{pq}} \left\|\chi_{(a,c)} g\right\|_{L_\omega^{p'q'}} \left(\int_a^c \omega(t)\, dt\right)^{1/p}$$

$$= \left(1 - \frac{1}{\mu(I)}\left\|\chi_{(a,c)} g\right\|_{L_\omega^{p'q'}} \left(\int_a^c \omega(t)\, dt\right)^{1/p}\right)\left\|\chi_{(a,c)}\Psi_c\right\|_{L_\omega^{pq}}$$

$$\equiv \left(1 - \frac{H(a,c)}{\mu(I)}\right)\left\|\chi_{(a,c)}\Psi_c\right\|_{L_\omega^{pq}},$$

where $H(a,c) := \left\|\chi_{(a,c)} g\right\|_{L_\omega^{p'q'}} \left(\int_a^c \omega(t)\, dt\right)^{1/p}$. By the absolute continuity of the norms in L_ω^{pq} and $L_\omega^{p'q'}$, we can find a point $c \in (a,b)$ such that $H(a,c) = \beta\mu(I)$ for any fixed $\beta \in (0, 1-\delta)$. Thus, by Theorem 1.1 applied to the interval (a,c), we have

$$C \geq (1-\beta) A(a,c). \tag{2.3}$$

Similar argumentation, but for f with supp$f \subseteq (c,b)$, leads to the estimate

$$C\left\|\chi_{(c,b)} f\right\|_{L_v^r} \geq \left(1 - \frac{W(c,b)}{\mu(I)}\right)\left\|\chi_{(c,b)}\Psi_c\right\|_{L_\omega^{pq}},$$

where $W(c,b) = \left(\int_c^b \omega(t)\, dt\right)^{1/p} \left\|\chi_{(c,b)} g\right\|_{L_\omega^{p'q'}}$.

We have

$$1 - \frac{W(c,b)}{\mu(I)} = \frac{1}{\mu(I)}(\mu(I) - W(c,b))$$

$$\geq \frac{1}{\mu(I)}\left((1-\delta)\left(\int_I \omega(t)\, dt\right)^{1/p} \left\|\chi_I g\right\|_{L_\omega^{p'q'}} - W(c,b)\right)$$

$$\geq \beta(1-\delta) - \frac{W(c,b)}{\mu(I)}.$$

If $c \to b$ then $W(c,b) \to 0$ and $\beta \to (1-\delta)$. Therefore, we may choose $c \in (a,b)$ so that

$$\frac{W(c,b)}{\mu(I)} \leq \frac{\beta(1-\delta)}{2}.$$

Since the operator is bounded, then

$$C \geq \frac{\beta(1-\delta)}{2} B(c,b). \tag{2.4}$$

By equating the right-hand sides of (2.3) and (2.4) we find β:

$$1-\beta=\frac{\beta(1-\delta)}{2} \qquad \Longrightarrow \qquad \beta=\frac{2}{3-\delta}.$$

Thus,

$$C \geq \max(A(a,c),B(c,b))$$

and the lower estimate is proved.

Sufficiency. In order to prove the upper estimate we use the boundedness of the operator and Hölder's inequality:

$$\Big\|\chi_I(F-F_I)\Big\|_{L^{pq}_\omega}=\Big\|\chi_I\Big(\Psi_c-\Psi_{c,I}\Big)\Big\|_{L^{pq}_\omega}$$

$$\leq \Big\|\chi_I\Psi_c\Big\|_{L^{pq}_\omega}+\frac{1}{\mu(I)}\Big\|\chi_I\Psi_c\Big\|_{L^{pq}_\omega}\Big\|\chi_I g\Big\|_{L^{p'q'}_\omega}\left(\int_I\omega(t)\,dt\right)^{1/p}$$

$$\leq \frac{2}{(1-\delta)}\Big\|\chi_I\Psi_c\Big\|_{L^{pq}_\omega}\leq\frac{2}{(1-\delta)}\left(\Big\|\chi_{[a,c]}\Psi_c\Big\|_{L^{pq}_\omega}+\Big\|\chi_{[c,b]}\Psi_c\Big\|_{L^{pq}_\omega}\right)$$

$$\ll \max(A(a,c),B(c,b))\left(\Big\|\chi_{(a,c)}f\Big\|_{L^r_v}+\Big\|\chi_{(c,b)}f\Big\|_{L^r_v}\right)$$

$$\ll \max(A(a,c),B(c,b))\Big\|\chi_I f\Big\|_{L^r_v}.$$

This immediately yields the required result. □

Going back to the definition of the sequence $\{c_k\}$ for $k=2,3,...,N-1$ we consider the operator

$$\mathcal{T}_I f(x)=\chi_I(x)(F(x)-F_I)$$

on a finite interval $I\subset[c_1,c_N]$. By Theorem 2.1 the operator norm continuously depends on the interval I. Therefore, we can choose intervals $I_k=[c_k,c_{k+1}]$, $k=1,...N-1$ so that

$$\|\mathcal{T}_{I_k}\|=\varepsilon,\ k=1,...N-2,\quad \|\mathcal{T}_{I_{N-1}}\|\leq\varepsilon, \tag{2.5}$$

where N depends on ε. Notice that the choice of the number δ and the measures $\mu(I_k)$ for each of the intervals, relevantly to the condition of the theorem, is not further essential.

For our further investigation we shall need the following technical lemma.

Lemma 2.1. ([3], [8]) *Let $1<p,q<\infty$ and $(0,\infty)=\cup I_k$, where $\{I_k\}$ is a sequence of disjoint measurable intervals.*

1) *If* $\max(p,q)\leq\alpha$ *then*

$$\sum_k\|\chi_{I_k}h\|^\alpha_{L^{pq}_\omega}\leq\|h\|^\alpha_{L^{pq}_\omega}. \tag{2.6}$$

2) *If* $\min(p,q)\geq\alpha$ *then,*

$$\|h\|^\alpha_{L^{pq}_\omega}\leq\sum_k\|\chi_{I_k}h\|^\alpha_{L^{pq}_\omega}. \tag{2.7}$$

Upper and lower estimates for the approximation numbers of the operator T are contained in the following

Theorem 2.2. *Let* $1 < p < r \le q < \infty$. *Assume that the operator* $T : L_v^r \to L_\omega^{pq}$ *of form* (1.1) *is compact. Given* $0 < \varepsilon < \|T\|$ *and integer* $N > 2$, *let intervals* $I_k = [c_k, c_{k+1}], k = 0, 1, ...N$ *be chosen so that conditions* (2.1) *and* (2.5) *are satisfied. Then*

$$\frac{1}{2}\varepsilon\,(N+1)^{\frac{1}{q}-\frac{1}{r}} \le a_{N+1}(T), \qquad a_N(T) \le \varepsilon.$$

Proof. The upper estimate. For $k = 1, 2, 3, ...N-1$ we define

$$F_k(x) = \chi_{I_k}(x) \int_{c_k}^{x} f(\tau)\, d\tau,$$

$$P_k f(x) = \chi_{I_k}(x) \left\{ Tf(x) - (F_k(x) - F_{k,I_k}) \right\},$$

where the operator $P = \sum_{k=1}^{N-1} P_k$ is linear bounded with *rank* $P \le N-1$. Then, by Jensen's inequality,

$$\|Tf - Pf\|_{L_\omega^{pq}}^q$$

$$\le \left\|\chi_{[0,c_1]} Tf\right\|_{L_\omega^{pq}}^q + \sum_{k=1}^{N-1} \left\|\chi_{I_k}(Tf - P_k f)\right\|_{L_\omega^{pq}}^q + \left\|\chi_{[c_N,\infty)} Tf\right\|_{L_\omega^{pq}}^q$$

$$\le \varepsilon^q \left\|\chi_{[0,c_1]} f\right\|_{L_v^r}^q + \sum_{k=1}^{N-1} \left\|\mathcal{T}_{I_k} f\right\|_{L_\omega^{pq}}^q + \varepsilon^q \left\|\chi_{[c_N,\infty)} f\right\|_{L_v^r}^q$$

$$\le \varepsilon^q \sum_{k=0}^{N} \left\|\chi_{I_k} f\right\|_{L_v^r}^q \le \varepsilon^q \left\|f\right\|_{L_v^r}^q.$$

Therefore, by the definition of the approximation numbers,

$$a_N(T) \le \varepsilon.$$

The lower estimate. Fix $\lambda \in (0,1)$ and define a sequence of functions $f_k \in L_v^r$, with $\operatorname{supp} f_k \subset I_k$, satisfying the inequalities

$$\frac{\left\|\chi_{I_i} F_i\right\|_{L_\omega^{pq}}}{\left\|f_i\right\|_{L_v^r}} \ge \lambda\varepsilon, \quad i = 0, N, \tag{2.8}$$

$$\frac{\left\|\chi_{I_k}\left(F_k - F_{k,I_k}\right)\right\|_{L_\omega^{pq}}}{\left\|f_k\right\|_{L_v^r}} \ge \lambda\varepsilon, \quad k = 1, 2, ..., N-1, \tag{2.9}$$

where

$$F_k(x) = \int_{c_k}^{x} f_k(\tau)\, d\tau, \quad k = 0, 1, ..., N.$$

Let $\widetilde{P}:L^r_v\to L^{pq}_\omega$ be a linear bounded operator with rank$\widetilde{P}\le N$. Then, by linear dependence of the functions $\widetilde{P}f_k$, $k=0,1,...,N$, there exist constants $\nu_0,\nu_1,\nu_2,...,\nu_N$ such that

$$\sum_{k=0}^{N}\nu_k\left(\widetilde{P}f_k\right)=\widetilde{P}\left(\sum_{k=0}^{N}\nu_k f_k\right)=0.$$

Put

$$f(\tau)=\sum_{k=0}^{N}\nu_k f_k(\tau)$$

and

$$F(x)=\int_0^x f(\tau)\,d\tau,\quad k=1,...,N-1,\quad x>0.$$

We have for all $x\in I_k$:

$$F(x)=\nu_k\int_0^{c_k}f(\tau)\,d\tau+\nu_k\int_{c_k}^{x}f(\tau)\,d\tau\equiv\mu_k+\nu_k F_k(x), \tag{2.10}$$

$k=1,...,N-1$, where the constants μ_k are defined by the right hand side.

We shall need the following inequality for our proof of the lower bound:

$$\|\chi_I(F-F_I)\|_{L^{pq}_\omega}\le 2\inf_{c\in R}\|\chi_I(F-c)\|_{L^{pq}_\omega}. \tag{2.11}$$

Indeed,

$$\begin{aligned}\|\chi_I(F-F_I)\|_{L^{pq}_\omega}&\le\|\chi_I(F-c-(F-c)_I)\|_{L^{pq}_\omega}\\&\le\|\chi_I(F-c)\|_{L^{pq}_\omega}+|(F-c)_I|\,\|\chi_I\|_{L^{pq}_\omega}\\&\le\|\chi_I(F-c)\|_{L^{pq}_\omega}+\frac{\left(\int_I\omega(x)\,dx\right)^{1/p}}{\mu(I)}\left|\int_I(F-c)g(x)\omega(x)\,dx\right|\\&\le\|\chi_I(F-c)\|_{L^{pq}_\omega}+\frac{\left(\int_I\omega(x)\,dx\right)^{1/p}}{\mu(I)}\|\chi_I(F-c)\|_{L^{pq}_\omega}\|\chi_I g\|_{L^{p'q'}_\omega}\\&\le\|\chi_I(F-c)\|_{L^{pq}_\omega}+\frac{1}{(1-\delta)}\|\chi_I(F-c)\|_{L^{pq}_\omega}\le 2\|\chi_I(F-c)\|_{L^{pq}_\omega}.\end{aligned}$$

We now obtain, by using (2.6) and with help of decomposition (2.10) and inequality (2.8):

$$\begin{aligned}\left\|Tf-\widetilde{P}f\right\|^q_{L^{pq}_\omega}&=\left\|Tf\right\|^q_{L^{pq}_\omega}\\&\ge\|\chi_{I_0}F_0\|^q_{L^{pq}_\omega}+\sum_{k=1}^{N-1}\left\|\chi_{I_k}F\right\|^q_{L^{pq}_\omega}+\left\|\chi_{I_N}F_N\right\|^q_{L^{pq}_\omega}\\&\ge(\lambda\varepsilon)^q\|\nu_0 f_0\|^q_{L^r_v}+\sum_{k=1}^{N-1}\left\|\chi_{I_k}\left(\nu_k F_k+\mu_k\right)\right\|^q_{L^{pq}_\omega}+(\lambda\varepsilon)^q\|\nu_N f_N\|^q_{L^r_v}\end{aligned}$$

(by (2.11))

$$\geq (\lambda\varepsilon)^q \|\nu_0 f_0\|^q_{L^r_v} + \left(\frac{1}{2}\right)^q \sum_{k=1}^{N-1} \Big\| \chi_{I_k}\big(\nu_k F_k - (\nu_k F_k)_{I_k}\big)\Big\|^q_{L^{pq}_\omega} + (\lambda\varepsilon)^q \|\nu_N f_N\|^q_{L^r_v}$$

$$= (\lambda\varepsilon)^q \|\nu_0 f_0\|^q_{L^r_v} + \left(\frac{1}{2}\right)^q \sum_{k=1}^{N-1} |\nu_k|^q \Big\| \chi_{I_k}\big(F_k - F_{k,I_k}\big)\Big\|^q_{L^{pq}_\omega} + (\lambda\varepsilon)^q \|\nu_N f_N\|^q_{L^r_v}$$

(by (2.9))

$$\geq (\lambda\varepsilon)^q \|\nu_0 f_0\|^q_{L^r_v} + \left(\frac{\lambda\varepsilon}{2}\right)^q \sum_{k=1}^{N-1} |\nu_k|^q \Big\| f_k \Big\|^q_{L^r_v} + (\lambda\varepsilon)^q \|\nu_N f_N\|^q_{L^r_v}$$

$$= (\lambda\varepsilon)^q \|\nu_0 f_0\|^q_{L^r_v} + \left(\frac{\lambda\varepsilon}{2}\right)^q \sum_{k=1}^{N-1} \Big\| \nu_k f_k \Big\|^q_{L^r_v} + (\lambda\varepsilon)^q \|\nu_N f_N\|^q_{L^r_v}$$

$$\geq \left(\frac{\lambda\varepsilon}{2}\right)^q \sum_{k=0}^{N} \Big\| \nu_k f_k \Big\|^q_{L^r_v}$$

$$\geq \left(\frac{\lambda\varepsilon}{2}\right)^q \left(\sum_{k=0}^{N} \Big\| \nu_k f_k \Big\|^r_{L^r_v}\right)^{q/r} (N+1)^{1-q/r}$$

$$= \left(\frac{\lambda\varepsilon}{2}\right)^q (N+1)^{1-q/r} \Big\| f \Big\|^q_{L^r_v}.$$

Hence

$$a_{N+1}(T) \geq \frac{1}{2}\lambda\varepsilon (N+1)^{1/q-1/r},$$

and, letting $\lambda \to 1$, we obtain the required estimate. □

3 Schatten–Neumann norm estimates of the Hardy operator

Let a sequence $\{\xi_n\}$, $n \in \mathbf{Z}$, be defined by the formula

$$U(\xi_n) = \int_0^{\xi_n} v^{1-r'}(t)\, dt = 2^{n+1}. \tag{3.1}$$

Moreover, let

$$J_n = (\xi_{n-1}, \xi_n), \tag{3.2}$$

and

$$\sigma_n = \left(\int_{\xi_{n-1}}^{\xi_n} v^{1-r'}(t)\, dt\right)^{1/r'} \left(\int_{\xi_n}^{\xi_{n+1}} \omega(t)\, dt\right)^{1/p}, \tag{3.3}$$

$$\sigma_n = 2^{n/r'} \left(\int_{\xi_n}^{\xi_{n+1}} \omega(t)\, dt\right)^{1/p}, \tag{3.4}$$

$$\int_{\xi_n}^{\xi_{n+1}} \omega(t)\,dt = \frac{\sigma_n^p}{2^{np/r'}}.$$

Since

$$\sigma_n \le \left(\int_0^{\xi_n} v^{1-r'}(t)\,dt\right)^{\frac{1}{r'}} \left(\int_{\xi_n}^{\xi_{n+1}} \omega(t)\,dt\right)^{\frac{1}{p}} = 2^{\frac{n+1}{r'}} \left(\int_{\xi_n}^{\xi_{n+1}} \omega(t)\,dt\right)^{\frac{1}{p}},$$

then

$$2^{np/r'}\left(\int_{\xi_n}^{\xi_{n+1}} \omega(t)\,dt\right) = \sigma_n^p \le 2^{(n+1)p/r'}\left(\int_{\xi_n}^{\xi_{n+1}} \omega(t)\,dt\right). \tag{3.5}$$

Lemma 3.1. *Let numbers $n_1, n_2, n_3 \in \mathbf{Z}$ be such that $n_1 < n_2 < n_3$, and points c_0 and c_1 of the partition $I_k = (c_k, c_{k+1})$ fall within the following intervals: $c_0 \in J_{n_1}$, $x_0 \in J_{n_2}$, $c_1 \in J_{n_3}$. Then*

$$\left(\int_{c_0}^{x_0} v^{1-r'}(t)\,dt\right)^{1/r'} \left(\int_{x_0}^{c_1} \omega(t)\,dt\right)^{1/p} \le 2^{\frac{2}{r'}+\frac{1}{p}} \max_{n_2-1\le n\le n_3-1} \sigma_n. \tag{3.6}$$

Proof. We have

$$\begin{aligned}
&\left(\int_{c_0}^{x_0} v^{1-r'}(t)\,dt\right)^{1/r'} \left(\int_{x_0}^{c_1} \omega(t)\,dt\right)^{1/p} \\
&\le \left(\int_{\xi_{n_1-1}}^{\xi_{n_2}} v^{1-r'}(t)\,dt\right)^{1/r'} \left(\int_{\xi_{n_2-1}}^{\xi_{n_3}} \omega(t)\,dt\right)^{1/p} \\
&= \Big[U(\xi_{n_2}) - U(\xi_{n_1-1})\Big]^{1/r'} \left(\int_{\xi_{n_2-1}}^{\xi_{n_3}} \omega(t)\,dt\right)^{1/p} \\
&\le \Big[U(\xi_{n_2})\Big]^{1/r'} \left(\int_{\xi_{n_2-1}}^{\xi_{n_3-1}} \omega(t)\,dt + \int_{\xi_{n_3-1}}^{\xi_{n_3}} \omega(t)\,dt\right)^{1/p} \\
&= 2^{(n_2+1)/r'} \left(\frac{\sigma_{n_2-1}^p}{2^{(n_2-1)p/r'}} + \frac{\sigma_{n_3-1}^p}{2^{(n_3-1)p/r'}}\right)^{1/p} \\
&\le 2^{(n_2+1)/r'} \max_{n_2-1\le n\le n_3-1} \sigma_n \left(\frac{2}{2^{(n_2-1)p/r'}}\right)^{1/p} \\
&= 2^{\frac{2}{r'}+\frac{1}{p}} \max_{n_2-1\le n\le n_3-1} \sigma_n.
\end{aligned}$$

□

Lemma 3.2. *Let $\gamma = \dfrac{r'p}{r'+p}$, $I_k = (c_k, c_{k+1})$, $x_k \in I_k$ and $\xi_{n_3-1} < c_1 < c_2 < \dots < c_l < \xi_{n_3}$. Then*

$$\sum_{k=1}^{l} \left(\int_{c_k}^{x_k} v^{1-r'}(t)\,dt\right)^{\gamma/r'} \left(\int_{x_k}^{c_{k+1}} \omega(t)\,dt\right)^{\gamma/p} \le 2^{\gamma/r'} \sigma_{n_3-1}^{\gamma}.$$

Proof. We obtain

$$\sum_{k=1}^{l}\Big(\int_{c_k}^{x_k} v^{1-r'}(t)\,dt\Big)^{\gamma/r'}\Big(\int_{x_k}^{c_{k+1}} \omega(t)\,dt\Big)^{\gamma/p}$$

$$\leq \sum_{k=1}^{l}\Big(\int_{I_k} v^{1-r'}(t)\,dt\Big)^{p/(r'+p)}\Big(\int_{I_k} \omega(t)\,dt\Big)^{r'/(p+r')}$$

(applying Hölder's inequality with exponents $\dfrac{r'+p}{p}$ and $\dfrac{r'+p}{r'}$)

$$\leq \left(\sum_{k=1}^{l}\int_{I_k} v^{1-r'}(t)\,dt\right)^{\frac{p}{p+r'}}\left(\sum_{k=1}^{l}\int_{I_k} \omega(t)\,dt\right)^{\frac{r'}{p+r'}}$$

$$\leq \left(\int_{\xi_{n_3-1}}^{\xi_{n_3}} v^{1-r'}(t)\,dt\right)^{\frac{\gamma}{r'}}\left(\int_{\xi_{n_3-1}}^{\xi_{n_3}} \omega(t)\,dt\right)^{\frac{\gamma}{p}}$$

$$= \Big[U(\xi_{n_3}) - U(\xi_{n_3-1})\Big]^{\gamma/r'}\left(\frac{\sigma^p_{n_3-1}}{2^{(n_3-1)p/r'}}\right)^{\gamma/p}$$

$$= \frac{\Big[2^{n_3+1} - 2^{n_3}\Big]^{\gamma/r'}}{2^{(n_3-1)\gamma/r'}}\sigma^{\gamma}_{n_3-1} = 2^{\gamma/r'}\sigma^{\gamma}_{n_3-1}.$$

□

Lemma 3.3. *Let $I=(a,b)$, $I\subset \mathbf{R}^+$ and*

$$A(I) = \sup_{a<x<b}\left(\int_x^b v^{1-r'}(t)\,dt\right)^{\frac{1}{r'}}\left(\int_a^x \omega(t)\,dt\right)^{\frac{1}{p}},$$

$$B(I) = \sup_{a<x<b}\left(\int_a^x v^{1-r'}(t)\,dt\right)^{\frac{1}{r'}}\left(\int_x^b \omega(t)\,dt\right)^{\frac{1}{p}}.$$

Then

$$A(\overline{J_n}\bigcup\overline{J_{n+1}}) \geq 4^{\frac{1}{r'}}\sigma_{n-1},$$

$$B(\overline{J_n}\bigcup\overline{J_{n+1}}) \geq \sigma_n.$$

Proof. We have

$$A(\overline{J_n}\bigcup\overline{J_{n+1}}) = \sup_{\xi_{n-1}<x<\xi_{n+1}}\left(\int_x^{\xi_{n+1}} v^{1-r'}(t)\,dt\right)^{\frac{1}{r'}}\left(\int_{\xi_{n-1}}^x \omega(t)\,dt\right)^{\frac{1}{p}}$$

$$\geq \left(\int_{\xi_n}^{\xi_{n+1}} v^{1-r'}(t)\,dt\right)^{\frac{1}{r'}}\left(\int_{\xi_{n-1}}^{\xi_n} \omega(t)\,dt\right)^{\frac{1}{p}}$$

$$= \Big[U(\xi_{n+1}) - U(\xi_n)\Big]^{1/r'} \left(\frac{\sigma_{n-1}^p}{2^{(n-1)p/r'}}\right)^{1/p}$$

$$= \frac{\Big[2^{n+2} - 2^{n+1}\Big]^{1/r'}}{2^{\frac{n-1}{r'}}} \cdot \sigma_{n-1} = 4^{\frac{1}{r'}} \sigma_{n-1},$$

and

$$B(\overline{J_n} \bigcup \overline{J_{n+1}}) = \sup_{\xi_{n-1} < x < \xi_{n+1}} \left(\int_{\xi_{n-1}}^{x} v^{1-r'}(t)\, dt\right)^{\frac{1}{r'}} \left(\int_{x}^{\xi_{n+1}} \omega(t)\, dt\right)^{\frac{1}{p}}$$

$$\geq \left(\int_{\xi_{n-1}}^{\xi_n} v^{1-r'}(t)\, dt\right)^{\frac{1}{r'}} \left(\int_{\xi_n}^{\xi_{n+1}} \omega(t)\, dt\right)^{\frac{1}{p}} = \sigma_n.$$

□

Lemma 3.4. *Let* $0 < a < b < \infty$*,* $I = (a, b) \subset \mathbf{R}^+$ *and a point* $c \in I$ *be chosen so that*

$$\|\mathcal{T}_I\| = \sup_{f \neq 0} \frac{\|\chi_I(F - F_I)\|_{L^{pq}_\omega}}{\|\chi_I f\|_{L^r_v}} \approx \max(A(a, c), B(c, b)),$$

where

$$A(a, c) = \sup_{a < s < c} \left(\int_{s}^{c} v^{1-r'}(t)\, dt\right)^{\frac{1}{r'}} \left(\int_{a}^{s} \omega(t)\, dt\right)^{\frac{1}{p}},$$

$$B(c, b) = \sup_{c < s < b} \left(\int_{c}^{s} v^{1-r'}(t)\, dt\right)^{\frac{1}{r'}} \left(\int_{s}^{b} \omega(t)\, dt\right)^{\frac{1}{p}}.$$

Define

$$D(I) = \max\Big(A(a, c), B(c, b)\Big),$$

and let $0 < \varepsilon < \|T\|$*,*

$$S_I(\varepsilon) = \{n \in \mathbf{Z} : \quad \overline{J_{n+1}} \subset I, \quad \sigma_n > \varepsilon\} \quad \text{and} \quad \text{card} S_I(\varepsilon) \geq 4.$$

Then $D(I) > \varepsilon$*.*

Proof. If

$$n_1 = \min\{n : \ n \in S_I(\varepsilon)\},$$
$$n_2 = \max\{n : \ n \in S_I(\varepsilon)\},$$

then $\overline{J_{n_1}} \bigcup \overline{J_{n_1+1}} \subset (a, c)$, and, by Lemma 3.3,

$$A(a, c) \geq A\Big(\overline{J_{n_1}} \bigcup \overline{J_{n_1+1}}\Big) \geq 4^{\frac{1}{p'}} \sigma_{n_1} > 4^{\frac{1}{p'}} \varepsilon > \varepsilon.$$

For the same reason, $\overline{J_{n_2-1}} \bigcup \overline{J_{n_2}} \subset (c, b)$ and

$$B(c, b) \geq B\Big(\overline{J_{n_2-1}} \bigcup \overline{J_{n_2}}\Big) \geq \sigma_{n_2-1} > \varepsilon.$$

Therefore,

$$D(I) = \max\Big(A(a, c), B(c, b)\Big) > \varepsilon.$$

□

Lemma 3.5. *Let* $0 < \varepsilon < \|T\|$ *and card* $S_I(\varepsilon) \geq 4$, *then*

$$\|\mathcal{T}_I\| > \varepsilon.$$

Proof. By Theorem 2.1 $\|\mathcal{T}_I\| \approx D(I)$. From here the result follows by Lemma 3.4. □

Lemma 3.6. *Let* $0 < \varepsilon < \|T\|$ *and* $N = N(\varepsilon)$ *be defined by formulas* (2.1) *and* (2.5). *Then*

$$card\ \{k \in \mathbf{Z}\ : \sigma_k > \varepsilon\} \leq 6N(\varepsilon).$$

Proof. We have

$$card\ \{k \in \mathbf{Z}\ : c_i \in \overline{J_k}\ \text{ for some } i,\ 1 \leq i \leq N\} \leq 2N. \tag{3.7}$$

For $k \in \mathbf{Z}$ out from the set (3.7) such that $\overline{J_k} \subset I_i = (c_i, c_{i+1})$ for $1 \leq i \leq N$, we obtain, by the choice of $\|\mathcal{T}_{I_k}\|$ and in view of Lemma 3.5, that

$$card\ \{k \in \mathbf{Z}\ :\ \overline{J_k} \subset I_i, \sigma_k > \varepsilon\} \leq 3.$$

Therefore,

$$\begin{aligned} card\ \{k \in \mathbf{Z}\ :\ \sigma_k > \varepsilon\} &= \sum_{i=0}^{N} card\ \{k \in \mathbf{Z}\ :\ \overline{J_k} \subset I_i, \sigma_k > \varepsilon\} + 2N \\ &\leq 3(N+1) + 2N \leq 6N. \end{aligned}$$

□

Lemma 3.7. *The following estimate is true for any* $t > 0$*:*

$$card\ \{k \in \mathbf{Z}\ : \sigma_k > t\} \leq 6\ \ card\ \left\{k \in \mathbf{N}\ : a_k(T) k^{\frac{1}{r} - \frac{1}{q}} \geq \frac{t}{2}\right\}.$$

Proof. By Theorem 2.2,

$$card\ \left\{k \in \mathbf{N}\ : a_k(T) k^{\frac{1}{r} - \frac{1}{q}} \geq \frac{1}{2}\varepsilon\right\} \geq N(\varepsilon).$$

Then we obtain, by Lemma 3.6,

$$card\ \left\{k \in \mathbf{Z}\ : \sigma_k > t\right\} \leq 6N(t) \leq 6\ \ card\ \{k \in \mathbf{N}\ : a_k(T) k^{\frac{1}{r} - \frac{1}{q}} \geq \frac{t}{2}\right\},$$

and the proof is complete. □

Consider the space $\ell^s_\omega(\mathbf{Z})$, $s > 1$, which consists of all sequences $\{x_k\}$ satisfying the condition $\|\{x_k\}\|_{\ell^s_\omega(\mathbf{Z})} < \infty$, where

$$\|\{x_k\}\|_{\ell^s_\omega(\mathbf{Z})} = \sup_{t>0} t (card\ \{\ k \in \mathbf{Z} : |x_k| > t\})^{1/s}.$$

Theorem 3.1. *For any* $s \in (1, \infty)$

$$\left\|\{\sigma_k\}\right\|^s_{\ell^s_\omega(\mathbf{Z})} \leq 6 \cdot 2^s \left\|\left\{a_k(T) k^{\frac{1}{r} - \frac{1}{q}}\right\}\right\|^s_{\ell^s_\omega(\mathbf{N})}.$$

Proof. Let $\left\{a_k(T)k^{\frac{1}{r}-\frac{1}{q}}\right\} \in \ell^s_\omega(\mathbf{N})$. Then, by Lemma 3.7,

$$card\ \left\{k \in \mathbf{Z}\ : \sigma_k > t\right\} \le 6\ \ card\ \left\{k \in \mathbf{N}\ : a_k(T)k^{\frac{1}{r}-\frac{1}{q}} \ge \frac{t}{2}\right\}$$

and, therefore,

$$\sup_{t>0} t^s card\ \left\{k \in \mathbf{Z}\ : \sigma_k > t\right\} \le 6 \sup_{t>0} t^s card\ \left\{k \in \mathbf{N}\ : a_k(T)k^{\frac{1}{r}-\frac{1}{q}} \ge \frac{t}{2}\right\}$$

$$\left\|\{\sigma_k\}\right\|^s_{\ell^s_\omega(\mathbf{Z})} \le 6 \cdot 2^s \left\|\left\{a_k(T)k^{\frac{1}{r}-\frac{1}{q}}\right\}\right\|^s_{\ell^s_\omega(\mathbf{N})}.$$

□

Theorem 3.2. *For any* $s \in (0,\infty)$

$$\left\|\{\sigma_k\}\right\|^s_{\ell^s(\mathbf{Z})} \le 6 \cdot 2^s \left\|\left\{a_k(T)k^{\frac{1}{r}-\frac{1}{q}}\right\}\right\|^s_{\ell^s(\mathbf{N})}.$$

Proof. We have

$$\left\|\{\sigma_k\}\right\|^s_{\ell^s(\mathbf{Z})} = s\int_0^\infty t^{s-1} card\{k \in \mathbf{Z}\ : \sigma_k > t\}\,dt$$

$$\le 6s\int_0^\infty t^{s-1} card\left\{k \in \mathbf{N}\ : a_k(T)k^{\frac{1}{r}-\frac{1}{q}} \ge \frac{t}{2}\right\}dt$$

$$= 6 \cdot 2^s \left\|\left\{a_k(T)k^{\frac{1}{r}-\frac{1}{q}}\right\}\right\|^s_{\ell^s(\mathbf{N})}.$$

□

Theorem 3.3. *Let* $1 < p < r \le q < \infty,\ \gamma = \dfrac{pr'}{p+r'}$, $s > \gamma$ *and* $T : L^r_v(R^+) \to L^{pq}_\omega(R^+)$ *be a compact operator of form* (1.1). *Then*

$$\left\|\{a_k(T)\}\right\|^s_{\ell^s_\omega(\mathbf{N})} \le C^s(p,r')\beta(s/\gamma)\left\|\{\sigma_k\}\right\|^s_{\ell^s_\omega(\mathbf{Z})},$$

$$\left\|\{a_k(T)\}\right\|^s_{\ell^s(\mathbf{N})} \le C^s(p,r')\beta(s/\gamma)\left\|\{\sigma_k\}\right\|^s_{\ell^s(\mathbf{Z})}.$$

Proof. Let $0 < \varepsilon < \|T\|$, $N = N(\varepsilon)$ be defined by formulas (2.1) and (2.5). Then for any c_k there exists a number j_k such that $c_k \subset \overline{J}_{j_k}$ and only the following cases are possible:

(1) $j_{k_0} < j_{k_0+1}$

(2) $j_k = j_{k+1} = ... = j_{k+m_k}$, $I_i \subset J_{j_k}$, $k \le i \le k+m_k$, $m_k > 1$.

We obtain, by Theorem 2.1 and Lemma 3.1,

$$(1)\ \ \varepsilon = \|\mathcal{T}_{I_{k_0}}\| \le C_1 D(I_{k_0}) \le C_1\Big(A(I_{k_0}) + B(I_{k_0})\Big) \le C \sup_{j_{k_0} \le j \le j_{k_0+1}} \sigma_j \equiv C\sigma_{j_k}$$

for some $j_k \in [j_{k_0}, j_{k_0+1}]$. We also have:

$$(2)\quad \varepsilon^\gamma m_k = \sum_{i=k}^{k+m_k} \|\mathcal{T}_{I_i}\|^\gamma \le C^\gamma \sigma_{j_k}^\gamma, \quad \text{where} \quad \gamma = \frac{pr'}{p+r'}.$$

Then

$$N(\varepsilon) = card\ \Big\{k:\ \sigma_{j_k} \ge \frac{\varepsilon}{C}\Big\} + \sum_{k:m_k>1} card\ \Big\{k:\ \sigma_{j_k} \ge \frac{\varepsilon m_k^{1/\gamma}}{C}\Big\}$$

$$\le \sum_{n=1}^\infty card\ \Big\{k:\ \sigma_k \ge \frac{n^{1/\gamma}\varepsilon}{C}\Big\}.$$

By Theorem 2.2,

$$card\ \{k \in \mathbf{N}\ : a_k > \varepsilon\} \le N(\varepsilon) + 1 \le 2N(\varepsilon).$$

Therefore,

$$\Big\|\{a_k\}\Big\|_{\ell^s_\omega(\mathbf{N})}^s = 2\sup_{t>0} t^s N(t)$$

$$\le 2\sup_{t>0} t^s \sum_{n=1}^\infty card\Big\{k \in \mathbf{Z}:\ \sigma_k \ge \frac{n^{1/r}t}{C}\Big\} = 2C^s\Big(\sum_{n=1}^\infty \frac{1}{n^{s/r}}\Big)\Big\|\{\sigma_k\}\Big\|_{\ell^s_\omega(\mathbf{Z})}^s,$$

and

$$\Big\|\{a_k(T)\}\Big\|_{\ell^s(\mathbf{N})}^s = s\int_0^\infty t^{s-1} card\{k \in \mathbf{N}\ : a_k(T) > t\}\, dt$$

$$\le s\int_0^\infty t^{s-1} 2N(t)\, dt \le 2s\int_0^\infty \sum_{n=1}^\infty t^{s-1} card\Big\{k \in \mathbf{Z}:\ \sigma_k \ge \frac{n^{1/\gamma}t}{C}\Big\}\, dt$$

$$= 2sC^s \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{s/\gamma}}\Big(\frac{tn^{s/\gamma}}{C}\Big)^{s-1} card\Big\{k:\ \sigma_k \ge \frac{n^{1/\gamma}t}{C}\Big\}\, d\Big(\frac{n^{s/\gamma}t}{C}\Big)$$

$$= 2C^s\Big(\sum_{n=1}^\infty \frac{1}{n^{s/\gamma}}\Big)\Big\|\{\sigma_k\}\Big\|_{\ell^s(\mathbf{Z})}^s \equiv C^s(p,q)\beta(s/\gamma)\Big\|\{\sigma_k\}\Big\|_{\ell^s(\mathbf{Z})}^s.$$

□

Let the sequence $\{\xi_k\}$, $k \in \mathbf{Z}$ be defined by formula (3.1), and $\{\eta_k\}$, $k \in \mathbf{Z}$ be such that

$$V(\eta_k) = \int_{\eta_k}^\infty \omega(t)\, dt = 2^{-k+1}.$$

Put

$$\delta_k = \Big(\int_{\eta_{k-1}}^{\eta_k} v^{1-r'}(t)\, dt\Big)^{1/r'}\Big(\int_{\eta_k}^{\eta_{k+1}} \omega(t)\, dt\Big)^{1/p}$$

$$= 2^{-k/p}\Big(\int_{\eta_{k-1}}^{\eta_k} v^{1-r'}(t)\, dt\Big)^{1/r'}$$

and denote

$$J_s = \left(\int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt \right)^{s/r'} \left(\int_x^\infty \omega(t)\,dt \right)^{\frac{s}{p}-1} \omega(x)\,dx \right)^{1/s},$$

$$J'_s = \left(\int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt \right)^{\frac{s}{r'}-1} \left(\int_x^\infty \omega(t)\,dt \right)^{s/p} v^{1-r'}(x)\,dx \right)^{1/s}.$$

Lemma 3.8. *Let $0 < s < \infty$, $1 < p < r \le q < \infty$ and $J_s < \infty$ ($J'_s < \infty$). Then $J'_s < \infty$ ($J_s < \infty$) and*

$$J_s = \left(\frac{p}{r'} \right)^{1/s} J'_s.$$

Proof. Since $0 \le J_s < \infty$, then

$$\lim_{t\to\infty} \left(\int_t^\infty \left(\int_0^x v^{1-r'}(t)\,dt \right)^{s/r'} \left(\int_x^\infty \omega(t)\,dt \right)^{\frac{s}{p}-1} \omega(x)\,dx \right)^{1/s} = 0.$$

This yields

$$\lim_{t\to\infty} \left(\int_0^t v^{1-r'}(t)\,dt \right)^{s/r'} \left(\int_t^\infty \omega(t)\,dt \right)^{s/p} = 0.$$

Therefore, we have, integrating by parts:

$$\begin{aligned}
\infty > J_s^s &= \frac{p}{s} \int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt \right)^{s/r'} \frac{s}{p} \left(\int_x^\infty \omega(t)\,dt \right)^{\frac{s}{p}-1} \omega(x)\,dx \\
&= \frac{p}{s} \int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt \right)^{s/r'} d\left(- \int_x^\infty \omega(t)\,dt \right)^{s/p} \\
&\ge \frac{p}{s} \int_0^\infty \left(\int_x^\infty \omega(t)\,dt \right)^{s/p} d\left(\int_0^x v^{1-r'}(t)\,dt \right)^{s/r'} \\
&= \frac{p}{s} \int_0^\infty \left(\int_x^\infty \omega(t)\,dt \right)^{s/p} \frac{s}{r'} \left(\int_0^x v^{1-r'}(t)\,dt \right)^{\frac{s}{r'}-1} v^{1-r'}(x)\,dx \\
&= \frac{p}{r'} \int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt \right)^{\frac{s}{r'}-1} \left(\int_x^\infty \omega(t)\,dt \right)^{s/p} v^{1-r'}(x)\,dx = \frac{p}{r'} J_s'^s.
\end{aligned}$$

Thus, $J_s \ge \left(\frac{p}{r'} \right)^{\frac{1}{s}} J'_s$ and, therefore, $J'_s < \infty$.

Conversely, let $J'_s < \infty$. Then

$$\lim_{t\to 0} \left(\int_0^t v^{1-r'}(t)\,dt \right)^{s/r'} \left(\int_t^\infty \omega(t)\,dt \right)^{s/p} = 0$$

and, by the same argumentation,

$$J'_s \ge \left(\frac{r'}{p} \right)^{\frac{1}{s}} J_s.$$

Therefore, $J_s < \infty$. □

Put

$$A_s = \left(\sum_k \sigma_k^s\right)^{1/s}, \quad B_s = \left(\sum_k \delta_k^s\right)^{1/s}.$$

Theorem 3.4. *Let $1 < p < r \le q < \infty$ and $0 < s < \infty$. Then*

$$A_s \approx B_s \approx J_s \approx J'_s.$$

Proof. We obtain, by Lemma 3.8, that $J_s \approx J'_s$.

Let $0 < s < \infty$. Then

$$A_s^s = \sum_k \sigma_k^s = \sum_k 2^{ks/r'}\left(\int_{\xi_k}^{\xi_{k+1}} \omega(t)\,dt\right)^{s/p} \le \sum_k 2^{ks/r'}\left(\int_{\xi_k}^{\infty} \omega(t)\,dt\right)^{s/p}.$$

Put

$$\sum_k 2^{ks/r'}\left(\int_{\xi_k}^{\infty} \omega(t)\,dt\right)^{s/p} \equiv \mathcal{A}_s^s.$$

We have

$$J_s^s = \sum_k \int_{\xi_k}^{\xi_{k+1}} \left(\int_0^x v^{1-r'}(t)\right)^{s/r'} \left(\int_x^\infty \omega(t)\right)^{\frac{s}{p}-1} \omega(x)\,dx$$

$$\ge \sum_k \int_{\xi_k}^{\xi_{k+1}} \left(\int_0^{\xi_k} v^{1-r'}(t)\,dt\right)^{s/r'} \left(\int_x^\infty \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)dx$$

$$= \sum_k \int_{\xi_k}^{\xi_{k+1}} 2^{(k+1)s/r'} \left(\int_x^\infty \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx$$

$$= 2^{s/r'}\frac{p}{s}\sum_k 2^{ks/r'} \int_{\xi_k}^{\xi_{k+1}} d\left[-\left(\int_x^\infty \omega(t)\,dt\right)^{s/p}\right]$$

$$= \frac{p}{s}2^{s/r'}\sum_k 2^{ks/r'}\left[\left(\int_{\xi_k}^\infty \omega(t)\,dt\right)^{s/p} - \left(\int_{\xi_{k+1}}^\infty \omega(t)\,dt\right)^{s/p}\right]$$

$$= \frac{p}{s}2^{s/r'}\left[\mathcal{A}_s^s - 2^{-s/r'}\sum_k 2^{s/r'}2^{ks/r'}\left(\int_{\xi_{k+1}}^\infty \omega(t)\,dt\right)^{s/p}\right]$$

$$= \frac{p}{s}2^{s/r'}\left[\mathcal{A}_s^s - 2^{-s/r'}\mathcal{A}_s^s\right] = \frac{p}{s}\left[2^{s/r'}-1\right]\mathcal{A}_s^s \ge \frac{p}{s}\left[2^{s/r'}-1\right]A_s^s,$$

that is

$$J_s^s \ge \frac{p}{s}\left[2^{s/r'}-1\right]A_s^s.$$

Therefore,

$$A_s = \left(\sum_k \sigma_k^s\right)^{1/s} \le \left[\frac{p}{s}\left(2^{s/r'}-1\right)\right]^{-1/s} J_s, \quad 0 < s < \infty.$$

In order to prove the reverse inequality we assume that $0 < s < \infty$ and $s \le p$. Then $\dfrac{s}{p} \le 1$, that is $\dfrac{s}{p} - 1 \le 0$, and if

$$\int_x^{\xi_{k+1}} \omega(t)\,dt \le \int_x^{\infty} \omega(t)\,dt$$

then

$$\left(\int_x^{\infty} \omega(t)\,dt\right)^{\frac{s}{p}-1} \le \left(\int_x^{\xi_{k+1}} \omega(t)\,dt\right)^{\frac{s}{p}-1}.$$

We have

$$\left(\int_{\xi_k}^{\xi_{k+1}} \omega(t)\,dt\right)^{\frac{s}{p}} = \frac{s}{p}\int_{\xi_k}^{\xi_{k+1}} \left(\int_x^{\xi_{k+1}} \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx$$

$$\ge \frac{s}{p}\int_{\xi_k}^{\xi_{k+1}} \left(\int_x^{\infty} \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx,$$

that is

$$\left(\int_{\xi_k}^{\xi_{k+1}} \omega(t)\,dt\right)^{\frac{s}{p}} \ge \frac{s}{p}\int_{\xi_k}^{\xi_{k+1}} \left(\int_x^{\infty} \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx.$$

Therefore,

$$\sum_k 2^{ks/r'}\left(\int_{\xi_k}^{\xi_{k+1}} \omega(t)\,dt\right)^{\frac{s}{p}} \ge \frac{s}{p}\sum_k 2^{ks/r'}\int_{\xi_k}^{\xi_{k+1}} \left(\int_x^{\infty} \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx.$$

Notice that

$$2^{(k+2)s/r'} = \left(\int_0^{\xi_{k+1}} v^{1-r'}(t)\,dt\right)^{s/r'} \ge \left(\int_0^{x} v^{1-r'}(t)\,dt\right)^{s/r'},$$

if $\xi_k \le x \le \xi_{k+1}$. Thus,

$$A_s^s \ge \frac{s}{p}\sum_k \int_{\xi_k}^{\xi_{k+1}} 2^{ks/r'}2^{2s/r'}2^{-2s/r'}\left(\int_x^{\infty} \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx$$

$$= \frac{s}{p}\sum_k 2^{-2s/r'}\int_{\xi_k}^{\xi_{k+1}} 2^{(k+2)s/r'}\left(\int_x^{\infty} \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx$$

$$\ge \frac{s}{p}2^{-2s/r'}\sum_k \int_{\xi_k}^{\xi_{k+1}} \left(\int_0^{x} v^{1-r'}(t)\,dt\right)^{s/r'}\left(\int_x^{\infty} \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx$$

$$= \frac{s}{p}2^{-2s/r'}\int_0^{\infty} \left(\int_0^{x} v^{1-r'}(t)\,dt\right)^{s/r'}\left(\int_x^{\infty} \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx = \frac{s}{p}2^{-2s/r'}J_s^s.$$

Finally, we obtain

$$A_s^s \ge \frac{s}{p}2^{-2s/r'}J_s^s,$$

$$2^{-2/r'}\left(\frac{s}{p}\right)^{1/s} J_s \le \left(\sum_k \sigma_k^s\right)^{1/s}, \quad \text{if } \ 0<s<\infty \text{ and } s\le p.$$

Consider the case $1<p<s<\infty$. We write

$$J_s^s = \sum_k \int_{\xi_k}^{\xi_{k+1}} \left(\int_0^x v^{1-r'}(t)\,dt\right)^{s/r'} \left(\int_x^\infty \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx$$

$$\le \sum_k \int_{\xi_k}^{\xi_{k+1}} \left(\int_0^{\xi_{k+1}} v^{1-r'}(t)\,dt\right)^{s/r'} \left(\int_x^\infty \omega(t)\,dt\right)^{\frac{s}{p}-1} \omega(x)\,dx$$

$$= \sum_k \int_{\xi_k}^{\xi_{k+1}} 2^{(k+2)s/r'} \left(\int_x^\infty \omega(t)\,dt\right)^{s/p-1} \omega(x)\,dx$$

$$= \frac{p}{s} 2^{2s/r'} \sum_k 2^{ks/r'} \left[\left(\int_{\xi_k}^\infty \omega(t)\,dt\right)^{s/p} - \left(\int_{\xi_{k+1}}^\infty \omega(t)\,dt\right)^{s/p}\right]$$

$$\le \frac{p}{s} 2^{2s/r'} \sum_k 2^{ks/r'} \left(\int_{\xi_k}^\infty \omega(t)\,dt\right)^{s/p}.$$

Let $\alpha = \dfrac{p}{2r'}$. We obtain, by applying Hölder's inequality with exponents $\dfrac{s}{p}$ and $\dfrac{s}{s-p}$, that

$$\int_{\xi_k}^\infty \omega(t)\,dt = \sum_{m\ge k} \int_{\xi_m}^{\xi_{m+1}} \omega(t)\,dt = \sum_{m\ge k} \left[2^{\alpha m} \int_{\xi_m}^{\xi_{m+1}} \omega(t)\,dt\right] 2^{-\alpha m}$$

$$\le \left[\sum_{m\ge k} 2^{\frac{\alpha m s}{p}} \left(\int_{\xi_m}^{\xi_{m+1}} \omega(t)\,dt\right)^{s/p}\right]^{p/s} \left[\sum_{m\ge k} 2^{-\alpha m\cdot\frac{s}{s-p}}\right]^{1-\frac{p}{s}}$$

$$= C_1 2^{-k\alpha} \left[\sum_{m\ge k} 2^{\frac{\alpha m s}{p}} \left(\int_{\xi_m}^{\xi_{m+1}} \omega(t)\,dt\right)^{s/p}\right]^{p/s},$$

where $C_1 = \dfrac{2^\alpha}{\left(2^{\frac{\alpha s}{s-p}}-1\right)^{\frac{s-p}{s}}}$. Therefore,

$$J_s^s \le \frac{p}{s} 2^{2s/r'} \sum_k 2^{ks/r'} C_1^{s/p} 2^{\frac{-k\alpha s}{p}} \sum_{m\ge k} 2^{\frac{\alpha m s}{p}} \left(\int_{\xi_m}^{\xi_{m+1}} \omega(t)\,dt\right)^{s/p}$$

$$= C_2 \sum_k 2^{\frac{ks}{2r'}} \sum_{m\ge k} 2^{\frac{ms}{2r'}} \left(\int_{\xi_m}^{\xi_{m+1}} \omega(t)\,dt\right)^{s/p},$$

where

$$C_2 = \frac{p}{s}\cdot \frac{2^{5s/2r'}}{\left(2^{\frac{ps}{2r'(s-p)}}-1\right)^{\frac{s}{p}-1}}.$$

We also have

$$C_2\sum_k 2^{\frac{ks}{2r'}}\sum_{m\geq k}2^{\frac{ms}{2r'}}\left(\int_{\xi_m}^{\xi_{m+1}}\omega(t)\,dt\right)^{s/p}=C_2\sum_m 2^{\frac{ms}{2r'}}\left(\int_{\xi_m}^{\xi_{m+1}}\omega(t)\,dt\right)^{s/p}\sum_{k\leq m}2^{\frac{ks}{2r'}}.$$

Since

$$\sum_{k\leq m}2^{\frac{ks}{2r'}}=\sum_{k=0}^{m}\left(2^{\frac{s}{2r'}}\right)^k=\frac{2^{\frac{ms}{2r'}}\cdot 2^{\frac{s}{2r'}}-1}{2^{\frac{s}{2r'}}-1},$$

then

$$C_2\sum_m 2^{\frac{ms}{2r'}}\omega(t)\,dt\Big)^{s/p}\frac{\left(2^{\frac{ms}{2p'}}\cdot 2^{\frac{s}{2p'}}-1\right)}{\left(2^{\frac{s}{2r'}}-1\right)}$$

$$\leq\frac{p}{s}\cdot\frac{2^{3s/2r'}}{\left(2^{\frac{ps}{2r'(s-p)}}-1\right)^{\frac{s}{p}-1}\left(2^{\frac{s}{2r'}}-1\right)}\sum_m 2^{\frac{ms}{r'}}\left(\int_{\xi_m}^{\xi_{m+1}}\omega(t)\,dt\right)^{s/p}\equiv C_3A_s^s,$$

where

$$C_3=\frac{p}{s}\cdot\frac{2^{3s/2r'}}{\left(2^{\frac{ps}{2r'(s-p)}}-1\right)^{\frac{s}{p}-1}\left(2^{\frac{s}{2r'}}-1\right)}.$$

Thus,

$$J_s\leq C_3^{1/s}A_s,\quad\text{if}\quad 1<p<s<\infty,$$

and, therefore,

$$\left(\sum_k\sigma_k^s\right)^{1/s}\approx\left(\int_0^\infty\left(\int_0^x v^{1-r'}(t)\,dt\right)^{s/r'}\left(\int_x^\infty\omega(t)\,dt\right)^{\frac{s}{p}-1}\omega(x)\,dx\right)^{1/s},$$

for all $0<s<\infty$.

Analogously, we can prove that

$$\left(\sum_k\delta_k^s\right)^{1/s}\leq\left[\frac{r'}{s}\left[2^{s/p}-1\right]\right]^{-1/s}J_s',\quad 0<s<\infty;$$

$$\left(\frac{s}{r'}\right)^{1/s}2^{-2/p}J_s'\leq\left(\sum_k\delta_k^s\right)^{1/s},\quad 0<s<\infty\text{ and }s\leq r';$$

$$\left(\frac{s}{r'}\right)^{1/s}\frac{\left(2^{\frac{r's}{2p(s-r')}}-1\right)^{\frac{s-r'}{sr'}}\left(2^{\frac{s}{2p}}-1\right)^{1/s}}{2^{2p}}J_s'\leq\left(\sum_k\delta_k^s\right)^{1/s},\qquad 1<r'<s<\infty.$$

Finally,

$$\left(\sum_k\delta_k^s\right)^{1/s}\approx\left(\int_0^\infty\left(\int_0^x v^{1-r'}(t)\,dt\right)^{\frac{s}{r'}-1}\left(\int_x^\infty\omega(t)\,dt\right)^{\frac{s}{p}}\omega(x)\,dx\right)^{1/s},$$

when $0<s<\infty$. □

Corollary 3.1. *Let* $1 < p < r \le q < \infty$, $\gamma = \dfrac{pr'}{p+r'}$ *and* $T : L_v^r(R^+) \to L_\omega^{pq}(R^+)$ *be a compact operator of form* (1.1).

If $0 < s \le p < \infty$, *then*

$$\frac{1}{2}\left(\frac{s}{6p}\right)^{1/s} 2^{\frac{-2}{r'}}\left(\int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt\right)^{s/r'}\left(\int_x^\infty \omega(t)\,dt\right)^{\frac{s}{p}-1}\omega(x)\,dx\right)^{1/s}$$
$$\le \left\|\left\{a_n(T)n^{\frac{1}{r}-\frac{1}{q}}\right\}\right\|^s_{\ell^s(\mathbf{N})}.$$

If $1 < p < s < \infty$, *then*

$$\frac{1}{2}\left(\frac{1}{6}\right)^{1/s} C\left(\int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt\right)^{s/r'}\left(\int_x^\infty \omega(t)\,dt\right)^{\frac{s}{p}-1}\omega(x)\,dx\right)^{1/s}$$
$$\le \left\|\left\{a_n(T)n^{\frac{1}{r}-\frac{1}{q}}\right\}\right\|^s_{\ell^s(\mathbf{N})}.$$

If $\gamma < s < \infty$, *then*

$$\left\|\{a_n(T)\}\right\|_{\ell^s(\mathbf{N})}$$
$$\le C(p,r')\beta^{1/s}(s/\gamma)\left(\int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt\right)^{s/r'}\left(\int_x^\infty \omega(t)\,dt\right)^{\frac{s}{p}-1}\omega(x)\,dx\right)^{1/s}.$$

Here the constant $C(p,r')$ *depends on* p *and* r' *only,*

$$C = \left(\frac{s}{p}\right)^{1/s}\frac{\left(2^{\frac{ps}{2r'(s-p)}}-1\right)^{\frac{1}{p}-\frac{1}{s}}\left(2^{\frac{s}{2r'}}-1\right)^{1/s}}{2^{\frac{3}{2r'}}}$$

and $\beta(s/\gamma) = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{s/\gamma}}$.

Corollary 3.2. *Let* $1 < p < r < \infty$, $1 < s < \infty$ *and the operator* $T : L_v^r(R^+) \to L_\omega^{pr}(R^+)$ *of form* (1.1) *be compact. Then*

$$\left(\sum_n a_n^s(T)\right)^{1/s} \approx \left(\int_0^\infty \left(\int_0^x v^{1-r'}(t)\,dt\right)^{s/r'}\left(\int_x^\infty \omega(t)\,dt\right)^{\frac{s}{p}-1}\omega(x)\,dx\right)^{1/s}.$$

References

[1] D.E. Edmunds, W.D. Evans, D.J. Harris *Approximation numbers of certain Volterra integral operators,* J. London Math. Soc. 37 (1988), no. 2, 471-489.

[2] D.E. Edmunds, W.D. Evans, D.J. Harris *Two-sided estimates of the approximation numbers of certain Volterra integral operators,* Studia Math. 124 (1997), no. 1, 59-80.

[3] D.E. Edmunds, P. Gurka, L. Pick *Compactness of Hardy-type integral operators in weighted Banach function spaces,* Studia Math. 109 (1994), 73-90.

[4] M.A. Lifshits, W. Linde *Approximation and entropy numbers of Volterra operators with application to Brownian motion,* Mem. Am. Math. Soc. 745 (2002), 1-87.

[5] E. Lomakina, V. Stepanov *On asymptotic behaviour of the approximation numbers and estimates of Schatten von Neumann norms of the Hardy–type integral operators,* Function spaces and application. Narosa Publishing Hause. New Delhi. (2000), 153-187.

[6] E. Lomakina, V. Stepanov *On the compactness and approximation numbers of Hardy type integral operators in Lorentz spases,* J. London Math. Soc. 53 (1996), no. 2, 369-382.

[7] E.N. Lomakina, V.D. Stepanov *On Hardy-type operators in Banach function spaces on the half-axis,* Dokl. Akad. Nauk. 359 (1998), no. 1, 21–23.

[8] E.T. Sawyer *Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator,* Trans. Amer. Math. Soc. 281 (1984), 329-337.

[9] A.A. Vasil'eva *Estimates for the widths of weighted Sobolev classes,* Sb. Math. 201 (2010), no. 7, 947–984.

Elena Lomakina
Department of Mathematics and Mathematical Methods in Economics
Khabarovsk State University of Economics and Law
134 Tikhookeanskaya St.,
Khabarovsk 680042, Russia

Department of Higher Mathematics
Far Eastern State Transport University
47 Seryshev St.,
Khabarovsk 680021, Russia

E-mail: lomakina.as@mail.ru

Received: 14.04.2015