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OPTIMAL DISTRIBUTED CONTROL FOR THE PROCESSES OF OSCILLATION DESCRIBED BY FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate the problem of distributed optimal control for the oscillation processes described by Fredholm integro-differential equations with partial derivatives when the function of the external source depends nonlinearly on the control parameters. We have developed an algorithm for finding approximate solutions of nonlinear optimization problems with arbitrary precision. The developed method of solving nonlinear optimization problems is constructive and can be used in applications.

1 Introduction

In applied problems many real processes are described by integro-differential equations with partial derivatives [8], [11], [12]. After the advent of the optimal control theory for systems with distributed parameters, many applied problems were investigated by the methods of the optimal control theory [1], [3], [5]. In developing mathematical research methods for applied problems the use of generalized solutions to boundary value problem appeared to be more convenient.

In this article we investigate the problem of unique solvability of nonlinear distributed optimal control for oscillation processes described by Fredholm integro-differential equations with partial derivatives. We established that the optimal control satisfies simultaneously two relations: of equality-type and inequality-type. In this case the relation of equality-type leads to a nonlinear integral equation, and the second relation is a differential inequality for the function of the external source.

Sufficient conditions for the existence of a unique solution of the optimization problem in the form of the triple $(u^0(t,x), V^0(t,x), J[u^0(t,x)])$ are found. Here $u^0(t,x)$ is the desired optimal control, $V^0(t,x)$ is a optimal process, $J[u^0(t,x)]$ is a minimum value of the functional.

We have developed an algorithm for finding approximate solutions of the boundary value problem and have proved its convergence.

2 Boundary value problem of the controlled process

The process of oscillation will be described by a scalar function $V(t,x)$, defined on the region $Q_T = Q \times (0,T]$, where Q is a region of the space R^n bounded by a piecewise smooth surface γ, which satisfies the integro-differential equation [8], [11], [12]

$$V_{tt} - AV = \lambda \int_0^T K(t,\tau)V(\tau,x)d\tau + f[t,x,u(t,x)], \quad x \in Q \subset R^n, \quad 0 < t \leq T, \quad (2.1)$$

on the boundary of Q satisfies the initial condition

$$V(0,x) = \psi_1(x), \quad V_t(0,x) = \psi_2(x), \quad x \in Q, \quad (2.2)$$

and the boundary condition

$$\Gamma V(t,x) \equiv \sum_{i,j=1,n}^{n} a_{ij}(x)V_{x_j}(t,x)cos(\delta,x_i) + a(x)V(t,x) = 0, \quad x \in \gamma, \quad 0 < t \leq T. \quad (2.3)$$

Here A is the elliptic operator defined by the formula

$$AV(t,x) = \sum_{i,j=1}^{n} (a_{ij}(x)V_{x_j}(t,x))_{x_i} - c(x)V(t,x),$$

$$a_{ij}(x) = a_{ji}(x), \quad \sum_{i,j=1}^{n} a_{ij}(x)\alpha_i\alpha_j \geq c_0 \sum_{i=0}^{n} \alpha_i^2, \quad c_0 > 0;$$

δ is a normal vector, emanating from the point $x \in \gamma$; $K(t,\tau)$ is a given function defined in the region $D = \{0 \leq t \leq 1, \ \ 0 \leq \tau \leq 1\}$ and satisfying the condition

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty, \quad (2.4)$$

i.e. $K(t,\tau)$ is an element of the Hilbert space $H(D)$;

$$\psi_1 \in H_1(Q), \quad \psi_2 \in H(Q), \quad f_u[t,x,u(t,x)] \neq 0, \quad \forall (t,x) \in Q_T, \quad (2.5)$$

$a(x) \geq 0$, $c(x) \geq 0$ are given measurable functions; $H_1(Q)$ is the Sobolev space of first-order; the function of the external source $f[t,x,u(t,x)]$ depends nonlinearly on the control functions $u(t,x) \in H(Q_T)$ and the set of allowable values of the control is bounded; λ is a parameter, T is a fixed moment of time and $\alpha > 0$ is a constant.

As is known [10], under conditions (2.5) problem (2.1) - (2.3) has no classical solutions. Therefore, we will use the notion of a generalized solution to problem (2.1) - (2.3).

We seek the solution to problem (2.1) - (2.3) in the form:

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t)z_n(x), \quad (2.6)$$

where $V_n(t) = \langle V(t,x), z_n(x)\rangle = \int_Q V(t,x)z_n(x)dx$ are the Fourier coefficients, $z_n(x)$ satisfy the boundary value problem [7]

$$Az(x) = -\lambda^2 z(x), \quad x \in Q, \quad \Gamma z(x) = 0, \quad x \in \gamma,$$

for each $n = 1, 2, 3, ...$ The system of the eigenfunctions $\{z_n(x)\}$ form a complete orthonormal system in the Hilbert space $H(Q)$, and the corresponding eigenvalues λ_n satisfy the following conditions $\lambda_n \le \lambda_{n+1}, \forall n = 1, 2, 3, ...,\quad \lim_{n\to\infty} \lambda_n = \infty.$

Definition 1. A function $V(t,x) \in H(Q_T)$ is called a generalized solution to problem (2.1) - (2.3) if it satisfies the initial condition in a weak sense, that is for any function $\phi_0 \in H(Q), \quad \phi_1 \in H(Q)$ we have the equalities:

$$\lim_{t\to+0} \int_Q V(t,x)\phi_0(x)dx = \int_Q \psi_1(x)\phi_0(x)dx,$$

$$\lim_{t\to+0} \int_Q V_t(t,x)\phi_1(x)dx = \int_Q \psi_2(x)\phi_1(x)dx,$$

and the Fourier coefficients $V_n(t)$ satisfy the following linear Fredholm integral equation of the second type

$$V_n(t) = \psi_{1n} cos\lambda_n t + \frac{1}{\lambda_n}\psi_{2n} sin\lambda_n t+ \tag{2.7}$$

$$+\frac{1}{\lambda_n}\int_0^t sin\lambda_n(t-\tau)\left(\lambda\int_0^T K(\tau,s)V_n(s)ds + f_n(\tau,u)\right)dx$$

where ψ_{1n}, ψ_{2n} and $F_n(t,u)$ are the Fourier coefficients of the functions $\psi_1,\ \psi_2,\ f(t,x,u(t,x))$ respectively.

To determine the Fourier coefficients $V_n(t)$ equation (2.7) can be rewritten as

$$V_n(t) = \lambda\int_0^T K_n(t,s)V_n(s)ds + a_n(t) \tag{2.8}$$

where

$$K_n(t,s) = \frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t-\tau)K(\tau,s)d\tau, \quad K_n(0,s) = 0, \tag{2.9}$$

$$a_n(t) = \psi_{1n}\cos\lambda_n t + \frac{1}{\lambda_n}\psi_{2n}\sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t-\tau)f_n(\tau,u(\tau))d\tau. \tag{2.10}$$

We find the solution to integral equation (2.8) by the formula [4], [9]:

$$V_n(t) = \lambda\int_0^T R_n(t,s,\lambda)a_n(s)ds + a_n(t) \tag{2.11}$$

where

$$R_n(t,s,\lambda) = \sum_{i=1}^{\infty}\lambda^{i-1}K_{n,i}(t,s), \quad n = 1,2,3,..., \tag{2.12}$$

is the resolvent of the kernel $K_n(t,s) \equiv K_{n,1}(t,s)$, and the iterated kernels $K_{n,i}(t,s)$ are defined by the formula

$$K_{n,i+1}(t,s) = \int_0^T K_n(t,\eta) K_{n,i}(\eta,s)d\eta \quad i = 1,2,3,..., \quad K_{n,1}(t,s) = K_n(t,s), \quad (2.13)$$

for each $n = 1,2,3,....$ We investigate the convergence of Neumann series (2.12). According to (2.9) and (2.13) by direct calculation the following estimates are established

$$|K_{n,i}(t,s)|^2 \leq \frac{T^{2i-1}}{(\lambda_n^2)^i} K_0^{i-1} \int_0^T K^2(\tau,s)d\tau, \quad i = 1,2,3,.... \quad (2.14)$$

Convergence of Neumann series (2.12) follows by the inequality

$$|R_n(t,s,\lambda)| \leq \sum_{i=1}^{\infty} |\lambda|^{i-1} |K_{n,i}(t,s)|$$

$$\leq \sum_{i=1}^{\infty} |\lambda|^{i-1} \left(\frac{T^{2i-1}}{(\lambda_n^2)^i} \right)^{1/2} (K_0^{i-1})^{1/2} \left(\int_0^T K^2(y,s)dy \right)^{1/2}$$

$$\leq \left(\int_0^T K^2(y,s)dy \right)^{1/2} \sum_{i=1}^{\infty} |\lambda|^{i-1} \frac{\sqrt{T}}{\sqrt{\lambda_n^2}} \left(\frac{T^{2(i-1)}}{(\lambda_n^2)^{i-1}} \right)^{1/2} (K_0^{1/2})^{i-1}$$

$$= \frac{\sqrt{T}}{\lambda_n} \left(\int_0^T K^2(y,s)ds \right)^{1/2} \sum_{i=1}^{\infty} \left(|\lambda| \frac{T\sqrt{K_0}}{\lambda_n} \right)^{i-1}$$

$$= \frac{\sqrt{T}}{\lambda_n} \left(\int_0^T K^2(y,s)ds \right)^{1/2} \frac{\lambda_n}{\lambda_n - |\lambda| T\sqrt{K_0}}$$

$$= \sqrt{T} \left(\int_0^T K^2(y,s)ds \right)^{1/2} \frac{1}{\lambda_n - |\lambda| T\sqrt{K_0}}.$$

It converges for the values of the parameter λ that satisfy the inequality

$$|\lambda| \frac{T}{\lambda_n} \sqrt{K_0} < 1.$$

By direct calculation we establish the following inequality

$$\int_0^T R_n^2(t,s,\lambda)ds \leq \int_0^T \int_0^T K^2(y,s)dyds \frac{1}{(\lambda_n - |\lambda| T\sqrt{K_0})^2} \quad (2.15)$$

$$= \frac{K_0 T}{(\lambda_n - |\lambda| T\sqrt{K_0})^2},$$

which will later be repeatedly used.

The Neumann series for values of the parameter λ satisfying $|\lambda| < \frac{1}{\sqrt{K_0 T}} \lambda_n \xrightarrow[n\to\infty]{} \infty$ converges absolutely for each $n = 1,2,3,...$, i.e. the radius of convergence increases when n is growing. In this case the resolvent $R_n(t,s,\lambda)$, as the sum of an absolutely convergent series, is a continuous function.

Note that Neumann series absolutely converges for any $n = 1, 2, 3, ...$ only for values of the parameter λ satisfying

$$|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}. \tag{2.16}$$

Thus, we find the solution to problem (2.1) - (2.3) by formula (2.6), where $V_n(t)$ is defined by (2.11) as the unique solution to integral equation (2.8). It is easy to verify that this solution satisfies initial condition (2.2).

Now we show that this solution is an element of the space $H(Q)$. Taking into account (2.9) and (2.10) by direct calculation we have the following inequality

$$\int_0^T \int_Q V^2(t,x)dxdt = \int_0^T \int_Q \left(\sum_{n=1}^{\infty} V_n(t)z_n(x)\right)^2 (t,x)dxdt$$

$$= \int_0^T \sum_{n=1}^{\infty} V_n^2(t)dt \le \int_0^T \sum_{n=1}^{\infty} \left(\lambda \int_0^T R_n(t,s,\lambda)a_n(s)ds + a_n(t)\right)^2 dt$$

$$\le 2\int_0^T \sum_{n=1}^{\infty} \left(\lambda^2 \int_0^T R_n^2(t,s,\lambda)ds \int_0^T a_n^2(s)ds + a_n^2(t)\right)dt$$

$$= 2\left\{\int_0^T \sum_{n=1}^{\infty} \lambda^2 \frac{K_0 T}{(\lambda_n - |\lambda|\, T\sqrt{K_0})^2} \int_0^T a_n^2(s)dsdt + \int_0^T \sum_{n=1}^{\infty} a_n^2(t)dt\right\}; \tag{2.17}$$

Based on the following calculations Further we shall take into account the following calculations:

$$1)\quad \sum_{n=1}^{\infty} \int_0^T a_n^2(s)ds$$

$$\le \sum_{n=1}^{\infty} \int_0^T \left(\psi_{1n} cos\lambda_n s + \frac{1}{\lambda_n}\psi_{2n} sin\lambda_n s + \frac{1}{\lambda_n}\int_0^t sin\lambda_n(s-\tau)f_n(\tau,u(\tau))d\tau\right)^2 ds$$

$$\le 3\sum_{n=1}^{\infty} \int_0^T \left(\psi_{1n}^2 cos^2\lambda_n s + \frac{1}{\lambda_n^2}\psi_{2n}^2 sin^2\lambda_n s\right.$$

$$\left. + \frac{1}{\lambda_n^2}\int_0^t sin^2\lambda_n(T-\tau)d\tau \int_0^T f_n^2[\tau,u]d\tau\right)ds$$

$$\le 3T\left(\sum_{n=1}^{\infty} \psi_{1n}^2 + \sum_{n=1}^{\infty} \frac{\psi_{2n}^2}{\lambda_n^2} + \sum_{n=1}^{\infty} \frac{T}{\lambda_n^2}\int_0^T f_n^2[\tau,u]d\tau\right)$$

$$\le 3T\left(\|\psi_1(x)\|_H^2 + \frac{1}{\lambda_1^2}\|\psi_2(x)\|_H^2 + \frac{T}{\lambda_1^2}\sum_{n=1}^{\infty}\int_0^T f_n^2[\tau,u]d\tau\right) < \infty;$$

$$2)\quad \sum_{n=1}^{\infty}\int_0^T a_n^2(t)dt=\sum_{n=1}^{\infty}\int_0^T \Bigg(\psi_{1n}cos\lambda_n t+\frac{1}{\lambda_n}\psi_{2n}sin\lambda_n t$$

$$+\frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t-\tau)f_n(\tau,u)d\tau\Bigg)^2dt\le 3\int_0^T\sum_{n=1}^{\infty}\Bigg(\psi_{1n}^2$$

$$+\frac{1}{\lambda_n^2}\psi_{2n}^2\sin^2\lambda_n t+\frac{1}{\lambda_n}\int_0^T\sin^2\lambda_n(T-\tau)d\tau\int_0^T f_n^2[\tau,u]d\tau\Bigg)dt$$

$$\le 3T\left(\|\psi_1(x)\|_H^2+\frac{1}{\lambda_1^2}\|\psi_2(x)\|_H^2+\frac{T}{\lambda_1^2}\sum_{n=1}^{\infty}\int_0^T f_n^2[\tau,u]d\tau\right)$$

$$=3T\left(\|\psi_1(x)\|_H^2+\frac{1}{\lambda_1^2}\|\psi_2(x)\|_H^2+\frac{T}{\lambda_1^2}\|f(t,x,u(t,x))\|_H^2\right);$$

$$3)\quad 2\left\{\int_0^T\sum_{n=1}^{\infty}\lambda^2\frac{K_0T}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}\int_0^T a_n^2(s)dsdt+\int_0^T\sum_{n=1}^{\infty}a_n^2(t)dt\right\}$$

$$\le 2T\{\frac{\lambda^2K_0T}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}3T\Bigg(\|\psi_1(x)\|_H^2+\frac{1}{\lambda_1^2}\|\psi_2(x)\|_H^2$$

$$+\frac{T}{\lambda_1^2}\|f(t,x,u(t,x))\|_H^2\Bigg)+3T\Bigg(\|\psi_1(x)\|_H^2+\frac{1}{\lambda_1^2}\|\psi_2(x)\|_H^2$$

$$+\frac{T}{\lambda_1^2}\|f(t,x,u(t,x))\|_H^2\Bigg)\}=6T^2\Bigg(\|\psi_1(x)\|_H^2+\frac{1}{\lambda_1^2}\|\psi_2(x)\|_H^2$$

$$+\frac{T}{\lambda_1^2}\|f(t,x,u(t,x))\|_H^2\Bigg)\left(1+\frac{\lambda^2K_0T}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}\right)<\infty.$$

Consequently, from (2.17) we obtain the relation

$$\int_0^T\int_Q V^2(t,x)dxdt<\infty,$$

i.e. $V(t,x)\in H(Q_T)$.

When defining the functions $V_n(t),\quad n=1,2,3,...$, by formulas (2.11)-(2.12), it is not always possible to find the resolvent $R_n(t,s,\lambda)$ explicitly. In practice, most often approximations of the resolvent are considered. The truncated series of the form

$$R_n^m(t,s,\lambda)=\sum_{i=1}^{m}\lambda^{i-1}K_{n,i}(t,s),\quad n=1,2,3,..., \tag{2.18}$$

is called m th approximation of the resolvent for each fixed $n=1,2,3,....$ The function $V_n^m(t)$ defined by the formula

$$V_n^m(t)=\lambda\int_0^T R_n^m(t,s,\lambda)a_n(s)ds+a_n(t),\quad n=1,2,3,..., \tag{2.19}$$

is called the m th approximation of the function $V_n(t)$ for each fixed $n = 1, 2, 3, ...$ According to formula (2.6), the m th approximation of the solution to boundary value problem (2.1)-(2.3) is found by the formula

$$V^m(t,x) = \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n^m(t,s,\lambda)a_n(s)ds + a_n(t)\right) z_n(x), \qquad (2.20)$$

where $V_n^m(t)$ have the form (2.19). We show that the approximate solution $V^m(t,x)$ to boundary-value problem (2.1)-(2.3) converges to the exact solution $V(t,x)$ in the norm of the space $H(Q_T)$. By taking into account (2.12), (2.14), (2.15), (2.18), (2.19) and the inequality

$$\int_{m+1}^{\infty} q^{x-1}dx = \frac{1}{q}\int_{m+1}^{\infty} q^x dx = q^{-1}\left(q^{m+1} + \left.\frac{q^x}{lnq}\right|_{m+1}^{\infty}\right)$$

$$= q^{-1}q^{m+1}\left(1 - \frac{1}{lnq}\right) = q^m\left(1 - \frac{1}{lnq}\right), \quad 0 < q < 1,$$

the convergence of the approximate solution $V_n^m(t)$ follows from the inequality

$$\|V(t,x) - V^m(t,x)\|_H^2$$

$$= \left(\int_0^T\int_Q\sum_{n=1}^{\infty}\lambda\int_0^T [R_n(t,s,\lambda) - R_n^m(t,s,\lambda)]a_n(s)ds z_n(x)\right)^2 dxdt$$

$$= \int_0^T\sum_{n=1}^{\infty}\left(\lambda\int_0^T [R_n(t,s,\lambda) - R_n^m(t,s,\lambda)]\, a_n(s)ds\right)^2 dt$$

$$= \int_0^T \lambda^2\sum_{n=1}^{\infty}\int_0^T [R_n(t,s,\lambda) - R_n^m(t,s,\lambda)]^2 ds\int_0^T a_n^2(s)dsdt$$

$$= \lambda^2\int_0^T\sum_{n=1}^{\infty}\int_0^T\left[\sum_{i=m+1}^{\infty}|\lambda|^{i-1}|K_{n,i}(t,s)|\right]^2 ds\int_0^T a_n^2(s)dsdt$$

$$\leq \lambda^2\int_0^T\sum_{n=1}^{\infty}\int_0^T\frac{T}{\lambda_n^2}\int_0^T K^2(y,s)dy$$

$$\times\left(\sum_{i=m+1}^{\infty}\left(|\lambda|\frac{T}{\lambda_n}\sqrt{K_0}\right)^{i-1}\right)^2 ds\int_0^T a_n^2(s)dsdt$$

$$\leq \lambda^2T^2K_0\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\left(|\lambda|\frac{T}{\lambda_n}\sqrt{K_0}\right)^{2m}\left(1 - \frac{1}{ln\frac{|\lambda|T\sqrt{K_0}}{\lambda_n}}\right)^2\int_0^T a_n^2(s)ds \qquad (2.21)$$

$$\leq \frac{\lambda^2T^2K_0}{\lambda_1^2}\left(|\lambda|\frac{T}{\lambda_1}\sqrt{K_0}\right)^{2m}\left(1 - \frac{1}{ln\frac{|\lambda|T\sqrt{K_0}}{\lambda_1}}\right)^2\sum_{n=1}^{\infty}\int_0^T a_n^2(s)ds$$

$$\leq C_3(\lambda)\left(|\lambda|\frac{T}{\lambda_1}\sqrt{K_0}\right)^{2m}\xrightarrow[m\to\infty]{}0,$$

where

$$C_3(\lambda)=\frac{\lambda^2T^2K_0}{\lambda_1^2}\left(1-\frac{1}{ln\frac{|\lambda|T\sqrt{K_0}}{\lambda_1}}\right)^2$$

$$\times 3T\left(\|\psi_1(x)\|_H^2+\frac{1}{\lambda_1^2}\|\psi_2(x)\|_H^2+\frac{T}{\lambda_1^2}\|f(t,x,u(t,x))\|_H^2\right).$$

3 Formulation of optimal control problem and conditions of optimality

We will consider the optimization problem in which it is required to minimize the integral functional

$$J[u(t,x)]=\int_Q\left\{[V(T,x)-\xi_1(x)]^2+[V_t(T,x)-\xi_2(x)]^2\right\}dx \tag{3.1}$$

$$+\beta\int_0^T\int_Q p^2[t,x,u(t,x)]dxdt,\quad \beta>0,$$

where $\xi_1\in H_1(Q)$, $\xi_2\in H(Q)$ are given functions; $p[t,x,u(t,x)]\in H(Q_T)$ is nonlinear and monotonic function with respect to the functional variable, defined on the set of solutions to problem (2.1) - (2.3). So we need to find the control $u^0(t,x)\in H(Q_T)$ which together with the corresponding solution $V^0(t,x)$ of boundary value problem (2.1) - (2.3) gives the smallest possible value of functional (3.1). In this case $u^0(t,x)$ is called the optimal control, and $V^0(t,x)$ the optimal process.

Since by condition (2.5) each control $u(t,x)$ uniquely defines the controlled process $V(t,x)$, then for the control $u(t,x)+\Delta u(t,x)$ corresponds the solution to boundary value problem (2.1)-(2.3) in the form $V(t,x)+\Delta V(t,x)$, where $\Delta V(t,x)$ is the increment corresponding to the increment $\Delta u(t,x)$. According to the procedure of application of the maximum principle [1], [3], the increment of functional (3.1) can be written as

$$\Delta J[u]=J[u+\Delta u]-J[u]=-\int_0^T\int_Q\Delta\Pi[t,x,V(t,x),\omega(t,x),u(t,x)]dxdt \tag{3.2}$$

$$+\int_Q\left[\Delta V^2(T,x)+\Delta V_t^2(T,x)\right]dx,$$

where

$$\Pi[t,x,V(t,x),\omega(t,x),u(t,x)]=f[t,x,u(t,x)]\omega(t,x)-\beta p^2[t,x,u(t,x)], \tag{3.3}$$

and the function $\omega(t,x)$ is a solution of the adjoint boundary value problem

$$\omega_{tt}-A\omega=\lambda\int_0^T K(\tau,t)\omega(\tau,x)d\tau,\quad x\in Q,\quad 0\leq t<T,$$

$$\omega(T,x)+2\left[V_t(T,x)-\xi_2(x)\right]=0,\quad \omega_t(T,x)-2\left[V(T,x)-\xi_1(x)\right]=0,$$

$$\Gamma\omega(t,x)=\sum_{i,j=1}^{n}(a_{ij}(x)\omega_{x_j}(t,x))_{x_i}-c(x)\omega(t,x)=0 \tag{3.4}$$

According to the maximum principle for systems with distributed parameters [1], [3], the optimal control is determined by the relations

$$\frac{2\beta p(t,x,u(t,x))p_u(t,x,u(t,x))}{f_u[t,u(t)]}=\omega(t,x), \tag{3.5}$$

$$f_u[t,x,u(t,x)]\left(\frac{p(t,x,u(t,x))p_u(t,x,u(t,x))}{f_u[t,x,u(t,x)]}\right)_u>0, \tag{3.6}$$

which are called *the optimality conditions.*

4 Solution of the adjoint boundary-value problem

We are looking for a solution of the adjoint boundary value problem (3.4) in the form of the series

$$\omega(t,x)=\sum_{n=1}^{\infty}\omega_n(t)z_n(x). \tag{4.1}$$

It is easy to verify that the Fourier coefficients $\omega_n(t)$ for each fixed $n=1,2,3,...$, satisfy the conditions

$$\omega_n^{''}(t)+\lambda_n^2\omega_n(t)=\lambda\int_0^T K(\tau,t)\omega_n(\tau)d\tau,$$

$$\omega_n(T)+2[V_n^{'}(T)-\xi_{2n}]=0,\quad \omega_n^{'}(T)-2[V_n(T)-\xi_{1n}]=0,$$

which can be converted to the linear non-homogeneous Fredholm integral equation of the second type

$$\omega_n(t)=-2\lambda\int_0^T B_n(s,t)\omega_n(s)ds+q_n(t), \tag{4.2}$$

where

$$B_n(s,t)=\frac{1}{\lambda_n}\int_t^T sin\lambda_n(\eta-t)K(s,\eta)d\eta\quad and\quad B_n(s,T)=0, \tag{4.3}$$

$$q_n(t)=-2\left\{[V_n^{'}(T)-\xi_{2n}]cos\lambda_n(T-t)+[V_n(T)-\xi_{1n}]\frac{1}{\lambda_n}sin\lambda_n(T-t)\right\}.$$

We finf the solution to equation (4.2) by using the formula [4], [9]

$$\omega_n(t)=\lambda\int_0^T W_n(s,t,\lambda)q_n(s)ds+q_n(t), \tag{4.4}$$

where the resolvent $W_n(s,t,\lambda)$ of the kernel $B_n(s,t)$ is given by

$$W_n(s,t,\lambda)=\sum_{i=1}^{\infty}\lambda^{i-1}B_{n,i}(s,t)\qquad i=1,2,3,..., \tag{4.5}$$

and the iterated kernels $B_{n,i}(s,t)$ are defined by the formula

$$B_{n,i+1}(s,t)=\int_0^T B_n(\tau,t)B_{n,i}(s,\tau)d\tau,\quad i=1,2,3,...,\quad B_{n,1}(s,t)=B_n(s,t). \tag{4.6}$$

for each fixed $n=1,2,3,....$ Now we investigate the convergence of Neumann series (4.5). According to (4.3) and (4.6) by direct calculation we get

$$|B_{n,i}(s,t)|^2=\frac{T^{2i-1}}{(\lambda_n^2)^i}K_0^{i-1}\int_0^T K^2(s,\eta)d\eta,\quad i=1,2,3,..., \tag{4.7}$$

$$\int_0^T |B_{n,i}(s,t)|^2\,ds\le\frac{T^{2i-1}}{(\lambda_n^2)^i}K_0^{i-1}\int_0^T\int_0^T K^2(s,\eta)d\eta=\frac{T^{2i-1}K_0^i}{(\lambda_n^2)^i}. \tag{4.8}$$

Convergence of Neumann series (4.5) follows by the inequality

$$W_n(s,t,\lambda)=\sum_{i=1}^{\infty}\lambda^{i-1}B_{n,i}(s,t)\le\sum_{i=1}^{\infty}|\lambda|^{i-1}\,|B_{n,i}(s,t)|$$

$$\le\sum_{i=1}^{\infty}|\lambda|^{i-1}\sqrt{\frac{T^{2i-1}}{(\lambda_n^2)^i}K^{i-1}}\left(\int_0^T K^2(s,\eta)d\eta\right)^{1/2}$$

$$\le\sqrt{T}\left(\int_0^T K^2(s,\eta)d\eta\right)^{1/2}\frac{1}{\lambda_n-|\lambda|T\sqrt{K_0}},$$

which converges for the values of the parameter λ that satisfy the inequality $|\lambda|\frac{T}{\lambda_n}\sqrt{K_0}<1$ for every $n=1,2,3,....$

By direct calculation we establish the following inequality

$$|W_n(s,t,\lambda)|\le\frac{\sqrt{T}}{\lambda_n^2-|\lambda|TK_0}\left(\int_0^T K^2(s,\eta)d\eta\right)^{1/2}$$

and

$$\int_0^T W_n^2(s,t,\lambda)ds\le\frac{TK_0}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}.$$

Thus, the solution of the adjoint boundary-value problem (3.4) we find by the formula

$$\omega(t,x)=\sum_{n=1}^{\infty}\left[\lambda\int_0^T W_n(s,t,\lambda)q_n(s)ds+q_n(t)\right]z_n(x). \tag{4.9}$$

It is easy to verify that $\omega(t,x)$ is an element of the space $H(Q_T)$. This follows by the inequality

$$\int_0^T\int_Q\omega^2(t,x)dxdt=\int_0^T\int_Q\left(\sum_{n=1}^{\infty}\omega_n(t)z_n(x)\right)^2(t,x)dxdt=\int_0^T\sum_{n=1}^{\infty}\omega_n^2(t)dt$$

$$=\int_0^T\sum_{n=1}^{\infty}\left(\lambda\int_0^T W_n(s,t,\lambda)q_n(s)ds+q_n(t)\right)^2dt$$

$$\leq 2\int_0^T \sum_{n=1}^{\infty}\left(\left(\lambda\int_0^T W_n(s,t,\lambda)q_n(s)ds\right)^2 + q_n^2(t)\right)dt$$

$$\leq 2\int_0^T \sum_{n=1}^{\infty}\left(\lambda^2\int_0^T W_n^2(s,t,\lambda)ds\int_0^T q_n^2(s)ds + q_n^2(t)\right)dt$$

$$\leq 2\int_0^T \sum_{n=1}^{\infty}\left(\lambda^2\frac{TK_0}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}\int_0^T q_n^2(s)ds + q_n^2(t)\right)dt$$

$$\leq 2\sum_{n=1}^{\infty}\left(1+\lambda^2\frac{T^2K_0}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}\right)\int_0^T q_n^2(s)ds$$

$$\leq 2\left(1+\lambda^2\frac{T^2K_0}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}\right)\sum_{n=1}^{\infty}\int_0^T q_n^2(s)ds$$

$$\leq 8\left(1+\lambda^2\frac{T^2K_0}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}\right)$$

$$\times\sum_{n=1}^{\infty}\int_0^T\left\{[V_n'(T)-\xi_{2n}]^2 cos^2\lambda_n(T-s) + [V_n(T)-\xi_{1n}]^2\frac{1}{\lambda_n^2}sin^2\lambda_n(T-s)\right\}ds$$

$$\leq 16\int_0^T\sum_{n=1}^{\infty}\left\{V_n'^2(T)+\xi_{2n}^2+\frac{1}{\lambda_n^2}\left(V_n^2(T)-\xi_{1n}^2\right)\right\}\left[1+\frac{\lambda^2T^2K_0}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}\right]dt<\infty$$

which holds because

$$\sum_{n=1}^{\infty}V_n'^2(T)<\infty,\quad \sum_{n=1}^{\infty}V_n^2(T)<\infty,$$

$$\sum_{n=1}^{\infty}\xi_{1n}^2=\|\xi_1(x)\|_H^2 \quad\text{and}\quad \sum_{n=1}^{\infty}\xi_{2n}^2=\|\xi_2(x)\|_H^2\,.$$

5 Nonlinear integral equation for optimal control

We find the optimal control according to optimality conditions (3.5) and (3.6). We substitute in (3.5) the solution of adjoint boundary value problem (3.4) defined by (4.9) and have equality

$$2\frac{\beta p(t,x,u(t,x))p_u(t,x,u(t,x))}{f_u[t,x,u(t,x)]} = -2\sum_{n=1}^{\infty}\left(\lambda\int_0^T W_n(s,t,\lambda)q_n(s)ds+q_n(t)\right)z_n(x).$$

According to (4.3) , (2.11), (2.10)-(2.12) we reduce this equality to the form

$$\frac{\beta p(t,x,u(t,x))p_u(t,x,u(t,x))}{f_u[t,x,u(t,x)]} =$$

$$=\sum_{n=1}^{\infty}L_n^*(t,\lambda)\left[h_n-\int_0^T G_n(\tau,\lambda)f_n[\tau,u]d\tau\right]z_n(x), \tag{5.1}$$

where

$$L_n^*(t,\lambda)=\{W_{1n}(t,\lambda),W_{2n}(t,\lambda)\},$$

$$W_{1n}(t,\lambda)=\cos\lambda_n(T-t)+\lambda\int_0^T W_n(s,t,\lambda)\cos\lambda_n(T-s)ds; \tag{5.2}$$

$$W_{2n}(t,\lambda)=\frac{1}{\lambda_n}\left(\sin\lambda_n(T-t)+\lambda\int_0^T W_n(s,t,\lambda)\sin\lambda_n(T-s)ds\right);$$

$$G_n(\tau,\lambda)=\{G_{1n}(\tau,\lambda),G_{2n}(\tau,\lambda)\};$$

$$G_{1n}(\tau,\lambda)=\cos\lambda_n(T-\tau)+\lambda\int_\tau^T R'_{nt}(T,s,\lambda)\sin\lambda_n(s-\tau)ds; \tag{5.3}$$

$$G_{2n}(\tau,\lambda)=\frac{1}{\lambda_n}\left(\sin\lambda_n(T-\tau)+\lambda\int_\tau^T R_n(T,s,\lambda)\sin\lambda_n(s-\tau)ds\right);$$

$$h_n=\{h_{1n},h_{2n}\};$$

$$h_{1n}=\xi_{2n}-\psi_{1n}\left[-\lambda_n\sin\lambda_nT+\lambda\int_0^T R'_{nt}(T,s,\lambda)\cos\lambda_n sds\right] \tag{5.4}$$

$$-\frac{\psi_{2n}}{\lambda_n}\left[\cos\lambda_nT+\lambda\int_0^T R'_{nt}(T,s,\lambda)\sin\lambda_n sds\right];$$

$$h_{2n}=\xi_{1n}-\psi_{1n}\left[\cos\lambda_nT+\lambda\int_0^T R_n(T,s,\lambda)\cos\lambda_n sds\right]$$

$$-\frac{\psi_{2n}}{\lambda_n}\left[\sin\lambda_nT+\lambda\int_0^T R_n(T,s,\lambda)\sin\lambda_n sds\right];$$

Thus, the optimal controlis defined as the solution of nonlinear integral equation (5.1), at the same time must satisfy condition (3.6). Condition (3.6) restricts the class of functions of external actions $f[t,x,u(t,x)]$. Therefore, we assume that the function $f(t,x,u(t,x))$ satisfies (3.6) for any control $u(t,x)\in H(Q_T)$ i.e. the optimization problem is considered in class $\{f(t,x,u(t,x))\}$ of functions satisfying (3.6).

We rewrite nonlinear integral equation (5.6) in the form

$$\frac{\beta p(t,x,u(t,x))p_u(t,x,u(t,x))}{f_u[t,u(t)]}$$
$$+\sum_{n=1}^{\infty}L_n^*(t,\lambda)\int_0^T\int_Q G_n(\tau,\lambda)f_n[\tau,y,u(\tau,y)]z_n(y)dyd\tau z_n(x) \tag{5.5}$$
$$=\sum_{n=1}^{\infty}L_n^*(t,\lambda)h_nz_n(x),$$

and we investigated its solvability.

Nonlinear integral equation (5.5) is solved following the work [2]. We set

$$\frac{\beta p(t,x,u(t,x))p_u(t,x,u(t,x))}{f_u[t,x,u(t,x)]}=l(t,x). \tag{5.6}$$

Lemma 5.1. *The function $l(t,x)$ is an element of the space $H(Q_T)$.*

Proof. By (2.5) we have the estimate

$$\sup_{(t,x)\in Q_T}\left|\frac{p_u(t,x,u(t,x))}{f_u(t,x,u(t,x))}\right|\le M,\quad \forall t\in[0,T].$$

As $p(t,x,u(t,x))\in H(Q_T)$ for any $u(t,x)\in H(Q_T)$ the statement of the lemma follows by the inequality

$$\int_0^T l^2(t,x)dxdt=\int_0^T\int_Q\beta^2\left(\frac{p(t,x,u(t,x))p_u(t,x,u(t,x))}{f_u(t,x,u(t,x))}\right)^2dxdt$$

$$\le\beta^2M^2\int_0^T\int_Q p^2(t,x,u(t,x))dxdt<\infty.$$

□

According to (3.6), the control $u(t,x)$ is uniquely determined by equality (5.5), i.e. there is a function φ such that

$$u(t,x)=\varphi(t,x,l(t,x),\beta). \tag{5.7}$$

By (5.5) and (5.6) we rewrite equation (5.1) in the form

$$l(t,x)+\int_0^T\int_Q L(t,\tau,x,y,\lambda)f[\tau,y,\varphi(\tau,y,l(\tau,y),\beta)]dydx=h(t,x) \tag{5.8}$$

where $L(t,\tau,x,y,\lambda)=\sum_{n=1}^{\infty}L_n^*(t,\lambda)G_n(\tau,\lambda)z_n(x)z_n(y)$;

$$h(t,x)=\sum_{n=1}^{\infty}L_n^*(t,\lambda)h_nz_n(x), \tag{5.9}$$

or in the operator form

$$l=N[l] \tag{5.10}$$

where $N[l]=N_0[l]+h$,

$$N_0[l]=-\int_0^T\int_Q L(t,\tau,x,y,\lambda)f[\tau,y,\varphi(\tau,y,l(\tau,y),\beta)]dyd\tau,\quad h=h(t,x).$$

Now we turn to the problem of unique solvability of operator equation (5.10).

Lemma 5.2. *The function $h(t,x)$ is an element of the space $H(Q_T)$.*

Proof. By direct calculation we establish the following inequality

$$\int_0^T\int_Q h^2(t,x)dxdt=\int_0^T\int_Q\left(\sum_{n=1}^{\infty}L_n^*(t,\lambda)h_nz_n(x)\right)^2dxdt$$

$$= \int_0^T \left(\sum_{n=1}^{\infty} L_n^*(t,\lambda) h_n \right)^2 dt = \int_0^T \sum_{n=1}^{\infty} |\langle L_n(t,\lambda), h_n \rangle|^2_{R^2} \, dt$$

$$\leq \int_0^T \sum_{n=1}^{\infty} \|L_n(t,\lambda)\|^2_{R^2} \|h_n\|^2_{R^2} \, dt \leq 4T \left[1 + \frac{\lambda^2 T^2 K_0}{(\lambda_1 - |\lambda| T \sqrt{K_0})^2} \right] \sum_{n=1}^{\infty} \|h_n\|^2_{R^2} .$$

Further, taking into account that $\psi_1 \in H_1(Q)$ and the following estimates

$$\int_0^T \left| R'_{nt}(t,s,\lambda) \right|^2 ds \leq \frac{T K_0 \lambda_n^2}{(\lambda_n - |\lambda| T \sqrt{K_0})^2},$$

$$\int_0^T |R_n(t,s,\lambda)|^2 \, ds \leq \frac{K_0 T}{(\lambda_n - |\lambda| T \sqrt{K_0})^2},$$

it is easy to show that

$$\sum_{n=1}^{\infty} \|h_n\|^2_{R^2} = \sum_{n=1}^{\infty} \left(h_{1n}^2 + h_{2n}^2 \right) < \infty.$$

From these inequalities it follows that $h(t,x) \in H(Q_T)$. □

Lemma 5.3. *The operator $N_0[l(t,x)]$ maps the space $H(Q_T)$ into itself, i.e. $N_0[l(t,x)]$ is an element of the space $H(Q_T)$ for any $l(t,x) \in H(Q_T)$.*

Proof. By direct calculation we have the inequalities

$$\int_0^T \int_Q N_0^2[l(t,x)] dx dt = \int_0^T \int_Q \left(- \int_0^T \int_Q L(t,\tau,x,y,\lambda) f[\tau,y,\varphi(\tau,y,l(\tau,y),\beta)] dy d\tau \right)^2 dx dt$$

$$= \int_0^T \left(\sum_{n=1}^{\infty} \left\langle L_n(t,\lambda), \int_0^T G_n(\tau,\lambda) f[\tau,y,\varphi(\tau,y,l(\tau,y),\beta)] d\tau \right\rangle \right)^2 dt$$

$$= \int_0^T \sum_{n=1}^{\infty} \left| \left\langle L_n(t,\lambda), \int_0^T G_n(\tau,\lambda) f[\tau,y,\varphi(\tau,y,l(\tau,y),\beta)] d\tau \right\rangle \right|^2 dt$$

$$\leq \int_0^T \sum_{n=1}^{\infty} \|L_n(t,\lambda)\|^2_{R^2} \left\| \int_0^T G_n(\tau,\lambda) f[\tau,y,\varphi(\tau,y,l(\tau,y),\beta)] d\tau \right\|^2_{R^2} dt$$

$$\leq \int_0^T \sum_{n=1}^{\infty} \|L_n(t,\lambda)\|^2_{R^2} \int_0^T \|G_n(\tau,\lambda)\|^2_{R^2} \, d\tau \int_0^T f_n^2[\tau,u] d\tau dt$$

$$\leq \int_0^T \sum_{n=1}^{\infty} 4 \left(1 + \frac{\lambda^2 T^2 K_0}{(\lambda_1 - |\lambda| T \sqrt{K_0})^2} \right) 2 \left(1 + \frac{\lambda^2 T^2 K_0}{(\lambda_1 - |\lambda| T \sqrt{K_0})^2} \right) T \int_0^T f_n^2[\tau,u] d\tau dt$$

$$= 8 \left(1 + \frac{\lambda^2 T^2 K_0}{(\lambda_1 - |\lambda| T \sqrt{K_0})^2} \right)^2 T^2 \sum_{n=1}^{\infty} \int_0^T f_n^2[\tau,u] d\tau$$

$$= 8 \left(1 + \frac{\lambda^2 T^2 K_0}{(\lambda_1 - |\lambda| T \sqrt{K_0})^2} \right)^2 T^2 \, \|f(t,x,\varphi[t,x,l(t,x),\beta])\|^2_{H(Q)} < \infty,$$

$$\|L_n(t,\lambda)\|^2_{R^2} \le 4\left(1+\frac{\lambda^2T^2K_0}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}\right);$$

$$\|G_n(t,\lambda)\|^2_{R^2} \le 2\left(1+\frac{1}{\lambda_n^2}\right)\left(1+\frac{\lambda^2T^2K_0}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}\right).$$

Hence the statement of the lemma follows. □

Lemma 5.4. *Suppose that the conditions*

$$\|f(t,x,u(t,x))-f(t,x,\bar u(t,x))\|_H \le f_0\|u-\bar u\|_H, \quad f_0>0 \tag{5.11}$$

and

$$\left\|\varphi[t,x,l,\beta]-\varphi[t,x,\bar l,\beta]\right\|_H \le \varphi_0(\beta)\left\|l(t)-\bar l(t)\right\|_H, \quad \varphi_0(\beta)>0 \tag{5.12}$$

are satisfied. Then if the condition

$$\gamma=\bar\gamma f_0\varphi_0(\beta)<1, \quad \bar\gamma=\sqrt{8}\left(1+\frac{\lambda T^2K_0}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}\right)T \tag{5.13}$$

is met, the operator $N_0[l]$ *is contractive.*

Proof. By direct calculations we have the inequality

$$\begin{aligned}
&\left\|N(l)-N(\bar l)\right\|_H=\left\|N_0(l)+h-N_0(\bar l)-h\right\|_H\\
&\le\bar\gamma\left\|f(t,x,\varphi[t,x,l(t,x),\beta])-f(t,x,\varphi[t,x,\bar l(t,x),\beta])\right\|_H\\
&\le\bar\gamma f_0\left\|\varphi[t,x,l(t,x),\beta]-\varphi[t,x,\bar l(t,x),\beta]\right\|_H\\
&\le\bar\gamma f_0\left\|\varphi[t,x,l(t,x),\beta]-\varphi[t,x,\bar l(t,x),\beta]\right\|_H\\
&\le\bar\gamma f_0\varphi(\beta)\left\|l(t,x)-\bar l(t,x)\right\|_H=\gamma\left\|l(t,x)-\bar l(t,x)\right\|_H,
\end{aligned}$$

from which follows the proof of the lemma. □

Theorem 5.1. *Suppose that conditions (2.4)-(2.5), (3.6), (5.2)-(5.4) are satisfied. Then operator equation (5.10) has a unique solution in the space* $H(Q_T)$.

Proof. According to Lemmas 4.1 and 4.2, operator equation (5.10) can be considered in the space $H(Q_T)$. According to Lemma 4.3 operator $N(l)$ is contractive. Since the Hilbert space $H(Q_T)$ is a complete metric space, by the theorem on contraction mappings [6] the operator $N(l)$ has a unique fixed point, i.e. operator equation (5.10) has a unique solution.

The solution of operator equation (5.10) can be found by the method of successive approximations, i.e. nth approximation of the solution is found by the formula

$$l_n(t,x)=N[l_{n-1}(t,x)], \quad n=1,2,3,...,$$

where $l_0(t,x)$ is an arbitrary element of the space $H(Q_T)$ and $h=l_0(t,x)$. We have the estimate [6]

$$\begin{aligned}
\left\|\bar l(t,x)-l_n(t,x)\right\| &= \frac{\gamma^n}{1-\gamma}\|N(l_0)-l_0\|_H\frac{\gamma^n}{1-\gamma}\|N(l_0)+h-l_0\|_H\\
&=\frac{\gamma^n}{1-\gamma}\|N[l_0(t,x)]\|_H
\end{aligned}$$

where $0<\gamma<1$ is the constraction constant. □

The exact solution $\bar{l}(t,x)$ can be found as the limit of the approximate solutions, i.e.

$$\bar{l}(t,x) = \lim_{n\to\infty} l_n(t,x).$$

Substituting this solution in (5.7) we find the required optimal control

$$u^0(t,x) = \varphi[t,x,\bar{l}(t,x),\beta]. \tag{5.14}$$

According to (2.6) we find the optimal process $V^0(t,x)$, i.e. the solution of boundary value problem (2.1)-(2.5) corresponding to the optimal control $u^0(t,x)$, by the formula

$$V^0(t,x) = \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,s,\lambda)a_n(s)ds + a_n(t)\right) z_n(x). \tag{5.15}$$

The minimum value of the functional (3.1) is calculated by the formula

$$J[u^0(t,x)] = \int_Q \left\{\left[V^0(T,x)-\xi_1(x)\right]^2 + \left[V_t^0(T,x)-\xi_2(x)\right]^2\right\} dx \tag{5.16}$$

$$+\beta\int_0^T\int_Q l^2(t,x,u^0(t,x))dxdt.$$

The found triple $(u^0(t,x), V^0(t,x), J[u^0(t,x)])$ is a solution to the nonlinear optimization problem.

6 An approximate solution of the optimization problem

In practice, it is not always possible to find the exact solution of equation (5.8), i.e. the limit function $\bar{l}(t,x)$. Therefore, in most cases the approximate solution $l_n(t,x)$ of (5.8) is considered, where the number n is determined by the inequality

$$\left\|\bar{l}(t,x) - l_n(t,x)\right\| \leq \frac{\gamma^n}{1-\gamma}\left\|N_0[l(t,x)]\right\|_H < \varepsilon \tag{6.1}$$

for a given $\varepsilon > 0$. By substituting the approximate solution $l_n(t,x)$ in (5.7) we find the nth approximation of the optimal control

$$u_n(t,x) = \varphi[t,x,l_n(t,x),\beta]. \tag{6.2}$$

Lemma 6.1. *Let the function $\varphi[t,x,l(t,x),\beta]$ satisfy condition (5.13). Then the nth approximate controls converges to the optimal control $u^0(t,x)$ by norm of the Hilbert space $H(Q_T)$ as $n\to\infty$.*

Proof. Lemma's assertion follows by the inequality

$$\left\|u^0(t,x) - u_n(t,x)\right\|_H = \left\|\varphi[t,x,\bar{l}(t,x),\beta] - \varphi[t,x,l_n(t,x),\beta]\right\|_H$$

$$\leq \varphi_0(\beta)\left\|\bar{l}(t,x) - l_n(t,x)\right\|_H \leq \varphi_0(\beta)\frac{\gamma^n}{1-\gamma}\left\|N_0[h(t,x)]\right\| \xrightarrow[n\to\infty]{} 0. \tag{6.3}$$

□

According to formulas (2.6), (2.10) (2.11) and supposing that $u = u^0(t,x)$ we find the optimal process $V^0(t,x)$. We will consider the following approximations :

$$V^m(t,x) = \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n^m(t,s,\lambda)a_n(s)ds + a_n(t)\right)z_n(x), \tag{6.4}$$

the mth approximation of the optimal process with respect the resolvent;

$$V_k^m(t,x) = \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n^m(t,s,\lambda)a_n^k(s)ds + a_n^k(t)\right)z_n(x), \tag{6.5}$$

where

$$a_n^k(t) = \psi_{1n}cos\lambda_n t + \frac{1}{\lambda_n}\psi_{2n}sin\lambda_n t$$
$$+\frac{1}{\lambda_n}\int_0^t sin\lambda_n(t-\tau)\int_Q f[\tau,y,u_k(\tau,y)]z_n(y)dyd\tau \tag{6.6}$$

the k,mth approximation of the optimal process corresponding to control $u_k(t,x)$;

$$V_k^{m,r}(t,x) = \sum_{n=1}^{r}\left(\lambda\int_0^T R_n^m(t,s,\lambda)a_n^k(s)ds + a_n^k(t)\right)z_n(x) \tag{6.7}$$

is the k,m,rth approximation of the optimal process which determined by a finite sum, i.e. approximation which is applicable in practice.

Investigation of the convergence of the optimal process will be carried out by the following scheme. We note that

$$\left\|V^0(t,x) - V_k^{m,r}(t,x)\right\|_H \leq \left\|V^0(t,x) - V^m(t,x)\right\|_H$$
$$+\left\|V^m(t,x) - V_k^m(t,x)\right\|_H + \left\|V_k^m(t,x) - V_k^{m,r}(t,x)\right\|_H, \tag{6.8}$$

and we prove that the following relations holds

$$\left\|V^0(t,x) - V^m(t,x)\right\|_H \xrightarrow[m\to\infty]{} 0; \tag{6.9}$$

$$\left\|V^m(t,x) - V_k^m(t,x)\right\|_H \xrightarrow[k\to\infty]{} 0, \tag{6.10}$$

for any fixed $m = 1,2,3,...$,

$$\left\|V_k^m(t,x) - V_k^{m,r}(t,x)\right\|_H \xrightarrow[r\to\infty]{} 0, \tag{6.11}$$

for any fixed $m,k = 1,2,3,....$

Then according to (6.9)-(6.11) by (6.8) we have

$$\left\|V^0(t,x) - V_k^{m,r}(t,x)\right\|_H \xrightarrow[m,k,r\to\infty]{} 0.$$

Relation (6.9) follows by the inequality :

$$\left\|V^0(t,x) - V^m(t,x)\right\|_H \leq \frac{\lambda T\sqrt{K_0}}{\lambda_1}\left(1 - \frac{1}{\ln\frac{|\lambda|T\sqrt{K_0}}{\lambda_1}}\right)^2$$

$$\times 3T\left(\|\psi_1(x)\|_H^2+\frac{1}{\lambda_1^2}\|\psi_2(x)\|_H^2+\frac{T}{\lambda_1^2}\|f(t,x,u(t,x))\|_H^2\right)^{1/2}$$
$$\times\left(|\lambda|\frac{T}{\lambda_1}\sqrt{K_0}\right)^m \xrightarrow[m\to\infty]{} 0$$

since $|\lambda|\frac{T}{\lambda_1}\sqrt{K_0}<1$.

Relation (6.10) follows by the inequality

$$\|V^m(t,x)-V_k^m(t,x)\|_H^2\le\int_0^T\int_Q\left(\sum_{n=1}^\infty\left[\lambda\int_0^T R_n^m(t,s,\lambda)a_n(s)ds+a_n(t)\right.\right.$$
$$\left.\left.-\lambda\int_0^T R_n^m(t,s,\lambda)a_n^k(s)ds-a_n^k(t)\right]z_n(x)\right)^2dxdt$$
$$=\int_0^T\sum_{n=1}^\infty\left[\lambda\int_0^T R_n^m(t,s,\lambda)\frac{1}{\lambda_n}\int_0^s sin(s-\tau)\right.$$
$$\times(\int_Q(f[\tau,y,u^0(\tau,y)]-f[\tau,y,u_k(\tau,y)])z_n(y)dy)d\tau ds$$
$$\left.+\frac{1}{\lambda_n}\int_0^s sin(s-\tau)\int_Q(f[\tau,y,u^0(\tau,y)]-f[\tau,y,u_k(\tau,y)])z_n(y)dyd\tau\right]^2dt$$
$$\le 2\int_0^T\sum_{n=1}^\infty\left[\lambda^2\int_0^T R_n^{m2}(t,s,\lambda)ds\int_0^T\int_0^s\frac{1}{\lambda_n^2}\int_0^s sin^2(s-\tau)d\tau\right.$$
$$\times\int_0^s\left(\int_Q\left(f[\tau,y,u^0(\tau,y)]-f[\tau,y,u_k(\tau,y)]\right)z_n(y)dy\right)^2d\tau ds$$
$$\left.+\frac{1}{\lambda_n^2}\int_0^t sin(t-\tau)d\tau\int_0^t(\int_Q[f(u^0)-f(u_k)]z_n(y)dy)^2d\tau\right]dt$$
$$\le 2\int_0^T\sum_{n=1}^\infty\left[\frac{\lambda^2K_0T^2}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}\frac{1}{\lambda_n^2}T\right.$$
$$\times\int_0^T\left(\int_Q\left(f[\tau,y,u^0(\tau,y)]-f[\tau,y,u_k(\tau,y)]\right)z_n(y)dy\right)^2d\tau$$
$$\left.+\frac{1}{\lambda_n^2}T\int_0^T\left(\int_Q\left(f[\tau,y,u^0(\tau,y)]-f[\tau,y,u_k(\tau,y)]\right)z_n(y)dy\right)^2d\tau\right]dt$$
$$\le 2T\frac{1}{\lambda_n^2}T\left(1+\frac{\lambda^2K_0T^2}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}\right)f_0^2\left\|u^0(t,x)-u_k(t,x)\right\|_H^2,$$

$$\|V^m(t,x)-V_k^m(t,x)\|_H$$
$$\le\frac{2T^2}{\lambda_n^2}\left(1+\frac{\lambda^2K_0T^2}{(\lambda_1-|\lambda|T\sqrt{K_0})^2}\right)^{1/2}f_0\left\|u^0(t,x)-u_k(t,x)\right\|_H\xrightarrow[k\to\infty]{}0.$$

Relation (6.11) follows by the inequality

$$\|V_k^m(t,x) - V_k^{m,r}(t,x)\|_H^2$$

$$\leq \int_0^T \sum_{n=r+1}^{\infty} \left(\left(\lambda \int_0^T R_n^m(t,s,\lambda) a_n^k(s) ds + a_n^k(t) \right) z_n(x) \right)^2 dx dt$$

$$= \int_0^T \sum_{n=r+1}^{\infty} \left(\lambda \int_0^T R_n^m(t,s,\lambda) [\psi_{1n} cos\lambda_n s \right.$$

$$+ \frac{1}{\lambda_n} \psi_{2n} sin\lambda_n s + \frac{1}{\lambda_n} \int_0^t sin\lambda_n(s-\tau) \int_Q f_n[\tau, y, u_k(\tau,y)] z_n(y) dy d\tau] ds$$

$$\left. + \psi_{1n} cos\lambda_n t + \frac{1}{\lambda_n} \psi_{2n} sin\lambda_n t + \frac{1}{\lambda_n} \int_0^t sin\lambda_n(t-\tau) \int_Q f_n[\tau, y, u_k(\tau,y)] z_n(y) dy d\tau \right)^2 dt$$

$$\leq \int_0^T \sum_{n=r+1}^{\infty} \left(\psi_{1n} \left[cos\lambda_n t + \lambda \int_0^T R_n^m(t,s,\lambda) cos\lambda_n s ds \right] \right.$$

$$+ \frac{\psi_{2n}}{\lambda_n} \left[sin\lambda_n t + \lambda \int_0^T R_n^m(t,s,\lambda) sin\lambda_n s ds \right]$$

$$+ \frac{1}{\lambda_n} \int_0^t sin\lambda_n(t-\tau) \int_Q f(\tau, y, u_k(\tau,y)) z_n(y) dy d\tau$$

$$\left. + \frac{1}{\lambda_n} \int_0^T (\lambda \int_\tau^T R_n^m(t,s,\lambda) sin\lambda_n(s-\tau) ds) \int_Q f(\tau, y, u_k(\tau,y)) z_n(y) dy d\tau \right)^2 dt$$

$$\leq 3 \int_0^T \sum_{n=r+1}^{\infty} \left(\psi_{1n}^2 [cos\lambda_n t + \lambda \int_0^T R_n^m(t,s,\lambda) cos\lambda_n s ds]^2 \right.$$

$$+ \frac{\psi_{2n}^2}{\lambda_n^2} [sin\lambda_n t + \lambda \int_0^T R_n^m(t,s,\lambda) sin\lambda_n s ds]^2$$

$$+ \frac{2}{\lambda_n^2} \left[\int_0^T sin^2\lambda_n(T-\tau) d\tau \int_0^T (\int_Q f(\tau, y, u_k(\tau,y)) z_n(y) dy \right]^2 d\tau$$

$$\left. + \int_0^T (\lambda \int_0^T R_n^2(t,s,\lambda) ds \int_0^T sin^2\lambda_n(s-0) ds) d\tau \int_0^T (\int_Q f(\tau, y, u_k(\tau,y)) z_n(y) dy)^2 d\tau \right) dt$$

$$\leq 6 \int_0^T \sum_{n=r+1}^{\infty} \left(\psi_{1n}^2 [cos^2\lambda_n t + \lambda^2 \int_0^T R_n^2(t,s,\lambda) ds \int_0^T cos^2\lambda_n s ds] \right.$$

$$+ \frac{\psi_{2n}^2}{\lambda_n^2} [sin^2\lambda_n t + \lambda^2 \int_0^T R_n^2(t,s,\lambda) ds \int_0^T sin^2\lambda_n s ds]$$

$$+ \frac{1}{\lambda_n^2} \left[\int_0^T sin^2\lambda_n(T-\tau) d\tau \int_0^T \lambda^2 \int_0^T R_n^2(t,s,\lambda) ds \int_0^T sin^2\lambda_n s ds d\tau \right]$$

$$\times \int_0^T (\int_Q f(\tau, y, u_k(\tau, y))z_n(y)dy)^2 d\tau \Big) dt$$

$$\leq 6T \sum_{n=r+1}^{\infty} \Bigg(\psi_{1n}^2 \left[1 + \frac{\lambda^2 T K_0}{(\lambda_n - |\lambda| T\sqrt{K_0})^2} T \right] + \frac{\psi_{2n}^2}{\lambda_n^2} \left[1 + \frac{\lambda^2 T K_0}{(\lambda_n - |\lambda| T\sqrt{K_0})^2} T \right]$$

$$+ \frac{1}{\lambda_n^2} \left[T + \frac{\lambda^2 T K_0}{(\lambda_n - |\lambda| T\sqrt{K_0})^2} T^2 \right] \int_0^T (\int_Q f(\tau, y, u_k(\tau, y))z_n(y)dy)^2 d\tau \Bigg)$$

$$\leq 6T \sum_{n=r+1}^{\infty} \Bigg(\psi_{1n}^2 \left[1 + \frac{\lambda^2 T K_0}{(\lambda_n - |\lambda| T\sqrt{K_0})^2} T \right] + \frac{\psi_{2n}^2}{\lambda_n^2} \left[1 + \frac{\lambda^2 T K_0}{(\lambda_n - |\lambda| T\sqrt{K_0})^2} T \right]$$

$$+ \frac{T}{\lambda_n^2} \left[1 + \frac{\lambda^2 T K_0}{(\lambda_n - |\lambda| T\sqrt{K_0})^2} T \right] \int_0^T (\int_Q f(\tau, y, u_k(\tau, y))z_n(y)dy)^2 d\tau \Bigg)$$

$$\leq 6T \left[1 + \frac{\lambda^2 T K_0}{(\lambda_{r+1} - |\lambda| T\sqrt{K_0})^2} T \right] \{ \sum_{n=r+1}^{\infty} \psi_{1n}^2 + \sum_{n=r+1}^{\infty} \frac{\psi_{2n}^2}{\lambda_n^2}$$

$$+ T \sum_{n=r+1}^{\infty} \frac{1}{\lambda_n^2} \int_0^T (\int_Q f(\tau, y, u_k(\tau, y))z_n(y)dy)^2 d\tau \}$$

$$\leq 6T \left[1 + \frac{\lambda^2 T K_0}{(\lambda_{r+1} - |\lambda| T\sqrt{K_0})^2} T \right] \{ \sum_{n=r+1}^{\infty} \psi_{1n}^2$$

$$+ \left(\|\psi_2(x)\|_H + T \|f(\tau, y, u_k(\tau, y))\|_H^2 \right) \sum_{n=r+1}^{\infty} \frac{1}{\lambda_n^2} \}$$

$$\leq 6T \left[1 + \frac{\lambda^2 T K_0}{(\lambda_{r+1} - |\lambda| T\sqrt{K_0})^2} T \right] \{ \sum_{n=r+1}^{\infty} \psi_{1n}^2$$

$$+ \left(\|\psi_2(x)\|_H + T \|f(\tau, y, u_k(\tau, y))\|_H^2 \right) \frac{1}{\pi^2} \frac{r+1}{r^2} \} \xrightarrow[r \to \infty]{} 0,$$

i.e. $\sum_{n=r+1}^{\infty} \psi_{1n}^2 \xrightarrow[r \to \infty]{} 0$ as a residual term of a convergent series.

Now we calculate the approximate value of the functional. In accordance with the approximations of the optimal process we will distinguish the follows types of the approximations of the minimum value of the functional:

$$J[u^0(t,x)] = \int_Q \left\{ [V^0(T,x) - \xi_1(x)]^2 + [V_t^0(T,x) - \xi_2(x)]^2 \right\} dx$$

$$+ \beta \int_0^T \int_Q p^2(t, x, u^0(t,x)) dx dt$$

is minimal value of the functional corresponding to the optimal control $u^0(t,x)$ and optimal process $V^0(t,x)$;

$$J_m[u^0(t,x)] = \int_Q \left\{ [V^m(T,x) - \xi_1(x)]^2 + [V_t^m(T,x) - \xi_2(x)]^2 \right\} dx$$

$$+\beta\int_0^T\int_Q p^2(t,x,u^0(t,x))dxdt$$

is minimal value of the functional corresponding to the optimal control $u^0(t,x)$ and mth approximation (with respect to resolvent) of the optimal process $V^0(t,x)$;

$$J_m[u_k(t,x)]=\int_Q\left\{[V_k^m(T,x)-\xi_1(x)]^2+[V_{kt}^m(T,x)-\xi_2(x)]^2\right\}dx$$

$$+\beta\int_0^T\int_Q p^2(t,x,u_k(t,x))dxdt$$

is minimal value of the functional corresponding kth approximation of optimal control and mth approximation of the optimal process;

$$J_m^r[u_k(t,x)]=\int_Q\left\{[V_k^{m,r}(T,x)-\xi_1(x)]^2+[V_{kt}^{m,r}(T,x)-\xi_2(x)]^2\right\}dx$$

$$+\beta\int_0^T\int_Q p^2(t,x,u_k(t,x))dxdt$$

is minimal value of the functional corresponding kth approximation of optimal control $u^0(t,x)$ and k,r,mth approximation of the optimal process $V^0(t,x)$. Investigation of convergence of the optimal process will be carried out as follows:

$$\left|J(u^0)-J_m^r(u_k)\right|\le\left|J(u^0)-J_m(u^0)\right|+\left|J_m(u^0)-J_m(u_k)\right|+\left|J_m(u_k)-J_m^r(u_k)\right|.$$

It is easy to prove that the following relations hold

$$\left|J(u^0)-J_m(u^0)\right|\xrightarrow[m\to\infty]{}0; \tag{6.12}$$

$$\left|J_m(u^0)-J_m(u_k)\right|\xrightarrow[k\to\infty]{}0, \tag{6.13}$$

for any fixed $m=1,2,3,...$;

$$|J_m(u_k)-J_m^r(u_k)|\xrightarrow[r\to\infty]{}0 \tag{6.14}$$

for any fixed values $m,k=1,2,3,....$

Then from (6.8) we have

$$\left|J(u^0)-J_m^r(u_k)\right|\xrightarrow[m,k,r\to\infty]{}0.$$

Relation (6.12) follows by the inequality

$$\left|J(u^0)-J_m(u^0)\right|=\int_Q\{\left(V^0(T,x)-\xi_1\right)^2+\left(V_t^0(T,x)-\xi_2\right)^2$$

$$-\left(V^m(T,x)-\xi_1\right)^2+\left(V_t^m(T,x)-\xi_2\right)^2\}dx$$

$$=\int_Q\{\left(V^0(T,x)+V^m(T,x)-2\xi_1\right)$$

$$\times \left(V^0(T,x) - V^m(T,x)\right) + \left(V_t^0(T,x) + V_t^m(T,x) - 2\xi_2\right)\left(V_t^0(T,x) - V_t^m(T,x)\right)\}$$
$$\leq \left\|V^0(T,x) + V^m(T,x) - 2\xi_1\right\|_H \left\|V^0(T,x) - V^m(T,x)\right\|_H$$
$$+ \left\|V_t^0(T,x) + V_t^m(T,x) - 2\xi_2\right\|_H \left\|V_t^0(T,x) - V_t^m(T,x)\right\|_H$$
$$\leq C_1 \left\|V^0(T,x) - V^m(T,x)\right\|_H + C_2 \left\|V_t^0(T,x) - V_t^m(T,x)\right\|_H \xrightarrow[m\to\infty]{} 0,$$

where

$$\left\|V^0(T,x)\right\|_H + \|V^m(T,x)\|_H + 2\,\|2\xi_1\|_H \leq C_1,$$
$$\left\|V_t^0(T,x)\right\|_H + \|V_t^m(T,x)\|_H + 2\,\|2\xi_2\|_H \leq C_2,$$

as

$$\left\|V^0(T,x) - V^m(T,x)\right\|_H \leq \sqrt{C_3(\lambda)}\left(|\lambda|\frac{T}{\lambda_1}\sqrt{K_0}\right)^m \xrightarrow[m\to\infty]{} 0,$$
$$\left\|V_t^0(T,x) - V_t^m(T,x)\right\|_H \leq \sqrt{C_4(\lambda)}\left(|\lambda|\frac{T}{\lambda_1}\sqrt{K_0}\right)^m \xrightarrow[m\to\infty]{} 0.$$

Concerning the first relation see equation (2.20) and the second relation is proved similarly.

Relation (6.13) follows by the inequality

$$\left|J_m(u^0) - J_m(u_k)\right| = \left|\beta\int_0^T\int_Q \left(p^2(t,x,u^0(t,x)) - p^2(t,x,u_k(t,x))\right)dxdt\right|$$
$$= \beta\int_0^T\int_Q \left(\left[p(t,x,u^0(t,x)) + p(t,x,u_k(t,x))\right]^2\right)^{1/2}$$
$$\times \left(\left[p(t,x,u^0(t,x)) - p(t,x,u_k(t,x))\right]^2\right)^{1/2} dxdt$$
$$\beta C_3 \left\|p(t,x,u^0(t,x)) - p(t,x,u_k(t,x))\right\|_H$$
$$= \beta C_3 p_0 \left\|u^0(t,x) - u_k(t,x)\right\|_H \xrightarrow[k\to\infty]{} 0, \quad m = 1,2,3,...,$$
$$C_3 = \int_0^T\int_Q \left(\left[p(t,x,u^0(t,x)) + p(t,x,u_k(t,x))\right]^2\right)^{1/2} dxdt.$$

Relation (6.14) follows by the inequality

$$|J_m(u_k) - J_m^r(u_k)| \leq \|V_k^m(T,x) + V_k^{m,r}(T,x) - 2\xi_1\|_H \|V_k^m(T,x) - V_k^{m,r}(T,x)\|_H$$
$$+ \|V_{kt}^m(T,x) + V_{kt}^{m,r}(T,x) - 2\xi_2\|_H \|V_{kt}^m(T,x) - V_{kt}^{m,r}(T,x)\|_H =$$
$$C_4 \|V_k^m(T,x) - V_k^{m,r}(T,x)\|_H + C_5 \|V_{kt}^m(T,x) - V_{kt}^{m,r}(T,x)\|_H \xrightarrow[r\to\infty]{} 0,$$

$$\|V_k^m(T,x)\|_H + \|V_k^{m,r}(T,x)\|_H + 2\,\|\xi_1\|_H \leq C_4,$$
$$\|V_{kt}^m(T,x)\|_H + \|V_{kt}^{m,r}(T,x)\|_H + 2\,\|\xi_2\|_H \leq C_5,$$

Thus we have proved that the approximate solution of nonlinear optimization problem converges to the exact solution $(u_k(t), V_k^{m,r}(t,x), J_m^r[u_k(t)])$ with respect to optimal control, optimal process and functional.

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