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ATANASSOV'S INTUITIONISTIC FUZZY TRANSLATIONS OF INTUITIONISTIC FUZZY SUBALGEBRAS AND IDEALS IN BCK/BCI-ALGEBRAS

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Abstract. In this paper, the concepts of intuitionistic fuzzy translation to intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebras are introduced. The notion of intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy subalgebras and ideals are introduced and several related properties are investigated. In this paper, the relationships between intuitionistic fuzzy translations and intuitionistic fuzzy extensions of intuitionistic fuzzy subalgebras and ideals are investigated.

1 Introduction

There exist two different intuitionistic fuzzy sets theories introduced by Atanassov (1983) and Takuti-Titani (1984), see [7, 8] and [34, 35]. Their common name was borrowed from intuitionistic logic which was well-established long before the above mentioned publications by Atanassov and Takeuti-Titani. But the essential difference is that Takeuti-Titani's intuitionistic fuzzy set theory [34] is developed within the scope of intuitionistic mathematics, while Atanassov's intuitionistic fuzzy sets theory is based upon different principles. The term "intuitionistic" in the paper is used only in Atanassov's sense.

The theory of intuitionistic fuzzy set is expected to play an important role in modern mathematics in general as it represents a generalization of fuzzy set. The notion of intuitionistic fuzzy set was defined by Atanassov [4, 5]. Then, several papers have been published by mathematicians to extend the classical mathematical concepts and fuzzy mathematical concepts to the case of intuitionistic fuzzy mathematics, for example see [1, 2, 9, 10, 11, 12, 13, 14, 16, 37].

The notion of BCK-algebras was proposed by Imai and Iseki [18] in 1966. In the same year, Iseki [19] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. In 1991, Xi [36] applied the concept of fuzzy sets to BCK-algebras. In 1993, Jun [23] and Ahmad [3] applied it to BCI-algebras. After that

Jun, Meng, Liu and several researchers investigated further properties of fuzzy BCK-algebras and fuzzy ideals (see [15, 17, 18, 20, 22, 25, 27, 28, 29, 31, 32]). Lee et al. [26] and Jun [24] discussed fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy subalgebras and ideals in BCK/BCI-algebras. They investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications. Recently, in [33], the authors have studied fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy H-ideals in BCK/BCI-algebras. In this paper, intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebras are discussed. Relations among intuitionistic fuzzy translations and intuitionistic fuzzy extensions of intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebras is also investigated.

2 Preliminaries

We first recall some basic concepts which are used to present the paper.

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms, for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We can define a partial ordering $\leqslant$ by $x \leqslant y$ if and only if $x * y = 0$.

If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra. Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:

(1) $(x * y) * z = (x * z) * y$,

(2) $((x * z) * (y * z)) * (x * y) = 0$,

(3) $x * 0 = x$,

(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0$.

Throughout this paper, X always means a BCK/BCI-algebra without any specification.

A non-empty subset S of X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$. A nonempty subset I of X is called an ideal of X if it satisfies

(I_1) $0 \in I$ and

(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

A fuzzy set $A = \{< x, \mu_A(x) >: x \in X\}$ in X is called a fuzzy subalgebra of X if it satisfies the inequality $\mu_A(x * y) \geqslant \min\{\mu_A(x), \mu_A(y)\}$, for all $x, y \in X$.

A fuzzy set $A = \{< x, \mu_A(x) >: x \in X\}$ in X is called a fuzzy ideal [3, 36] of X if it satisfies (i) $\mu_A(0) \geqslant \mu_A(x)$, and (ii) $\mu_A(x) \geqslant min\{\mu_A(x * y), \mu_A(y)\}$, for all $x, y \in X$.

An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in X is called an intuitionistic fuzzy subalgebra [25] of X if it satisfies the following two conditions

(i) $\mu_A(x * y) \geqslant \min\{\mu_A(x), \mu_A(y)\}$ and

(ii) $\nu_A(x * y) \leqslant \max\{\nu_A(x), \nu_A(y)\}$, for all $x, y \in X$.

An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in X is called an intuitionistic fuzzy ideal [25] of X if it satisfies

(i) $\mu_A(0) \geqslant \mu_A(x)$, $\nu_A(0) \leqslant \nu_A(x)$,

(ii) $\mu_A(x) \geqslant \min\{\mu_A(x*y), \mu_A(y)\}$ and
(iii) $\nu_A(x) \leqslant \max\{\nu_A(x*y), \nu_A(y)\}$, for all $x, y \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy subset $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Throughout this paper, we take $\mathfrak{T} := \inf\{\nu_A(x) | x \in X\}$ for any intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of X.

3 Translations of intuitionistic fuzzy subalgebras

Definition 1. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and let $\alpha \in [0, \mathfrak{T}]$. An object having the form $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is called an intuitionistic fuzzy α-translation of A if $(\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha$ and $(\nu_A)^T_\alpha(x) = \nu_A(x) - \alpha$, for all $x \in X$.

Theorem 3.1. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subalgebras of X and let $\alpha \in [0, \mathfrak{T}]$. Then, the intuitionistic fuzzy α-translation A^T_α of A is an intuitionistic fuzzy subalgebras of X.*

Proof. Let $x, y \in X$. Then,

$$\begin{aligned} (\mu_A)^T_\alpha(x*y) &= \mu_A(x*y) + \alpha \geqslant \min\{\mu_A(x), \mu_A(y)\} + \alpha \\ &= \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\} \\ &= \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\} \\ \text{and} \quad (\nu_A)^T_\alpha(x*y) &= \nu_A(x*y) - \alpha \leqslant \max\{\nu_A(x), \nu_A(y)\} - \alpha \\ &= \max\{\nu_A(x) - \alpha, \nu_A(y) - \alpha\} \\ &= \max\{(\nu_A)^T_\alpha(x), (\nu_A)^T_\alpha(y)\}. \end{aligned}$$

Hence, the intuitionistic fuzzy α-translation A^T_α of A is an intuitionistic fuzzy subalgebras of X. □

Theorem 3.2. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation A^T_α of A is an intuitionistic fuzzy subalgebras of X for some $\alpha \in [0, \mathfrak{T}]$. Then, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra of X.*

Proof. Assume that $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy subalgebras of X for some $\alpha \in [0, \mathfrak{T}]$. Let $x, y \in X$, we have

$$\begin{aligned} \mu_A(x*y) + \alpha &= (\mu_A)^T_\alpha(x*y) \geqslant \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\} \\ &= \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\} \\ &= \min\{\mu_A(x), \mu_A(y)\} + \alpha \\ \text{and} \quad \nu_A(x*y) - \alpha &= (\nu_A)^T_\alpha(x*y) \leqslant \max\{(\nu_A)^T_\alpha(x), (\nu_A)^T_\alpha(y)\} \\ &= \max\{\nu_A(x) - \alpha, \nu_A(y) - \alpha\} \\ &= \max\{\nu_A(x), \nu_A(y)\} - \alpha \end{aligned}$$

which implies that $\mu_A(x*y) \geqslant \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x*y) \leqslant \max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$. Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebras of X. □

Definition 2. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy subsets of X. If $A \leqslant B$ i.e, $\mu_A(x) \leqslant \mu_B(x)$ and $\nu_A(x) \geqslant \nu_B(x)$ for all $x \in X$, then we say that B is an intuitionistic fuzzy extension of A.

Definition 3. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy subsets of X. Then, B is called an intuitionistic fuzzy S-extension of A if the following assertions are valid:

(i) B is an intuitionistic fuzzy extension of A.

(ii) If A is an intuitionistic fuzzy subalgebra of X, then B is an intuitionistic fuzzy subalgebra of X.

From the definition of intuitionistic fuzzy α-translation, we get $(\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha$ and $(\nu_A)^T_\alpha(x) = \nu_A(x) - \alpha$ for all $x \in X$. Therefore, we have the following theorem.

Theorem 3.3. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and $\alpha \in [0, \mathfrak{T}]$. Then, the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy S-extension of A.*

The converse of the Theorem 3.3 is not true in general as seen in the following example.

Example 1. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.7	0.2	0.5	0.3	0.6
ν_A	0.2	0.7	0.4	0.6	0.3.

Then, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra of X. Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_B	0.76	0.23	0.55	0.38	0.64
ν_B	0.19	0.63	0.37	0.55	0.26

Then, B is an intuitionistic fuzzy S-extension of A. But it is not the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A for all $\alpha \in [0, \mathfrak{T}]$.

Clearly, the intersection of intuitionistic fuzzy S-extensions of an intuitionistic fuzzy subalgebra A of X is an intuitionistic fuzzy S-extension of A. But the union of intuitionistic fuzzy S-extensions of an intuitionistic fuzzy subalgebra A of X is not an intuitionistic fuzzy S-extension of A as seen in the following example.

Example 2. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.8	0.6	0.2	0.5	0.4
ν_A	0.2	0.4	0.7	0.5	0.6

Then, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra of X. Let $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be intuitionistic fuzzy subsets of X defined by

X	0	1	2	3	4
μ_B	0.8	0.7	0.3	0.6	0.5
ν_B	0.1	0.3	0.6	0.4	0.4

and

X	0	1	2	3	4
μ_C	0.85	0.60	0.40	0.55	0.65
ν_C	0.15	0.40	0.50	0.45	0.35

respectively.

Then, B and C are intuitionistic fuzzy S-extensions of A. Obviously, the union $B \cup C$ is an intuitionistic fuzzy extension of A, but it is not an intuitionistic fuzzy S-extension of A since $\mu_{B\cup C}(4 * 1) = \mu_{B\cup C}(3) = 0.6 \ngeqslant 0.65 = \min\{0.65, 0.7\} = \min\{\mu_{B\cup C}(4), \mu_{B\cup C}(1)\}$ and $\nu_{B\cup C}(4 * 1) = \nu_{B\cup C}(3) = 0.3 \nleqslant 0.35 = \max\{0.35, 0.3\} = \max\{\nu_{B\cup C}(4), \nu_{B\cup C}(1)\}$.

For an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X, $\alpha \in [0, \mathfrak{T}]$ and $t, s \in [0, 1]$ with $t \geqslant \alpha$, let

$$\begin{aligned} U_\alpha(\mu_A; t) : &= \{x \in X \mid \mu_A(x) \geqslant t - \alpha\} \\ \text{and} \quad L_\alpha(\nu_A; s) : &= \{x \in X \mid \nu_A(x) \leqslant s + \alpha\}. \end{aligned}$$

If A is an intuitionistic fuzzy subalgebra of X, then it is clear that $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are subalgebra of X for all $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $t \geqslant \alpha$. But, if we do not give a condition that A is an intuitionistic fuzzy subalgebra of X, then $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are not subalgebra of X as seen in the following example.

Example 3. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra in Example 2 and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.81	0.73	0.68	0.57	0.64
ν_A	0.17	0.25	0.32	0.41	0.35

Since $\mu_A(4*1) = 0.57 \ngeq 0.64 = \min\{\mu_A(4), \mu_A(1)\}$ and $\nu_A(4*1) = 0.41 \nleq 0.35 = \max\{\nu_A(4), \nu_A(1)\}$, therefore, $A = (\mu_A, \nu_A)$ is not an intuitionistic fuzzy subalgebra of X.

For $\alpha = 0.15$ and $t = 0.75$, we obtain $U_\alpha(\mu_A; t) = \{0, 1, 2, 4\}$ which is not a subalgebra of X since $4 * 1 = 3 \notin U_\alpha(\mu_A; t)$.

For $\alpha = 0.15$ and $s = 0.25$, we obtain $L_\alpha(\nu_A; s) = \{0, 1, 2, 4\}$ which is not a subalgebra of X since $4 * 1 = 3 \notin L_\alpha(\nu_A; s)$.

Theorem 3.4. *For $\alpha \in [0, \mathfrak{T}]$, let $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ be the intuitionistic fuzzy α-translation of $A = (\mu_A, \nu_A)$. Then, the following assertions are equivalent:*

(i) $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy subalgebra of X.

(ii) $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are subalgebra of X for $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geqslant \alpha$.

Proof. Assume that A^T_α is an intuitionistic fuzzy subalgebra of X. Then, $(\mu_A)^T_\alpha$ and $(\nu_A)^T_\alpha$ are fuzzy subalgebra of X. Let $x, y \in X$ such that $x, y \in U_\alpha(\mu_A; t)$ and $t \in Im(\mu_A)$ with $t \geqslant \alpha$. Then, $\mu_A(x) \geqslant t - \alpha$ and $\mu_A(y) \geqslant t - \alpha$ i.e., $(\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha \geqslant t$ and $(\mu_A)^T_\alpha(y) = \mu_A(y) + \alpha \geqslant t$. Since $(\mu_A)^T_\alpha$ is a fuzzy subalgebra of X, therefore, we have

$$\mu_A(x * y) + \alpha = (\mu_A)^T_\alpha(x * y) \geqslant \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\} \geqslant t$$

that is, $\mu_A(x * y) \geqslant t - \alpha$ so that $x * y \in U_\alpha(\mu_A; t)$. Again let $x, y \in X$ such that $x, y \in L_\alpha(\nu_A; s)$ and $s \in Im(\nu_A)$. Then, $\nu_A(x) \leqslant s + \alpha$ and $\nu_A(y) \leqslant s + \alpha$ i.e., $(\nu_A)^T_\alpha(x) = \nu_A(x) - \alpha \leqslant s$ and $(\nu_A)^T_\alpha(y) = \nu_A(y) - \alpha \leqslant s$. Since $(\nu_A)^T_\alpha$ is a fuzzy subalgebra of X, it follows that

$$\nu_A(x * y) - \alpha = (\nu_A)^T_\alpha(x * y) \leqslant \max\{(\nu_A)^T_\alpha(x), (\nu_A)^T_\alpha(y)\} \leqslant s$$

that is, $\nu_A(x * y) \leqslant s + \alpha$ so that $x * y \in L_\alpha(\nu_A; s)$. Therefore, $U_\alpha(\mu; t)$ and $L_\alpha(\nu_A; s)$ are subalgebra of X.

Conversely, suppose that $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are subalgebra of X for $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geqslant \alpha$. If there exists $a, b \in X$ such that $(\mu_A)^T_\alpha(a * b) < \beta \leqslant \min\{(\mu_A)^T_\alpha(a), (\mu_A)^T_\alpha(b)\}$, then $\mu_A(a) \geqslant \beta - \alpha$ and $\mu_A(b) \geqslant \beta - \alpha$ but $\mu_A(a*b) < \beta - \alpha$. This shows that $a \in U_\alpha(\mu_A; t)$ and $b \in U_\alpha(\mu_A; t)$ but $a * b \notin U_\alpha(\mu_A; t)$. This is a contradiction, and therefore $(\mu_A)^T_\alpha(x * y) \geqslant \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$ for all $x, y \in X$.

Again assume that there exist $c, d \in X$ such that $(\nu_A)^T_\alpha(c * d) > \delta \geqslant \max\{(\nu_A)^T_\alpha(c), (\nu_A)^T_\alpha(d)\}$. Then, $\nu_A(c) \leqslant \delta + \alpha$ and $\nu_A(d) \leqslant \delta + \alpha$ but $\nu_A(c*d) > \delta + \alpha$. Hence, $c \in L_\alpha(\nu_A; s)$ and $d \in L_\alpha(\nu_A; s)$ but $c * d \notin L_\alpha(\nu_A; s)$. This is impossible and therefore $(\nu_A)^T_\alpha(x * y) \leqslant \max\{(\nu_A)^T_\alpha(x), (\nu_A)^T_\alpha(y)\}$ for all $x, y \in X$. Consequently $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy subalgebra of X. □

Theorem 3.5. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subalgebra of X and let $\alpha, \beta \in [0, \mathfrak{T}]$. If $\alpha \geqslant \beta$, then the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy S-extension of the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A.*

Proof. It is straightforward. □

For every intuitionistic fuzzy subalgebra $A = (\mu_A, \nu_A)$ of X and $\beta \in [0, \mathfrak{T}]$, the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A is an intuitionistic fuzzy subalgebra of X. If $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy S-extension of A^T_β, then there exists $\alpha \in [0, \mathfrak{T}]$ such that $\alpha \geqslant \beta$ and $B \geqslant A^T_\alpha$ that is $\mu_B(x) \geqslant (\mu_A)^T_\alpha$ and $\nu_B(x) \leqslant (\nu_A)^T_\alpha$ for all $x \in X$. Hence, we have the following theorem.

Theorem 3.6. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subalgebra of X and let $\beta \in [0, \mathfrak{T}]$. For every intuitionistic fuzzy S-extension $B = (\mu_B, \nu_B)$ of the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A, there exists $\alpha \in [0, \mathfrak{T}]$ such that $\alpha \geqslant \beta$ and B is an intuitionistic fuzzy S-extension of the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A.*

Let us illustrate Theorem 3.6 using the following example.

Example 4. Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	2	3	4
1	1	0	2	3	4
2	2	2	0	4	3
3	3	3	4	0	2
4	4	4	3	2	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.71	0.63	0.58	0.47	0.47
ν_A	0.26	0.35	0.41	0.52	0.52

Then, A is an intuitionistic fuzzy subalgebra of X and $\mathfrak{T} = 0.26$. If we take $\beta = 0.15$, then the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A is given by

X	0	1	2	3	4
$(\mu_A)^T_\beta$	0.86	0.78	0.73	0.62	0.62
$(\nu_A)^T_\beta$	0.11	0.20	0.26	0.37	0.37

Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_B	0.89	0.83	0.77	0.68	0.68
ν_B	0.07	0.15	0.19	0.28	0.28

Then, B is clearly an intuitionistic fuzzy subalgebra of X which is an intuitionistic fuzzy S-extension of the intuitionistic fuzzy β-translation A^T_β of A. But B is not an intuitionistic fuzzy α-translation of A for all $\alpha \in [0, \mathfrak{T}]$. If we take $\alpha = 0.17$ then $\alpha = 0.17 > 0.15 = \beta$ and the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is given as follows:

X	0	1	2	3	4
$(\mu_A)^T_\alpha$	0.88	0.80	0.75	0.64	0.64
$(\nu_A)^T_\alpha$	0.09	0.18	0.24	0.35	0.35

Note that $B(x) \geqslant A^T_\alpha(x)$ that is $\mu_B(x) \geqslant (\mu_A)^T_\alpha$ and $\nu_B(x) \leqslant (\nu_A)^T_\alpha$ for all $x \in X$, and hence B is an intuitionistic fuzzy S-extension of the intuitionistic fuzzy α-translation A^T_α of A.

Definition 4. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and $\gamma \in [0, 1]$. An object having the form $A^m_\gamma = ((\mu_A)^m_\gamma, (\nu_A)^m_\gamma)$ is called an intuitionistic fuzzy γ-multiplication of A if $(\mu_A)^m_\gamma(x) = \mu_A(x) \cdot \gamma$ and $(\nu_A)^m_\gamma(x) = \nu_A(x) \cdot \gamma$, for all $x \in X$.

For any intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X, an intuitionistic fuzzy 0-multiplication $A^m_0 = ((\mu_A)^m_0, (\nu_A)^m_0)$ of A is an intuitionistic fuzzy subalgebra of X.

Theorem 3.7. *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra of X, then the intuitionistic fuzzy γ-multiplication of A is an intuitionistic fuzzy subalgebra of X for all $\gamma \in [0, 1]$.*

Proof. It is straightforward. □

Theorem 3.8. *If $A = (\mu_A, \nu_A)$ is any intuitionistic fuzzy subset of X, then the following assertions are equivalent:*

(i) A is an intuitionistic fuzzy subalgebra of X.

(ii) For all $\gamma \in (0, 1]$, A^m_γ is an intuitionistic fuzzy subalgebra of X.

Proof. Necessity follows from Theorem 3.7. For sufficient part let $\gamma \in (0, 1]$ be such that $A^m_\gamma = ((\mu_A)^m_\gamma, (\nu_A)^m_\gamma)$ is an intuitionistic fuzzy subalgebra of X. Then, for all $x, y \in X$, we have

$$\begin{aligned} \mu_A(x * y).\gamma &= (\mu_A)^m_\gamma(x * y) \geqslant \min\{(\mu_A)^m_\gamma(x), (\mu_A)^m_\gamma(y)\} \\ &= \min\{\mu_A(x).\gamma, \mu_A(y).\gamma\} \\ &= \min\{\mu_A(x), \mu_A(y)\}.\gamma \\ \text{and} \qquad \nu_A(x * y).\gamma &= (\nu_A)^m_\gamma(x * y) \leqslant \max\{(\nu_A)^m_\gamma(x), (\nu_A)^m_\gamma(y)\} \\ &= \max\{\nu_A(x).\gamma, \nu_A(y).\gamma\} \\ &= \max\{\nu_A(x), \nu_A(y)\}.\gamma. \end{aligned}$$

Therefore, $\mu_A(x * y) \geqslant \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leqslant \max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$ since $\gamma \neq 0$. Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra of X. □

4 Translations of intuitionistic fuzzy ideals

Theorem 4.1. *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideals of X, then the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideals of X, for all $\alpha \in [0, \mathfrak{T}]$.*

Proof. Let $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideals of X and $\alpha \in [0, \mathfrak{T}]$. Then, $(\mu_A)^T_\alpha(0) = \mu_A(0) + \alpha \geqslant \mu_A(x) + \alpha = (\mu_A)^T_\alpha(x)$ and $(\nu_A)^T_\alpha(0) = \nu_A(0) - \alpha \leqslant \nu_A(x) - \alpha =$

$(\nu_A)^T_\alpha(x)$ for all $x \in X$. Now,

$$\begin{aligned} (\mu_A)^T_\alpha(x) &= \mu_A(x)+\alpha \geqslant \min\{\mu_A(x*y),\mu_A(y)\}+\alpha \\ &= \min\{\mu_A(x*y)+\alpha,\mu_A(y)+\alpha\} \\ &= \min\{(\mu_A)^T_\alpha(x*y),(\mu_A)^T_\alpha(y)\} \\ \text{and}\quad (\nu_A)^T_\alpha(x) &= \nu_A(x)-\alpha \leqslant \max\{\nu_A(x*y),\nu_A(y)\}-\alpha \\ &= \max\{\nu_A(x*y)-\alpha,\nu_A(y)-\alpha\} \\ &= \max\{(\nu_A)^T_\alpha(x*y),(\nu_A)^T_\alpha(y)\} \end{aligned}$$

for all $x, y \in X$. Hence, the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideals of X. □

Theorem 4.2. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideals of X for some $\alpha \in [0, \mathfrak{T}]$. Then, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideal of X.*

Proof. Assume that $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy ideals of X for some $\alpha \in [0, \mathfrak{T}]$. Let $x, y \in X$, we have

$$\begin{aligned} \mu_A(0)+\alpha = (\mu_A)^T_\alpha(0) &\geqslant (\mu_A)^T_\alpha(x) = \mu_A(x)+\alpha \\ \nu_A(0)-\alpha = (\nu_A)^T_\alpha(0) &\leqslant (\nu_A)^T_\alpha(x) = \nu_A(x)-\alpha \end{aligned}$$

which implies $\mu_A(0) \geqslant \mu_A(x)$ and $\nu_A(0) \leqslant \nu_A(x)$. Now we have

$$\begin{aligned} \mu_A(x)+\alpha &= (\mu_A)^T_\alpha(x) \geqslant \min\{(\mu_A)^T_\alpha(x*y),(\mu_A)^T_\alpha(y)\} \\ &= \min\{\mu_A(x*y)+\alpha,\mu_A(y)+\alpha\} \\ &= \min\{\mu_A(x*y),\mu_A(y)\}+\alpha \\ \text{and}\quad \nu_A(x)-\alpha &= (\nu_A)^T_\alpha(x) \leqslant \max\{(\nu_A)^T_\alpha(x*y),(\nu_A)^T_\alpha(y)\} \\ &= \max\{\nu_A(x*y)-\alpha,\nu_A(y)-\alpha\} \\ &= \max\{\nu_A(x*y),\nu_A(y)\}-\alpha \end{aligned}$$

which implies that $\mu_A(x) \geqslant \min\{\mu_A(x*y), \mu_A(y)\}$ and $\nu_A(x) \leqslant \min\{\nu_A(x*y), \nu_A(y)\}$ for all $x, y \in X$. Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideals of X. □

Lemma 4.1. [25] *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy ideal of X. If $x \leqslant y$, then $\mu_A(x) \geqslant \mu_A(y)$ and $\nu_A(x) \leqslant \nu_A(y)$ that is, μ_A is order-reversing and ν_A is order-prereversing.*

Theorem 4.3. *Let $\alpha \in [0, \mathfrak{T}]$ and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy ideal of X. If X is a BCK-algebra, then the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy subalgebra of X.*

Proof. Since $x * y \leqslant x$ for all $x, y \in X$, then by Lemma 4.1 we have $\mu_A(x * y) \geqslant \mu_A(x)$ and $\nu_A(x * y) \leqslant \nu_A(x)$. Now

$$\begin{aligned}(\mu_A)^T_\alpha(x * y) &= \mu_A(x * y) + \alpha \geqslant \mu_A(x) + \alpha\\ &= \min\{\mu_A(x * y), \mu_A(y)\} + \alpha\\ &\geqslant \min\{\mu_A(x), \mu_A(y)\} + \alpha\\ &= \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\}\\ &= \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}\\ \text{and} \quad (\nu_A)^T_\alpha(x * y) &= \nu_A(x * y) - \alpha \leqslant \nu_A(x) - \alpha\\ &= \max\{\nu_A(x * y), \nu_A(y)\} - \alpha\\ &\leqslant \max\{\nu_A(x), \nu_A(y)\} - \alpha\\ &= \max\{\nu_A(x) - \alpha, \nu_A(y) - \alpha\}\\ &= \max\{(\nu_A)^T_\alpha(x), (\nu_A)^T_\alpha(y)\}.\end{aligned}$$

Hence, $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy subalgebra of X. □

The following example shows that if X is a BCI-algebra, then Theorem 4.3 is not true.

Example 5. Consider the direct product $X := Y \times \mathbb{Z}$ where $(Y, *, 0)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers. Let $P = Y \times \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers.

Define an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X as follows:

$$\mu_A(x) = \begin{cases} 0.6 & \text{if} \quad x \in P \\ 0.2 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_\text{A}(\text{x}) = \begin{cases} 0.3 & \text{if} \quad x \in P \\ 0.5 & \text{otherwise} \end{cases}$$

Then, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideal of X and $\mathfrak{T} = 0.3$. For $\alpha \in [0, \mathfrak{T}]$, we have

$$\begin{aligned}(\mu_A)^T_\alpha((0,2) * (0,4)) &= (\mu_A)^T_\alpha((0,-2)) = \mu_A((0,-2)) + \alpha = 0.2 + \alpha\\ &< 0.6 + \alpha = \min\{\mu_A((0,2)), \mu_A((0,4))\} + \alpha\\ &= \min\{\mu_A((0,2)) + \alpha, \mu_A((0,4)) + \alpha\}\\ &= \min\{(\mu_A)^T_\alpha((0,2)), (\mu_A)^T_\alpha((0,4))\}\\ \text{and} \quad (\nu_A)^T_\alpha((0,1) * (0,2)) &= (\nu_A)^T_\alpha((0,-1)) = \nu_A((0,-1)) - \alpha = 0.5 - \alpha\\ &> 0.3 - \alpha = \max\{\nu_A((0,1)), \nu_A((0,2))\} - \alpha\\ &= \max\{\nu_A((0,1)) - \alpha, \nu_A((0,2)) - \alpha\}\\ &= \max\{(\nu_A)^T_\alpha((0,1)), (\nu_A)^T_\alpha((0,2))\}.\end{aligned}$$

Theorem 4.4. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideals of X for all $\alpha \in [0, \mathfrak{T}]$. If $(x * a) * b = 0$ for all $a, b, x \in X$, then $(\mu_A)^T_\alpha(x) \geqslant \min\{(\mu_A)^T_\alpha(a), (\mu_A)^T_\alpha(b)\}$ and $(\nu_A)^T_\alpha(x) \leqslant \max\{(\nu_A)^T_\alpha(a), (\nu_A)^T_\alpha(b)\}$.*

Proof. Let $a, b, x \in X$ be such that $(x * a) * b = 0$. Then,

$$\begin{aligned}
(\mu_A)^T_\alpha(x) &\geqslant \min\{(\mu_A)^T_\alpha(x * a), (\mu_A)^T_\alpha(a)\} \\
&\geqslant \min\{\min\{(\mu_A)^T_\alpha((x * a) * b), (\mu_A)^T_\alpha(b)\}, (\mu_A)^T_\alpha(a)\} \\
&= \min\{\min\{(\mu_A)^T_\alpha(0), (\mu_A)^T_\alpha(b)\}, (\mu_A)^T_\alpha(a)\} \\
&= \min\{(\mu_A)^T_\alpha(b), (\mu_A)^T_\alpha(a)\} \text{ since } (\mu_A)^T_\alpha(0) \geqslant (\mu_A)^T_\alpha(b) \\
&= \min\{(\mu_A)^T_\alpha(a), (\mu_A)^T_\alpha(b)\} \\
\text{and} \quad (\nu_A)^T_\alpha(x) &\leqslant \max\{(\nu_A)^T_\alpha(x * a), (\nu_A)^T_\alpha(a)\} \\
&\leqslant \max\{\max\{(\nu_A)^T_\alpha((x * a) * b), (\nu_A)^T_\alpha(b)\}, (\nu_A)^T_\alpha(a)\} \\
&= \max\{\max\{(\nu_A)^T_\alpha(0), (\nu_A)^T_\alpha(b)\}, (\nu_A)^T_\alpha(a)\} \\
&= \max\{(\nu_A)^T_\alpha(b), (\nu_A)^T_\alpha(a)\} \text{ since } (\nu_A)^T_\alpha(0) \leqslant (\nu_A)^T_\alpha(b) \\
&= \max\{(\nu_A)^T_\alpha(a), (\nu_A)^T_\alpha(b)\}.
\end{aligned}$$

□

Corollary 4.1. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideals of X for all $\alpha \in [0, \mathfrak{T}]$. If $(\cdots((x * a_1) * a_2) * \cdots) * a_n = 0$ for all $x, a_1, a_2, \cdots, a_n \in X$, then $(\mu_A)^T_\alpha(x) \geqslant \min\{(\mu_A)^T_\alpha(a_1), (\mu_A)^T_\alpha(a_2), \cdots, (\mu_A)^T_\alpha(a_n)\}$ and $(\nu_A)^T_\alpha(x) \leqslant \max\{(\nu_A)^T_\alpha(a_1), (\nu_A)^T_\alpha(a_2), \cdots, (\nu_A)^T_\alpha(a_n)\}$.*

Definition 5. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy subsets of X. Then, B is called an intuitionistic fuzzy ideal extension of A if the following assertions are valid:

(i) B is an intuitionistic fuzzy extension of A.

(ii) If A is an intuitionistic fuzzy ideal of X, then B is an intuitionistic fuzzy ideal of X.

Theorem 4.5. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and $\alpha \in [0, \mathfrak{T}]$. Then, the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideal extension of A.*

The converse of Theorem 4.5 is not true in general as seen in the following example.

Example 6. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.8	0.6	0.4	0.7	0.2
ν_A	0.2	0.4	0.5	0.3	0.7

Then, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideal of X. Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_B	0.82	0.63	0.47	0.75	0.24
ν_B	0.17	0.35	0.49	0.24	0.66

Then, B is an intuitionistic fuzzy ideal extension of A. But it is not the intuitionistic fuzzy α-translation $A_\alpha^T = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ of A for all $\alpha \in [0, \mathfrak{T}]$.

Clearly, the intersection of intuitionistic fuzzy ideal extensions of an intuitionistic fuzzy ideal A of X is an intuitionistic fuzzy ideal extension of A. But the union of intuitionistic fuzzy ideal extensions of an intuitionistic fuzzy ideal A of X is not an intuitionistic fuzzy ideal extension of A as seen in the following example.

Example 7. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.71	0.53	0.53	0.53	0.53
ν_A	0.25	0.45	0.45	0.45	0.45

Then, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideal of X. Let $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be intuitionistic fuzzy subsets of X defined by

X	0	1	2	3	4
μ_B	0.80	0.73	0.56	0.56	0.73
ν_B	0.15	0.26	0.41	0.41	0.26

and

X	0	1	2	3	4
μ_C	0.82	0.71	0.74	0.71	0.71
ν_C	0.13	0.24	0.26	0.24	0.24

respectively. Then, B and C are intuitionistic fuzzy ideal extensions of A. Obviously, the union $B \cup C$ is an intuitionistic fuzzy extension of A, but it is not an intuitionistic fuzzy ideal extension of A since $\mu_{B\cup C}(3) = 0.71 \ngeqslant 0.73 = \min\{0.74, 0.73\} = \min\{\mu_{B\cup C}(3 * 1), \mu_{B\cup C}(1)\}$ and $\nu_{B\cup C}(3) = 0.26 \nleqslant 0.24 = \max\{0.23, 0.24\} = \max\{\nu_{B\cup C}(3 * 1), \nu_{B\cup C}(1)\}$.

If A is an intuitionistic fuzzy ideal of X, then it is clear that $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are ideals of X for all $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $t \geqslant \alpha$. But, if we do not give a condition that A is an intuitionistic fuzzy ideal of X, then $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are not ideals of X as seen in the following example.

Example 8. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra in Example 7 and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.75	0.53	0.68	0.68	0.68
ν_A	0.25	0.45	0.30	0.30	0.30

Since $\mu_A(1) = 0.53 \ngeqslant 0.68 = \min\{\mu_A(1*3), \mu_A(3)\}$ and $\nu_A(1) = 0.45 \nleqslant 0.30 = \max\{\nu_A(1*3), \nu_A(3)\}$, therefore, $A = (\mu_A, \nu_A)$ is not an intuitionistic fuzzy ideal of X.

For $\alpha = 0.15$ and $t = 0.72$, we obtain $U_\alpha(\mu_A; t) = \{0, 2, 3, 4\}$ which is not a ideal of X since $4 * 3 = 1 \notin U_\alpha(\mu_A; t)$.

For $\alpha = 0.15$ and $s = 0.23$, we obtain $L_\alpha(\nu_A; s) = \{0, 2, 3, 4\}$ which is not a ideal of X since $4 * 3 = 1 \notin L_\alpha(\nu_A; s)$.

Theorem 4.6. *For $\alpha \in [0, \mathfrak{T}]$, let $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ be the intuitionistic fuzzy α-translation of $A = (\mu_A, \nu_A)$.Then, the following assertions are equivalent:*

(i) $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy ideal of X.

(ii) $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are ideal of X for $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geqslant \alpha$.

Proof. Suppose that $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy ideal of X. Then, $(\mu_A)^T_\alpha$ and $(\nu_A)^T_\alpha$ are fuzzy ideal of X. Let $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geqslant \alpha$.

Since $(\mu_A)^T_\alpha(0) \geqslant (\mu_A)^T_\alpha(x)$ for all $x \in X$, we have $\mu_A(0) + \alpha = (\mu_A)^T_\alpha(0) \geqslant (\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha \geqslant t$ for $x \in U_\alpha(\mu_A; t)$. Hence, $0 \in U_\alpha(\mu_A; t)$. Let $x, y \in X$ such that $x * y, y \in U_\alpha(\mu_A; t)$. Then, $\mu_A(x * y) \geqslant t - \alpha$ and $\mu_A(y) \geqslant t - \alpha$ i.e., $(\mu_A)^T_\alpha(x*y) = \mu_A(x*y) + \alpha \geqslant t$ and $(\mu_A)^T_\alpha(y) = \mu_A(y) + \alpha \geqslant t$. Since $(\mu_A)^T_\alpha$ is a fuzzy ideal of X, therefore, we have $\mu_A(x) + \alpha = (\mu_A)^T_\alpha(x) \geqslant \min\{(\mu_A)^T_\alpha(x*y), (\mu_A)^T_\alpha(y)\} \geqslant t$ that is, $\mu_A(x) \geqslant t - \alpha$ so that $x \in U_\alpha(\mu_A; t)$. Therefore, $U_\alpha(\mu_A; t)$ is a fuzzy ideal of X.

Again since $(\nu_A)^T_\alpha(0) \leqslant (\nu_A)^T_\alpha(x)$ for all $x \in X$, we have $\nu_A(0) - \alpha = (\nu_A)^T_\alpha(0) \leqslant (\nu_A)^T_\alpha(x) = \nu_A(x) - \alpha \leqslant s$ for $x \in L_\alpha(\nu_A; s)$. Hence, $0 \in L_\alpha(\nu_A; s)$. Let $x, y \in X$ such that $x * y, y \in L_\alpha(\nu_A; s)$. Then, $\nu_A(x * y) \leqslant s + \alpha$ and $\nu_A(y) \leqslant s + \alpha$ i.e., $(\nu_A)^T_\alpha(x * y) = \nu_A(x * y) - \alpha \leqslant s$ and $(\nu_A)^T_\alpha(y) = \nu_A(y) - \alpha \leqslant s$. Since $(\nu_A)^T_\alpha$ is a fuzzy ideal of X, therefore, we have $\nu_A(x) - \alpha = (\nu_A)^T_\alpha(x) \leqslant \max\{(\nu_A)^T_\alpha(x*y), (\nu_A)^T_\alpha(y)\} \leqslant s$ that is, $\nu_A(x) \leqslant s + \alpha$ so that $x \in L_\alpha(\nu_A; s)$. Hence, $L_\alpha(\nu_A; s)$ is a fuzzy ideal of X.

Conversely, suppose that $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are ideals of X for $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geqslant \alpha$. If there exists $p \in X$ such that $(\mu_A)^T_\alpha(0) < \lambda \leqslant (\mu_A)^T_\alpha(p)$, then $\mu_A(p) \geqslant \lambda - \alpha$ but $\mu_A(0) < \lambda - \alpha$. This shows that $p \in U_\alpha(\mu_A; t)$ and $0 \notin U_\alpha(\mu_A; t)$. This is a contradiction, and $(\mu_A)^T_\alpha(0) \geqslant (\mu_A)^T_\alpha(x)$ for all $x \in X$. Again if there exists $q \in X$ such that $(\nu_A)^T_\alpha(0) > \varphi \geqslant (\nu_A)^T_\alpha(q)$, then $\nu_A(q) \leqslant \varphi + \alpha$ but $\nu_A(0) > \varphi + \alpha$. This shows that $q \in L_\alpha(\nu_A; s)$ and $0 \notin L_\alpha(\nu_A; s)$. This is a contradiction, and $(\nu_A)^T_\alpha(0) \leqslant (\nu_A)^T_\alpha(x)$ for all $x \in X$.

Let $a, b \in X$ be such that $(\mu_A)^T_\alpha(a) < \beta \leqslant \min\{(\mu_A)^T_\alpha(a * b), (\mu_A)^T_\alpha(b)\}$, then $\mu_A(a * b) \geqslant \beta - \alpha$ and $\mu_A(b) \geqslant \beta - \alpha$ but $\mu_A(a) < \beta - \alpha$. This shows that $a * b \in U_\alpha(\mu_A; t)$ and $b \in U_\alpha(\mu_A; t)$ but $a \notin U_\alpha(\mu_A; t)$. This is a contradiction, and therefore $(\mu_A)^T_\alpha(x) \geqslant \min\{(\mu_A)^T_\alpha(x * y), (\mu_A)^T_\alpha(y)\}$ for all $x, y \in X$.

Again assume that there exist $c, d \in X$ such that $(\nu_A)^T_\alpha(c) > \delta \geqslant \max\{(\nu_A)^T_\alpha(c * d), (\nu_A)^T_\alpha(d)\}$. Then, $\nu_A(c * d) \leqslant \delta + \alpha$ and $\nu_A(d) \leqslant \delta + \alpha$ but $\nu_A(c) > \delta + \alpha$. Hence, $c * d \in L_\alpha(\nu_A; s)$ and $d \in L_\alpha(\nu_A; s)$ but $c \notin L_\alpha(\nu_A; s)$. This is impossible and therefore $(\nu_A)^T_\alpha(x) \leqslant \max\{(\nu_A)^T_\alpha(x * y), (\nu_A)^T_\alpha(y)\}$ for all $x, y \in X$. Consequently $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy ideal of X. □

Theorem 4.7. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy ideal of X and let $\alpha, \beta \in [0, \mathfrak{T}]$. If $\alpha \geqslant \beta$, then the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideal extension of the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A.*

Proof. It is straightforward. □

For every intuitionistic fuzzy ideal $A = (\mu_A, \nu_A)$ of X and $\beta \in [0, \mathfrak{T}]$, the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A is an intuitionistic fuzzy ideal of X. If $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy ideal extension of A^T_β, then there exists $\alpha \in [0, \mathfrak{T}]$ such that $\alpha \geqslant \beta$ and $B \geqslant A^T_\alpha$ that is $\mu_B(x) \geqslant (\mu_A)^T_\alpha$ and $\nu_B(x) \leqslant (\nu_A)^T_\alpha$ for all $x \in X$. Hence, we have the following theorem.

Theorem 4.8. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy ideal of X and let $\beta \in [0, \mathfrak{T}]$. For every intuitionistic fuzzy ideal extension $B = (\mu_B, \nu_B)$ of the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A, there exists $\alpha \in [0, \mathfrak{T}]$ such that $\alpha \geqslant \beta$ and B is an intuitionistic fuzzy ideal extension of the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A.*

Let us illustrate the Theorem 4.8 using the following example.

Example 9. Let $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	3	3
1	1	0	1	4	3
2	2	2	0	3	3
3	3	3	3	0	0
4	4	3	4	1	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.68	0.26	0.52	0.45	0.26
ν_A	0.31	0.72	0.45	0.53	0.72

Then, A is an intuitionistic fuzzy ideal of X and $\mathfrak{T} = 0.31$. If we take $\beta = 0.21$, then the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A is given by

X	0	1	2	3	4
$(\mu_A)^T_\beta$	0.89	0.47	0.73	0.66	0.47
$(\nu_A)^T_\beta$	0.10	0.51	0.24	0.32	0.51

Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_B	0.92	0.51	0.78	0.72	0.51
ν_B	0.07	0.47	0.19	0.26	0.47

Then, B is clearly an intuitionistic fuzzy ideal of X which is an intuitionistic fuzzy ideal extension of the intuitionistic fuzzy β-translation A^T_β of A. But B is not an intuitionistic fuzzy α-translation of A for all $\alpha \in [0, \mathfrak{T}]$. If we take $\alpha = 0.17$ then $\alpha = 0.23 > 0.21 = \beta$ and the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is given as follows:

X	0	1	2	3	4
$(\mu_A)^T_\alpha$	0.91	0.49	0.75	0.68	0.49
$(\nu_A)^T_\alpha$	0.08	0.49	0.22	0.30	0.49

Note that $B(x) \geqslant A^T_\alpha(x)$ that is $\mu_B(x) \geqslant (\mu_A)^T_\alpha$ and $\nu_B(x) \leqslant (\nu_A)^T_\alpha$ for all $x \in X$, and hence B is an intuitionistic fuzzy ideal extension of the intuitionistic fuzzy α-translation A^T_α of A.

For any intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X, an intuitionistic fuzzy 0-multiplication $A^m_0 = ((\mu_A)^m_0, (\nu_A)^m_0)$ of A is an intuitionistic fuzzy ideal of X.

Theorem 4.9. *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideal of X, then the intuitionistic fuzzy γ-multiplication of A is an intuitionistic fuzzy ideal of X for all $\gamma \in [0, 1]$.*

Proof. It is straightforward. □

Theorem 4.10. *If $A = (\mu_A, \nu_A)$ is any intuitionistic fuzzy subset of X, then the following assertions are equivalent:*

(i) A is an intuitionistic fuzzy ideal of X.

(ii) for all $\gamma \in (0, 1]$, A^m_γ is an intuitionistic fuzzy ideal of X.

Proof. Necessity follows from Theorem 4.9. For sufficient part let $\gamma \in (0, 1]$ be such that $A^m_\gamma = ((\mu_A)^m_\gamma, (\nu_A)^m_\gamma)$ is an intuitionistic fuzzy ideal of X. Then, for all $x, y \in X$, we have

$$\begin{aligned} \mu_A(x).\gamma &= (\mu_A)^m_\gamma(x) \geqslant \min\{(\mu_A)^m_\gamma(x * y), (\mu_A)^m_\gamma(y)\} \\ &= \min\{\mu_A(x * y).\gamma, \mu_A(y).\gamma\} \\ &= \min\{\mu_A(x * y), \mu_A(y)\}.\gamma \\ \text{and} \quad \nu_A(x).\gamma &= (\nu_A)^m_\gamma(x) \leqslant \max\{(\nu_A)^m_\gamma(x * y), (\nu_A)^m_\gamma(y)\} \\ &= \max\{\nu_A(x * y).\gamma, \nu_A(y).\gamma\} \\ &= \max\{\nu_A(x * y), \nu_A(y)\}.\gamma. \end{aligned}$$

Therefore, $\mu_A(x) \geqslant \min\{\mu_A(x * y), \mu_A(y)\}$ and $\nu_A(x) \leqslant \max\{\nu_A(x * y), \nu_A(y)\}$ for all $x, y \in X$ since $\gamma \neq 0$. Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideal of X. □

5 Conclusions and future work

In this paper, intuitionistic fuzzy translation of intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebra is introduced and investigated some of their useful properties. The relationships between intuitionistic fuzzy translations and intuitionistic fuzzy extensions of intuitionistic fuzzy subalgebras and ideals has been constructed.

It is our hope that this work would other foundations for further study of the theory of BCK/BCI-algebras. In our future study of fuzzy structure of BCK/BCI-algebra, may be the following topics should be considered: (i) to find intuitionistic fuzzy translation of intuitionistic fuzzy positive implicative, intuitionistic fuzzy implicative and intuitionistic fuzzy commutative ideals in BCK/BCI-algebra and their relationship, (ii) to find intuitionistic fuzzy translation of intuitionistic fuzzy H-ideals, a-ideals and p-ideals in BCK/BCI-algebra, (iii) to find the relationship between intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals, a-ideals and p-ideals in BCK/BCI-algebra.

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