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APPLICATIONS OF ANTICOMPACT SETS TO ANALOGS OF DENJOY-YOUNG-SAKS AND LEBESGUE THEOREMS

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Abstract. We consider the problem of transfer of the Denjoy-Young-Saks theorem on derivates to infinite-dimensional Banach spaces and the problem of nondifferentiability of indefinite Pettis integral in infinite-dimensional Banach spaces. Our approach is based on the concept of an anticompact set proposed by us earlier. We prove an analog of the Denjoy-Young-Saks theorem on derivates in Banach spaces which have anticompact sets. Also in such spaces we obtain an analog of the Lebesgue theorem. This result states that each indefinite Pettis integral is differentiable almost everywhere in the topology of special Hilbert space generated by some anticompact set in the original space.

1 Introduction

In this paper we prove new versions of the Denjoy-Young-Saks theorem on derivates and the Lebesgue theorem about differentiability of indefinite Pettis integral in infinite-dimensional Banach spaces.

There is a well-known consequence of the Denjoy-Young-Saks theorem [1, 2], which describes the derivates of any real function $f : [a; b] \to \mathbb{R}$.

Theorem 1.1. *For almost all $x \in [a; b]$ either f is differentiable at x, or there is an infinite derivate f at the point x.*

In [3] we obtained a generalization of Theorem 1.1 to the case of almost everywhere separable mappings $F : [a; b] \to E$, where E is a Frechet space (in particular, Banach space) for usual derivates (limit points of $\frac{F(x+t)-F(x)}{t}$ as $t \to 0$ in the topology of the corresponding space).

Let E be a Banach space with the norm $\|\cdot\|$. Denote by $\widehat{\partial} F(x)$ any derivate of mapping F at the point $x \in [a; b]$ (i.e. $\widehat{\partial} F(x) = \lim\limits_{k\to\infty} \frac{F(x+h_k)-F(x)}{h_k}$ for some sequence $h_k \to 0$). Say that the mapping F has an infinite derivate $\widehat{\partial} F(x) = \infty$ at point x, if $\lim\limits_{k\to\infty} \left\| \frac{F(x+h_k)-F(x)}{h_k} \right\| = +\infty$ for some sequence $h_k \to 0$.

Theorem 1.2. (see [3]). *Let $F : [a; b] \to E$ be almost everywhere separable-valued on $[a; b]$. Then almost everywhere on $[a; b]$ one of the following conditions is true:*

(i) *there exists a derivate $\widehat{\partial}F(x) = \infty$;*

(ii) *for some sequence $h_k \to 0$ the sequence $\frac{\triangle F(x,h_k)}{h_k}$ does not contain convergent subsequence;*

(iii) *all derivates of F at the point x are finite and coincide, i.e. there exists the derivative $F'(x)$.*

Note that the condition *(ii)* in Theorem 1.2 can not be omitted and in [3] an example of a mapping which satisfies this condition everywhere on $[a; b]$ is constructed. In the present paper we obtain an analog of Theorem 1.2 for mappings in Banach spaces without this condition. However, differentiability and derivates will be taken not in the topology of the original space, but in the topology of a new special Banach space.

Further, there are many analogs of the classical Lebesgue integral for mappings in infinite-dimensional Banach space. The most well-known and widely used is the Bochner integral, since it preserves almost all properties of the Lebesgue integral [4, 5]. However, the class of Bochner integrable mappings is not wide enough for many applications [5, 6, 7].

In this regard, together with the Bochner integral, other concepts of integral actively studied and used for the mappings in infinite-dimensional Banach spaces [5, 6, 7]. In particular there is a well-known theory of Pettis integral [4] — [6]. The class of Pettis integrable mappings is significantly wider then the class of Bochner integrable mappings. But the Pettis integral loses many significant properties of the Bochner integral. For example, each indefinite Bochner integral $F : I = [a; b] \to E$ (E is a Banach space) $F(x) = (B)\int\limits_a^x f(t)dt$ $(a \leqslant x \leqslant b)$ preserves the property of differentiability almost everywhere on $[a; b]$. We consider the indefinite Pettis integrals, i.e. the mappings $F : I = [a; b] \to E$ (here E is a Banach space) of the form

$$F(x) = (P)\int\limits_a^x f(t)dt, \quad a \leqslant x \leqslant b, \tag{1.1}$$

where f is assumed Pettis integrable on any Lebesgue measurable subset $e \subset I$. As is shown in [8] (remark to Theorem 1), for arbitrary infinite-dimensional Banach space E there exists a strongly measurable and Pettis integrable mapping $f : I \to E$ such that

$$\lim_{h\to 0}\left\|\frac{1}{h}(P)\int\limits_x^{x+h} f(t)dt\right\| = \infty \quad \forall t \in I,$$

which implies (weak and usual) nondifferentiability of the mapping F of (1.1) everywhere on I. It means that the problem of finding conditions under which F of (1.1) is differentiable almost everywhere on I is natural and relevant.

The work [9] was devoted to the problem of nondifferentiability of the indefinite Pettis integral, where conditions of strong differentiability almost everywhere for Pettis integral of mappings in Frechet spaces (1.1) were obtained. To this end, two new characteristics of strongly measurable and Pettis integrable mappings were introduced, namely, *a weak integral boundedness* and *σ-compact measurability.* These results showed that indefinite Pettis integral can be differentiable only for sufficiently narrow and specific classes of mappings, that only outlined the problem. There is a natural problem to obtain an analogue of Lebesgue theorem for an arbitrary Pettis integrable mapping.

The main idea of this paper is to propose an approach to these problems, based on the concept of *anticompact set* in Banach spaces, introduced earlier in [10]. The idea of this approach is as follows. The concept of compact set in topological vector spaces is well-known. In contradiction with the finite-dimensional case for infinite-dimensional spaces the class of compact sets is essentially narrower than the class of bounded closed sets. In view of this fact we consider any closed bounded set in as a precompact set, but probably in some other Banach space (this space must be convenient enough).

The present paper is organized as follows. Section 2 is devoted to useful facts of theory of anticompact sets in Banach space [10, 11]. We recall the concept of anticompact set [10] and results [11] concerning to description of the class of Banach spaces with anticompact sets.

Section 3 is devoted to analogue of the Denjoy-Young-Saks theorem on derivates in Banach spaces having anticompact sets (see Theorem 3.2). On the base of Theorem 3.2 we consider the problem of nondifferentiability of Lipschitz mappings and obtain Corollary 3.1.

At last, in the fourth section of the paper we obtain analog of the Lebesgue theorem for an arbitrary Pettis integrable mapping in the class of Banach spaces having anticompact sets. For any Pettis integrable mapping we prove Bochner integrability in space $E_{C'}$, generated by some anticompact set C' in E (see Theorem 4.1). As a consequence, differentiability almost everywhere in $E_{C'}$ of indefinite Pettis integral is obtained (see Theorem 4.2 and Corollary 4.1).

2 Anticompact sets in Banach spaces

Recall the concept of an anticompact set proposed by us in [10]. Denote by $\Omega_{ac}(E)$ the set of all closed absolutely convex subsets of Banach space E. Here and throughout $p_{C'}(\cdot)$ means the Minkowski functional of absolutely convex set $C' \subset E$.

Definition 1. Say that the set $C' \in \Omega_{ac}(E)$ is *anticompact* in E, if:

(i) $p_{C'}(a) = 0 \Longleftrightarrow a = 0$ in (or $\bigcap\limits_{\lambda>0} \lambda \cdot C' = \{0\}$);

(ii) any bounded subset of E is contained and precompact in the space $E_{C'} = (span\ C',\ p_{C'}(\cdot))$; here $E_{C'}$ is completed with respect to the norm $\|\cdot\|_{C'} = p_{C'}(\cdot)$.

Denote by $\mathcal{C}'(E)$ the class of anticompact subsets of Banach space E.

In previous works [10, 11] some examples of anticompact sets in a separable Hilbert and Banach spaces are constructed. It turns out that the case of a Hilbert space is universal in some sense, since anticompact sets exist only in Banach spaces, linearly injectively and continuously embedded in a separable Hilbert space. In [11] it was shown

Lemma 2.1. *A Banach space E has anticompact sets if and only if there exists a linear injective compact operator $A : E \to F$, where F is some Banach space.*

The previous result have yielded more verifiable criterion.

Theorem 2.1. *A Banach space E has anticompact sets if and only if there exists a linear continuous injective operator $A : E \to \ell_2$.*

In ([11], Corollary 1) some another sufficiently simple description of the class of Banach spaces with anticompact sets was suggested.

Theorem 2.2. *The Banach space E has anticompact sets if and only if there exists countable total subset of linear continuous functionals on E.*

In particular it means that each separable Banach space has anticompact sets.

3 Analog of the Denjoy-Young-Saks theorem on derivates in Banach spaces

In this section we obtain an analog of the Denjoy-Young-Saks theorem on derivates in the class of Banach spaces having anticompact sets. We start with the definition of previously studied compact subdifferential (see, for example [3]), that we'll need later. Let $U(0)$ denotes an arbitrary closed absolutely convex neighborhood of zero in a Banach space E.

Definition 2. Let $\{B_\delta\}_{\delta>0}$ be decreasing on embeddings as $\delta \to +0$ system of closed convex subsets of a Banach space E, $B \subset E$. We say that the set $B = \bigcap\limits_{\delta \to +0} B_\delta$ is *K-limit* of a system $\{B_\delta\}_{\delta>0}$ for $\delta \to +0 : B = K - \lim\limits_{\delta \to +0} B_\delta$, if:

$$\forall U = U(0) \subset E \; \exists \delta = \delta_U > 0 : \; (0 < \delta < \delta_U) \Rightarrow (B_\delta \subset B + U(0)) \; .$$

It follows from Definition 2 that the set B is closed and convex. Let $I \subset \mathbb{R}$ be some real segment, $\overline{conv} A$ be a closed convex hull of the set A and we consider a mapping $F : I = [a; b] \to E$.

Definition 3. Let $x \in I$, $\delta > 0$. Partial K-subdifferential of the mapping F at the point x_0, corresponding to a given $\delta > 0$, is the closed convex set

$$\partial_K F(x_0, \delta) = \overline{conv} \left\{ \frac{F(x_0 + h) - F(x_0)}{h} \;\middle|\; 0 <\mid h \mid< \delta \right\} \; .$$

Definition 4. Say that the mapping $F : I \to E$ is compact subdifferentiable at the point $x_0 \in I$, if there exists K-limit of partial K-subdifferentials

$$\partial_K F(x_0) = K - \lim_{\delta \to +0} \partial_K F(x_0, \delta) \; .$$

Obtained in such a way set $\partial_K F(x_0)$ is called compact subdifferential of the mapping F at the point x_0.

If the mapping F is differentiable at x_0 in the usual sense, then it is compact subdifferentiable, moreover $\partial_K F(x_0) = \{F'(x_0)\}$. At the same time, there are compact subdifferentiable mappings, that have no the usual derivative (see [3]). The following auxiliary result was obtained in [3].

Theorem 3.1. *If the mapping $F : [a; b] \to E$ is almost everywhere separable-valued and compact subdifferentiable on $[a; b]$, then it is almost everywhere differentiable on $[a; b]$.*

On the base of Theorem 3.1, we can easily obtain the following analog of the Denjoy-Youg-Saks theorem on derivates.

Theorem 3.2. *Let $F : [a; b] \to E$, E has countable total set of linear continuous functionals. Then for all anticompact set $C' \in \mathcal{C}'E$ almost everywhere on $[a; b]$ one of the following conditions holds:*

(i) *the mapping $F : I \to E$ has a derivate $\widehat{\partial} F(x) = \infty$;*

(ii) *the mapping $F : I \to E_{C'}$ has a derivative $F'(x)$.*

Proof. Note first that at any point $x_0 \in [a; b]$, where condition (i) fails, the sets $\partial_K F(x_0, \delta)$ are bounded for sufficiently small $\delta > 0$, i.e. $\partial_K F(x_0, \delta)$ are precompact in any space $E_{C'}$, $C' \in \mathcal{C}'(E)$. Therefore there is a compact subdifferential of the mapping $F : [a; b] \to E_{C'}$ at the point x_0. It remains only to apply Theorem 3.1. □

Remark 1. Note that the assertion of the previous result will be true for the mappings in any Banach space if we impose on F restriction, requiring the existence of a countable total set of continuous linear functionals for some subspace $E_0 \subset E$, containing $F([a; b])$. In particular, any continuous mapping $F : I \to E$ will be of this type.

Consider the class of Lipschitz mappings $F : [a; b] \to E$, where E is an arbitrary Banach space. It is known that such mappings can be nowhere differentiable in infinite-dimensional case [4]. Theorem 3.2 and continuity of F imply

Corollary 3.1. *If $F : [a; b] \to E$ satisfies Lipschitz condition, then there is a separable subspace $E_0 \subset E$ such that $F([a; b]) \subset E_0$ and for every anticompact set $C' \in \mathcal{C}'(E_0)$ it holds:*

(i) *$F : I \to E_{C'}$ is compact subdifferentiable everywhere on I;*

(ii) *$F : I \to E_{C'}$ is differentiable almost everywhere on I.*

Remark 2. Question on belonging of $\partial_K F(x)$ and $F'(x)$ to the original space E is still open and requires separate consideration. We only note that for some $C' \in \mathcal{C}'(E)$ $F'(x)$ remains in the original space at almost all $x \in I$, if F is representable as an indefinite Pettis integral in E. It follows from Corollary 4.1.

4 Analog of the Lebesgue theorem on differentiability of indefinite Pettis integral

This section is devoted to analog of the Lebesgue theorem for the indefinite Pettis integral (1.1) in special class of Banach spaces. Recall, that the class of Pettis integrable mappings for which an indefinite Pettis integral (1.1) is almost everywhere differentiable, is quite specific (see [9]). Moreover it is well known that even in the case of differentiability of the integral its derivative at any point may not coincide with the integrand mapping (see [7]). There is a natural problem of finding an analog of the Lebesgue theorem on differentiability of Pettis integral in a more universal way. In this section of the paper we obtain this result by using the concept of anticompact set in Banach spaces. We start with the following result which states Bochner integrability for all Pettis integrable mappings in spaces having anticompact sets.

Theorem 4.1. *Let E be a Banach space with anticompact sets. If $f : I = [a; b] \to E$ is Pettis integrable, then $\exists\, C' \in \mathcal{C}'(E)$ such that f is Bochner integrable in space $E_{C'}$.*

Proof. 1) According to the Theorem 2.1, every space E, satisfying the condition of the Theorem 4.1, can be injectively continuously embedded in a separable Hilbert space $H \cong \ell_2$. Let $\varphi : E \to H$ be a corresponding injective continuous embedding. Then $\varphi(E) \subset H$ and $H^* \subset (\varphi(E))^* \cong E^*$ (here A^* is a dual space to the Banach space A). Therefore weak integrability of $f : I \to E$ means weak integrability of $\varphi(f) : I \to H$, i.e. it is sufficiently to consider the case $E = H$.

2) So, $E = H \cong \ell_2$. It is known that

$$h \in H \Leftrightarrow h = (h_1, h_2, ..., h_n, ...) : \sum_{k=1}^{\infty} |h_k|^2 < \infty.$$

We denote by $\ell_k(h) = h_k \ \forall k \in \mathbb{N}$ ($\ell_k \in H^*$). By virtue of the Pettis integrability of $f : I \to H$ the mapping $f_k = \ell_k(f) : I \to \mathbb{R}$ is Lebesgue integrable, i.e.

$$\int_a^b |f_k(t)|dt = C_k < \infty.$$

Without loss of generality $C_k \neq 0$. Note that, in view of separability of H, weak measurability of f implies strongly measurability of f. We prove that for all $f : I \to H$ it is sufficient to prove an existence of $C' \in \mathcal{C}'(H)$ such that $||f(t)||_{H_{C'}} : I \to \mathbb{R}$ is measurable and

$$\int_a^b ||f(t)||_{H_{C'}} dt < +\infty.$$

Let $C' = \left\{ h = (h_1, h_2, ..., h_n, ...) \in H \Big| \sum_{k=1}^{\infty} \frac{h_k^2}{k^4 C_k^2} \leqslant 1 \right\}$. It is clear that $H_{C'}$ is separable Hilbert space with the norm $||h||_{H_{C'}} = \left(\sum_{k=1}^{\infty} \frac{h_k^2}{k^4 C_k^2} \right)^{\frac{1}{2}}$. We note that

$$||f(t)||_{H_{C'}} = \lim_{n\to\infty} \left(\frac{|f_1(t)|^2}{C_1^2} + \frac{|f_2(t)|^2}{2^4 C_2^2} + ... + \frac{|f_n(t)|^2}{n^4 C_n^2} \right)^{\frac{1}{2}} \quad \forall t \in [a; b],$$

i.e. $f : I \to H_{C'}$ is measurable in the space $H_{C'}$ in view of the weak measurability and separability f. Further, $\forall n \in \mathbb{N}$

$$\int_a^b \left(\frac{|f_1(t)|^2}{C_1^2} + \frac{|f_2(t)|^2}{2^4 C_2^2} + ... + \frac{|f_n(t)|^2}{n^4 C_n^2} \right)^{\frac{1}{2}} dt < \int_a^b \left(\frac{|f_1(t)|}{C_1} + \frac{|f_2(t)|}{2^2 C_2} + ... + \frac{|f_n(t)|}{n^2 C_n} \right) dt =$$

$$= \frac{1}{C_1} \int_a^b |f_1(t)| dt + \frac{1}{2^2 C_2} \int_a^b |f_2(t)| dt + ... + \frac{1}{n^2 C_n} \int_a^b |f_n(t)| dt =$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} < 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} + ... = K < +\infty.$$

Applying Fatou's theorem, we get

$$\int_a^b ||f(t)||_{H_{C'}} dt \leqslant \overline{\lim_{n\to\infty}} \int_a^b \left(\frac{|f_1(t)|^2}{C_1^2} + ... + \frac{|f_n(t)|^2}{n^4 C_n^2} \right)^{\frac{1}{2}} dt \leqslant K < +\infty.$$

Hence, f is weakly measurable in $H_{C'}$ by virtue of $H^*_{C'} \subset H^*$,

$$\int_a^b ||f(t)||_{H_{C'}} dt < +\infty$$

and therefore $f : I \to H_{C'}$ is Bochner integrable, that completes the proof.

□

The properties of the Bochner integral, as well as the previous result imply the following analog of the Lebesgue theorem on differentiability of indefinite Pettis integral on the upper bound.

Theorem 4.2. *Let E be a Banach space with anticompact sets. If $f : I = [a; b] \to E$ is Pettis integrable, then there exists an anticompact set $C' \in \mathcal{C}'(E)$ such that*

$$\lim_{h\to 0} \frac{1}{h} \int_{t_0}^{t_0+h} ||f(t) - f(t_0)||_{E_{C'}} dt = 0$$

for almost all $t_0 \in I = [a; b]$.

Corollary 4.1. *If $K \in E$ is a fixed constant, then for every Pettis integrable mapping $f : [a; b] \to E$ there exists such an anticompact set $C' \in \mathcal{C}'(E)$, that the mapping $F(x) = K + (P) \int_a^x f(t) dt$ $(a \leqslant x \leqslant b)$ is almost everywhere differentiable in $E_{C'}$. In this case $F'_{E_{C'}}(x) = f(x)$ at almost all $x \in [a; b]$.*

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