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STURM COMPARISON THEOREMS FOR HALF-LINEAR EQUATIONS WITH A DAMPING TERM

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Abstract. In this paper, first we establish a Picone-type inequality for a pair of half-linear equations with a damping term. Next we prove some Sturm-type comparison theorems via the Picone-type inequality.

1 Introduction

Most of the classical results in oscillation theory are formulated for solutions of self adjoint Sturm-Liouville equations of the form

$$\ell_1 u := -(p_1(x)u')' + p_0(x)u = 0, \tag{1.1}$$

$$L_1 v := -(P_1(x)v')' + P_0(x)v = 0, \tag{1.2}$$

where p_0, p_1, P_0, P_1 are real valued continuous functions and p_1 and P_1 are positive on an appropriate interval. The starting point for this theory is the well known comparison theorem of C. Sturm [18] discovered in 1836.

Theorem 1.1. (Sturm Comparison Theorem). *Suppose that* $p_1(x) \equiv P_1(x)$, $P_0(x) \leqslant p_0(x)$, *and* $P_0(x) \neq p_0(x)$ *for* $x \in [x_1, x_2] \subset \mathbb{R}$. *If* x_1 *and* x_2 *are consecutive zeros of a nontrivial real solution* u *of* (1.1), *then every real solution* v *of* (1.2) *has a zero in* (x_1, x_2).

In 1909, Picone [17] modified Sturm's theorem as follows.

Theorem 1.2. (Sturm-Picone Theorem). *Suppose that* $0 < P_1(x) \leqslant p_1(x)$ *and* $P_0(x) \leqslant p_0(x)$ *for* $x \in [x_1, x_2]$. *If* x_1 *and* x_2 *are consecutive zeros of a nontrivial real solution* u *of* (1.1), *then every real solution* v *of* (1.2) *has one of the following properties:*
(i) v *has a zero in* (x_1, x_2)
or
(ii) v *is a constant multiple of* u.

Note that Theorem 1.2 is a special case of Leighton's theorem [14]. For a detailed study and earlier developments of this subject, we refer the reader to the books [13, 19].

The original proof by Picone was based on using the identity

$$\frac{d}{dx}\left[\frac{u}{v}\left(vp_1u' - uP_1v'\right)\right] = (p_0 - P_0)u^2 + (p_1 - P_1)u'^2 + P_1\left(u' - \frac{u}{v}v'\right)^2 + u\ell_1 u - \frac{u^2}{v}L_1 v \tag{1.3}$$

which holds for all real-valued functions u and v defined on $[x_1, x_2]$ such that u, v, p_1u' and P_1v' are differentiable on $[x_1, x_2]$ and $v(x) \neq 0$ for $x \in [x_1, x_2]$.

Identity (1.3) has proved to be a useful tool not only for comparing equations (1.1) and (1.2), but also for establishing Wirtinger type inequalities for second order linear ordinary differential equations and for finding lower bounds for the eigenvalues of the associated eigenvalue problems. It was generalized to high-order ordinary differential operators as well as to partial differential operators of elliptic type [4, 6, 10, 12, 13, 25, 26].

Comparison theorems analogues to the above ones were obtained for the differential equations

$$\widetilde{\ell}_1 u := -(p_1(x)u')' + q_0(x)u' + p_0(x)u = 0, \tag{1.4}$$

$$\widetilde{L}_1 v; = -(P_1(x)v')' + Q_0(x)v' + P_0(x)v = 0, \tag{1.5}$$

where the functions p_k, q_k, P_k, Q_k are of class C $(k = 0, 1)$ and p_1, P_1 are positive on an appropriate interval $I \subset \mathbb{R}$.

Starting with the Picone identity (1.3) and making use of (1.4) and (1.5), we get

$$\frac{d}{dx}\left[\frac{u}{v}\left(vp_1u' - uP_1v'\right)\right] = (p_0 - P_0)u^2 + (q_0 - Q_0)uu' + Q_0u(u' - \frac{u}{v}v') + (p_1 - P_1)u'^2 + P_1\left(u' - \frac{u}{v}v'\right)^2 + u\widetilde{\ell}_1 u - \frac{u^2}{v}\widetilde{L}_1 v \tag{1.6}$$

whenever $v(x) \neq 0$ on I. In the case $p_1(x) \geqslant P_1(x)$ on I by using "completing the square" method in the right hand side of (1.6) we have the following inequality

$$\frac{d}{dx}\left[\frac{u}{v}\left(vp_1u' - uP_1v'\right)\right] \geqslant \left(p_0 - P_0 - \frac{(q_0 - Q_0)^2}{4(p_1 - P_1)} - \frac{Q_0^2}{4P_1}\right)u^2 + u\widetilde{\ell}_1 u - \frac{u^2}{v}\widetilde{L}_1 v.$$

This last inequality readily yields the following generalization of Theorem 1.2.

Theorem 1.3. [13] *If x_1, x_2 are consecutive zeros of a nontrivial solution u of* (1.4) *and if*

(i) $0 < P_1(x) \leqslant p_1(x)$,

(ii) $0 < P_1(x) < p_1(x)$ *whenever* $q_0(x) \neq Q_0(x)$,

(iii) $P_0(x) + \dfrac{(q_0(x) - Q_0(x))^2}{4(P_1(x) - p_1(x))} + \dfrac{Q_0^2(x)}{4P_1(x)} \leqslant p_0(x)$

for $x \in [x_1, x_2]$, *then every solution* v *of* (1.5) *has a zero in* (x_1, x_2).

It is of interest to note a variation of (1.6) which is based on a device due to Picard [13] and leads to different versions of Theorem 1.3.

For any differentiable function f, we have

$$\frac{d}{dx}(u^2 f) = 2uu'f + u^2 f'.$$

Adding this to (1.6) yields

$$\frac{d}{dx}\left[\frac{u}{v}\left(vp_1u' - uP_1v'\right) + u^2 f\right] = \left(p_0 - P_0 + f'\right)u^2 + \left(q_0 - Q_0 + 2f\right)uu'$$
$$+ Q_0 u\Big(u' - \frac{u}{v}v'\Big) + \Big(p_1 - P_1\Big)u'^2 + P_1\Big(u' - \frac{u}{v}v'\Big)^2 + u\widetilde{\ell}_1 u - \frac{u^2}{v}\widetilde{L}_1 v,$$

where one can seek to complete the square for various choice of f. Assuming q_0 and Q_0 are of class C^1, the choice $f = \frac{Q_0 - q_0}{2}$ yields the inequality

$$\frac{d}{dx}\left[\frac{u}{v}\left(vp_1u' - uP_1v'\right) + u^2\frac{Q_0 - q_0}{2}\right] \geqslant \left[p_0 - P_0 - \frac{q_0' - Q_0'}{2} - \frac{Q_0^2}{4P_1}\right]u^2 + \Big(p_1 - P_1\Big)u'^2$$
$$+ u\widetilde{\ell}_1 u - \frac{u^2}{v}\widetilde{L}_1 v.$$

and the following result.

Theorem 1.4. [13] *Assume that* q_0 *and* Q_0 *are of class* $C^1[x_1, x_2]$. *If* x_1, x_2 *are consecutive zeros of a nontrivial solution* u *of* (1.4) *and if*

(i) $0 < P_1(x) \leqslant p_1(x)$,

(ii) $P_0(x) + \frac{q_0'(x) - Q_0'(x)}{2} + \frac{Q_0^2(x)}{4P_1(x)} \leqslant p_0(x)$

for $x \in [x_1, x_2]$, *then every solution* v *of* (1.5) *has a zero in* (x_1, x_2).

Note that other special choices of function f yield further variations along these lines.

The Sturm-Picone theorem is extended in several directions, see, S. Ahmad and A.C. Lazer [1] and S. Ahmad [2] for linear systems, E. Muller-Pfeiffer [16] for non-selfadjoint differential equations, J. Tyagi and V. Raghavenda [23] for implicit differential equations, W. Allegretto [5] for degenerate elliptic equations, C. Zhang and S. Sun [27] for linear equations on time scales, J. Jaroš, and T. Kusano [10] for half linear equations, [21, 24] for nonlinear equations. There is also a good amount of interest in the qualitative theory of partial differential equations to determine whether the given equation is oscillatory or not. In this direction, the Sturm-Picone theorem plays an important role [3, 5, 6, 8, 11, 12, 19, 20, 22, 25, 26].

In 1999, J. Jaros and T. Kusano [10] generalized Picone's identity (1.3) to the class of nonlinear second order differential equations of the form,

$$\ell_\alpha u := \Big(p_1(x)\varphi(u')\Big)' + p_0(x)\varphi(u) = 0, \tag{1.7}$$

$$L_\alpha v := \Big(P_1(x)\varphi(v')\Big)' + P_0(x)\varphi(v) = 0, \tag{1.8}$$

where $\varphi(s) := |s|^{\alpha-1}s$, $\alpha > 0$, p_1 and the functions p_0, P_1, P_0 are defined as before. The above equations are also called half-linear or sometimes homogeneous of degree α. They established a suitable Picone-type identity as follows

$$\frac{d}{dt}\left\{\frac{u}{\varphi(v)}\left(\varphi(v)p_1\varphi(u') - \varphi(u)P_1\varphi(v')\right)\right\} = (p_1 - P_1)\,|u'|^{\alpha+1} + (P_0 - p_0)\,|u|^{\alpha+1}$$
$$+ P_1\left[|u'|^{\alpha+1} + \alpha\left|\frac{uv'}{v}\right|^{\alpha+1} - (\alpha+1)u'\varphi\Big(\frac{uv'}{v}\Big)\right] + u\ell_\alpha u - u\frac{\varphi(u)}{\varphi(v)}L_\alpha v. \tag{1.9}$$

By using the above identity, they obtained the following comparison result which is an extension of Theorem 1.2 to the class of half linear equations (1.7) and (1.8).

Theorem 1.5. *Suppose that* $0 < P_1(x) \leqslant p_1(x)$ *and* $p_0(x) \leqslant P_0(x)$ *for* $x \in [x_1, x_2]$. *If* x_1, x_2 *are consecutive zeros of a nontrivial real solution* u *of* (1.7), *then every solution* v *of* (1.8) *has one of the following properties:*
(i) v *has a zero in* (x_1, x_2)
or
(ii) v *is a constant multiple of* u.

A natural question now arises: Is it possible to generalize the above Sturm comparison result to the half-linear equations with a damping term? The objective of this paper is to give an affirmative answer. In this paper we consider a class of half-linear equations with a damping term of the form

$$\ell u := \Big(p_1(x)\varphi(u')\Big)' + q_0(x)\varphi(u') + p_0(x)\varphi(u) = 0, \tag{1.10}$$

$$Lv := \Big(P_1(x)\varphi(v')\Big)' + Q_0(x)\varphi(v') + P_0(x)\varphi(v) = 0, \tag{1.11}$$

on a bounded open interval $x_1 < x < x_2$ where p_1, q_0, p_0, P_1, Q_0 and P_0 are real valued continuous functions and $P_1(x) > 0$, $p_1(x) > 0$ on $[x_1, x_2]$. These equations define the differential operators ℓ and L. The domain D_ℓ of ℓ is defined as the set of all real valued functions $u \in C^1[x_1, x_2]$ such that $p_1\varphi(u') \in C^1(x_1, x_2)$ and D_L is the analogue of D_ℓ with P_1 replacing p_1. A solution of (1.10) is a function $u \in D_\ell$ satisfying $\ell u = 0$ at every point in (x_1, x_2). We know that (1.10) (or (1.11)) possesses such a solution [7].

Equation (1.10) can be reduced to form (1.7)

$$\Big(\mu(x)\varphi(u')\Big)' + \mu(x)p_0(x)\varphi(u) = 0,$$

where $\mu(x) = e^{\int_{x_1}^{x} \frac{p'(t)+q_0(t)}{p_1(t)}dt}$, if the function p_1 is continuously differentiable on $[x_1, x_2]$ but in the case non-differentiability of p_1 this is impossible. For this reason there is an advantage in obtaining direct comparison theorems for solutions of (1.10) and (1.11). Besides the obvious practical advantage of eliminating the need for the above integrating factor, there is incentive in developing methods which can be generalized to n-dimensions (i.e. partial differential equations).

2 Picone-type inequalities and comparison theorems

We will need the following lemmas, in order to prove our results:

Lemma 2.1. [9, 11] *If $a, b \in \mathbb{R}$ and $\alpha > 0$, then*

$$a\varphi(a) + \alpha b\varphi(b) - (\alpha+1)a\varphi(b) \geqslant 0, \tag{2.1}$$

where equality holds if and only if $a = b$.

The following lemma is of basic importance for our later considerations:

Lemma 2.2. (Picone-type Inequality). *Assume that $I \subset \mathbb{R}$ is closed interval, q_0 and Q_0 are of class $C^1(I)$, and*

$$(\alpha+1)P_1(x) > |Q_0(x)|$$

for $x \in I$. If $u \in D_\ell(I_0)$ and $v \in D_L(I_0)$ for some non-degenerate subinterval $I_0 \subset I$ and $v(x) \neq 0$ for $x \in I_0$, then on I_0

$$\begin{aligned}
&\frac{d}{dx}\left\{\frac{u}{\varphi(v)}\left(\varphi(v)p_1(x)\varphi(u') - \varphi(u)P_1(x)\varphi(v')\right) - \frac{Q_0(x)-q_0(x)}{\alpha+1}|u|^{\alpha+1}\right\}\\
&\quad\geqslant \left(p_1(x) - \frac{P_1^{\alpha+1}(x)}{\left(P_1(x) - \frac{|Q_0(x)|}{\alpha+1}\right)^{\alpha}} - \frac{\alpha}{\alpha+1}|Q_0(x)| - |Q_0(x)-q_0(x)|\right)|u'|^{\alpha+1}\\
&\quad+\left(P_0(x) - p_0(x) - \frac{Q_0'(x)-q_0'(x)}{\alpha+1} - \frac{2}{\alpha+1}|Q_0(x)| - |Q_0(x)-q_0(x)|\right)|u|^{\alpha+1}\\
&\quad+\left(P_1(x) - \frac{|Q_0(x)|}{\alpha+1}\right)F\left(\frac{P_1(x)u'}{P_1(x) - \frac{|Q_0(x)|}{\alpha+1}},\ \frac{uv'}{v}\right) + u\ell u - u\frac{\varphi(u)}{\varphi(v)}Lv,
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
F\left(\frac{P_1(x)u'}{P_1(x) - \frac{|Q_0(x)|}{\alpha+1}},\ \frac{uv'}{v}\right) &= \left|\frac{P_1(x)u'}{P_1(x) - \frac{|Q_0(x)|}{\alpha+1}}\right|^{\alpha+1}\\
&+\alpha\left|\frac{uv'}{v}\right|^{\alpha+1} - (\alpha+1)\frac{P_1(x)u'}{P_1(x) - \frac{|Q_0(x)|}{\alpha+1}}\varphi\left(\frac{uv'}{v}\right).
\end{aligned}$$

Proof. Let f be any differentiable function. We easily see that

$$\frac{d}{dx}\left\{\frac{u}{\varphi(v)}\left[\varphi(v)p_1(x)\varphi(u') - \varphi(u)P_1(x)\varphi(v')\right] - f(x)|u|^{\alpha+1}\right\}$$
$$= \Big(P_0(x)-p_0(x)\Big)|u|^{\alpha+1}+p_1(x)|u'|^{\alpha+1}-q_0(x)\varphi(u')u+Q_0(x)u\varphi\Big(\frac{uv'}{v}\Big)+\alpha P_1(x)\left|\frac{uv'}{v}\right|^{\alpha+1}$$
$$-(\alpha+1)P_1(x)u'\varphi\Big(\frac{uv'}{v}\Big) - f'(x)|u|^{\alpha+1} - (\alpha+1)f(x)\varphi(u)u' + u\ell u - u\frac{\varphi(u)}{\varphi(v)}Lv.$$

Next we rewrite the right hand side as follows

$$\frac{d}{dx}\left\{\frac{u}{\varphi(v)}\left[\varphi(v)p_1(x)\varphi(u') - \varphi(u)P_1(x)\varphi(v')\right] - f(x)|u|^{\alpha+1}\right\}$$
$$= \Big(P_0(x) - p_0(x) - f'(x)\Big)|u|^{\alpha+1} + \Big(Q_0(x) - q_0(x) - (\alpha+1)f(x)\Big)u\varphi(u')$$
$$- Q_0(x)u\Big(\varphi(u') - \varphi\Big(\frac{uv'}{v}\Big)\Big) + p_1(x)|u'|^{\alpha+1} + \alpha P_1(x)\left|\frac{uv'}{v}\right|^{\alpha+1}$$
$$- \Big(\alpha+1\Big)P_1(x)u'\varphi\Big(\frac{uv'}{v}\Big) - (\alpha+1)f(x)(u'\varphi(u) - u\varphi(u')) + u\ell u - u\frac{\varphi(u)}{\varphi(v)}Lv.$$

Using Young's inequality, Lemma 2.1 and straightforward calculations, this equality implies the following inequality

$$\frac{d}{dx}\left\{\frac{u}{\varphi(v)}\left[\varphi(v)p_1(x)\varphi(u') - \varphi(u)P_1(x)\varphi(v')\right] - f(x)|u|^{\alpha+1}\right\}$$
$$\geqslant \Big(P_1(x) - \frac{|Q_0(x)|}{\alpha+1}\Big)F\Big(\frac{P_1(x)u'}{P_1(x) - \frac{|Q_0(x)|}{\alpha+1}},\ \ \frac{uv'}{v}\Big)$$
$$+ \Big(p_1(x) - \frac{P_1^{\alpha+1}(x)}{(P_1(x) - \frac{|Q_0(x)|}{\alpha+1})^{\alpha}} - \frac{\alpha}{\alpha+1}|Q_0(x)| - |f(x)|(\alpha+1)\Big)|u'|^{\alpha+1}$$
$$+ \Big(P_0(x) - p_0(x) - f'(x) - \frac{2}{\alpha+1}|Q_0(x)| - (\alpha+1)|f(x)|\Big)|u|^{\alpha+1}$$
$$+ \Big(Q_0(x) - q_0(x) - (\alpha+1)f(x)\Big)u\varphi(u') + u\ell u - u\frac{\varphi(u)}{\varphi(v)}Lv.$$

Choosing $f(x) = \dfrac{Q_0(x) - q_0(x)}{\alpha+1}$, we get the desired inequality (2.2).

□

Our first result is based on inequality (2.2). Let U be the set of all real-valued functions $\eta \in C^1[x_1, x_2]$, where x_1 and x_2 are consecutive zeros of η. Define the functional $J : U \to R$ by

$$J(\eta) = \int_{x_1}^{x_2}\left\{\left(\frac{P_1^{\alpha+1}(x)}{\Big(P_1(x) - \frac{|Q_0(x)|}{\alpha+1}\Big)^{\alpha}} + \frac{2\alpha}{\alpha+1}|Q_0(x)|\right)|\eta'|^{\alpha+1}\right.$$
$$\left.-\Big(P_0(x) - \frac{3}{\alpha+1}|Q_0(x)|\Big)|\eta|^{\alpha+1}\right\}dx,$$

where $(\alpha+1)P_1(x) > |Q_0(x)|$.

Theorem 2.1. (Wirtinger-type Inequality). *Assume that* q_0 *and* Q_0 *are of class* $C^1[x_1, x_2]$ *and* $(\alpha+1)P_1(x) > |Q_0(x)|$ *and on* $[x_1, x_2]$. *If there exists a solution* v *of* $Lv = 0$ *such that* $v(x) \neq 0$ *on* (x_1, x_2), *then for all* $\eta \in U$,

$$J(\eta) > 0 \tag{2.3}$$

except for the case $v(x_1) = v(x_2) = 0$, $\eta(x) = Kv(x)e^{-\int_{x_1}^{x} \frac{|Q_0(s)|}{(\alpha+1)P_1(s)} \frac{dv(s)}{v(s)}}$, *where* K *is a nonzero constant.*

Proof. By Picone's inequality (2.2) applied to the case $p_1(x) \equiv P_1(x)$, $q_0(x) \equiv Q_0(x)$, $p_0(x) \equiv P_0(x)$, and $u(x) \equiv \eta(x)$ we obtain

$$\begin{aligned}
&\frac{d}{dx}\left\{\eta P_1(x)\varphi(\eta') - \eta\frac{\varphi(\eta)}{\varphi(v)}P_1(x)\varphi(v')\right\}\\
&\geqslant \Big(P_1(x) - \frac{P_1^{\alpha+1}(x)}{(P_1(x) - \frac{|Q_0(x)|}{\alpha+1})^{\alpha}} - \frac{\alpha}{\alpha+1}|Q_0(x)|\Big)|\eta'|^{\alpha+1}\Big) - \frac{2}{\alpha+1}|Q_0(x)||\eta|^{\alpha+1}\\
&+ \Big(P_1(x) - \frac{|Q_0(x)|}{\alpha+1}\Big)F\Big(\frac{P_1(x)\eta'}{P_1(x) - \frac{|Q_0(x)|}{\alpha+1}},\ \frac{\eta v'}{v}\Big) + \eta L\eta - \eta\frac{\varphi(\eta)}{\varphi(v)}Lv.
\end{aligned} \tag{2.4}$$

Now, using the fact that v is a solution of $Lv = 0$ and

$$\begin{aligned}
\frac{d}{dx}(\eta P_1\varphi(\eta')) &= P_1|\eta'|^{\alpha+1} - Q_0\eta\varphi(\eta') - P_0|\eta|^{\alpha+1} + \eta L\eta\\
&\leqslant P_1|\eta'|^{\alpha+1} + |Q_0|\big(\frac{|\eta|^{\alpha+1}}{\alpha+1} + \alpha\frac{|\eta'|^{\alpha+1}}{\alpha+1}\big) - P_0|\eta|^{\alpha+1} + \eta L\eta,
\end{aligned}$$

we get

$$\begin{aligned}
&-\frac{d}{dx}\big(\eta\frac{\varphi(\eta)}{\varphi(v)P_1\varphi(v')}\big) + \Big(\frac{P_1^{\alpha+1}(x)}{(P_1(x) - \frac{|Q_0(x)|}{\alpha+1})^{\alpha}} + \frac{2\alpha}{\alpha+1}|Q_0(x)|\Big)|\eta'|^{\alpha+1} - \Big(P_0(x)\\
&-\frac{3}{\alpha+1}|Q_0(x)|\Big)|\eta|^{\alpha+1} \geqslant \Big(P_1(x) - \frac{|Q_0(x)|}{\alpha+1}\Big)F\Big(\frac{P_1(x)\eta'(x)}{P_1(x) - \frac{|Q_0(x)|}{\alpha+1}}, \frac{\eta(x)v'(x)}{v}\Big).
\end{aligned} \tag{2.5}$$

If both $v(x_1) \neq 0$ and $v(x_2) \neq 0$, then, integrating (2.5) from x_1 to x_2 and using Lemma 2.2, we obtain the desired inequality (2.3).

If $v(x_1) = 0$, then due to the fact that zeros of nontrivial solutions of second-order half-linear equations are simple (see, for example [[15], Lemma 2.3]) $v'(x_1)$ must be a nonzero finite value. Since obviously, $\lim_{x\to x_1^+} P_1(x)\eta(x)\varphi(v'(x)) = 0$ and also $\lim_{x\to x_1^+} \varphi\Big(\frac{\eta(x)}{v(x)}\Big) = \varphi\Big(\lim_{x\to x_1^+} \frac{\eta'(x)}{v'(x)}\Big) < \infty$ by the L'Hospital rule, we have

$$\lim_{x\to x_1^+} P_1(x)\eta(x)\frac{\varphi(v'(x))\varphi(\eta(x))}{\varphi(v(x))} = 0.$$

Similarly

$$\lim_{x\to x_2^-} P_1(x)\eta(x)\frac{\varphi(v'(x))\varphi(\eta(x))}{\varphi(v(x))}=0,$$

if $v(x_2)=0$.

Thus, integrating (2.5) over the interval $[x_1+\varepsilon, x_2-\varepsilon]$, letting $\varepsilon\to 0^+$ and using Lemma 2.1, we again obtain (2.3).

In the case $F\Big(\frac{P_1(x)\eta'(x)}{P_1(x)-\frac{|Q_0(x)|}{\alpha+1}}, \frac{\eta(x)v'(x)}{v}\Big)=0$ or equivalently $\frac{P_1\eta'}{P_1-\frac{|Q_0|}{\alpha+1}}\equiv\frac{\eta v'}{v}$ or equivalently $\eta(x)=Kv(x)e^{-\int_{x_1}^{x}\frac{|Q_0(s)|}{(\alpha+1)P_1(s)}\frac{dv(s)}{v(s)}}$, where K is a constant, we have $J(\eta)=0$, but $\eta\in U$ if and only if $v(x_1)=v(x_2)=0$ and $K\neq 0$. □

By Theorem 2.1 we immediately have the following Corollary which is a straightforward extension of Leighton's variational type lemma for the damped half-linear equations.

Corollary 2.1. *Assume that q_0 and Q_0 are of class $C^1[x_1,x_2]$ and $(\alpha+1)P_1(x)>|Q_0(x)|$ on $[x_1,x_2]$. If there exists an $\eta\in U$ such that*

$$J(\eta)\leqslant 0,$$

then every solution v of $Lv=0$ has a zero in (x_1,x_2) except for the case $v(x_1)=v(x_2)=0$ and $\eta(x)=Kv(x)e^{-\int_{x_1}^{x}\frac{|Q_0(s)|}{(\alpha+1)P_1(s)}\frac{dv(s)}{v(s)}}$, where K is a nonzero constant.

Now, along with the equation

$$Lv=0,$$

consider also the equation

$$\ell u=0$$

and define, for $\eta\in U$,

$$V(\eta):=\int_{t_1}^{t_2}\left(p_1(x)-\frac{P_1^{\alpha+1}(x)}{\Big(P_1(x)-\frac{|Q_0(x)|}{\alpha+1}\Big)^{\alpha}}-\frac{\alpha}{\alpha+1}|Q_0(x)|-|Q_0(x)-q_0(x)|\right)|\eta'|^{\alpha+1}$$
$$+\left(P_0(x)-p_0(x)-\frac{Q_0'(x)-q_0'(x)}{\alpha+1}-\frac{2}{\alpha+1}|Q_0(x)|-|Q_0(x)-q_0(x)|\right)|\eta|^{\alpha+1}. \quad (2.6)$$

The following comparison theorem is an extension of Leighton's Theorem for (1.10) and (1.11).

Theorem 2.2. (Leighton-type Comparison Theorem). *Assume that q_0 and Q_0 are of class $C^1[x_1,x_2]$ and $(\alpha+1)P_1(x)>|Q_0(x)|$ on $[x_1,x_2]$. If there exists an $u\in U$ such that $\ell u=0$ and*

$$V(u)\geqslant 0, \quad (2.7)$$

then every solution v of $Lv=0$ has a zero in (x_1,x_2) except for the case $v(x_1)=v(x_2)=0$ and $u(x)=Kv(x)e^{-\int_{x_1}^{x}\frac{|Q_0(s)|}{(\alpha+1)P_1(s)}\frac{dv(s)}{v(s)}}$, where K is a nonzero constant.

Proof. Assume to the contrary that $Lv = 0$ has a solution which is nonzero on (x_1, x_2). As in the proof of Theorem 2.1 we can show that the function

$$\frac{u}{\varphi(v)}\left[\varphi(v)p_1(x)\varphi(u') - \varphi(u)P_1(x)\varphi(v')\right] - \left(\frac{Q_0(x) - q_0(x)}{\alpha+1}\right)|u|^{\alpha+1}$$

tends to zero as $x \to x_1^+$ or $x \to x_2^-$, regardless of $v(x_1) = 0$ or $v(x_1) \neq 0$ ($v(x_2) = 0$ or $v(x_2) \neq 0$). Thus, integrating (2.2) from $x_1 + \varepsilon$ to $x_2 - \varepsilon$, letting $\varepsilon \to 0^+$ and using Lemma 2.1, we obtain

$$V(u) < 0$$

which contradicts (2.7) except for the case $V(u) = 0$ if $p_0 \equiv P_0$, $q_0 \equiv Q_0$, $p_1 \equiv P_1$ and $P_1u' = \left(P_1 - \frac{|Q_0|}{\alpha+1}\right)\frac{uv'}{v}$, or equivalently $u(x) = Kv(x)e^{-\int_{x_1}^{x}\frac{|Q_0(s)|}{(\alpha+1)P_1(s)}\frac{dv(s)}{v(s)}}$, where K is a nonzero constant and $v(x_1) = v(x_2) = 0$. □

Corollary 2.2. (Sturm-Picone Comparison Theorem). *Assume that q_0 and Q_0 are of class $C^1[x_1, x_2]$ and $(\alpha+1)P_1(x) > |Q_0(x)|$ on $[x_1, x_2]$. If*

(i) $p_1(x) \geqslant \frac{P_1^{\alpha+1}(x)}{\left(P_1(x) - \frac{|Q_0(x)|}{\alpha+1}\right)^{\alpha}} + \frac{\alpha}{\alpha+1}|Q_0(x)| + |Q_0(x) - q_0(x)|,$

(ii) $P_0(x) \geqslant p_0(x) + \frac{Q_0'(x) - q_0'(x)}{\alpha+1} + \frac{2}{\alpha+1}|Q_0(x)| + |Q_0(x) - q_0(x)|$

on $[x_1, x_2]$ and there exists an $u \in U$ such that $\ell u = 0$, then any solution v of $Lv = 0$ either has a zero in (x_1, x_2) or $v(x) = Ku(x)e^{\int_{x_1}^{x}\frac{P_1(s)}{P_1(s) - \frac{|Q_0(s)|}{\alpha+1}}\frac{du(s)}{u(s)}}$, where K is a nonzero constant.

Remark 1. If in addition to *(ii)* and *(iii)* in Corollary 2.2 we suppose that on any non-degenerate subinterval I_0 of $[x_1, x_2]$ neither *(ii)* nor *(iii)* becomes an identity, then the later possibility is excluded and any solution v of of $Lv = 0$ must have a zero in (x_1, x_2).

Remark 2. In the special cases, the above results give the well-known results. Indeed, when we take $\alpha = 1$, $Q_0(x) \equiv q_0(x) \equiv 0$ Corollary 2.2 reduces to Theorem 1.2. When we take $Q_0(x) \equiv q_0(x) \equiv 0$ Corollary 2.2 gives Theorem 1.5. But in the case linear equation with a damping term, that is $\alpha = 1$, we cannot obtain Theorem 1.3 from Corollary 2.2. Hence our results are a partial extension of the results that are known in the literature. The cases in which $\alpha = 1$ or the fundamental assumption $(\alpha+1)P_1(x) > |Q_0(x)|$ is not satisfied are left as open problems for the researchers.

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References

[1] S. Ahmad, A.C. Lazer, *A new generalization of the Sturm comparison theorem to self adjoint systems,* Proc. Amer. Math. Soc. 68 (1978), 185–188.

[2] S. Ahmad, *On Sturmian theory for second order systems,* Proc. Amer. Math. Soc. 87 (1983), 661–665.

[3] W. Allegretto, *Sturm type theorems for solutions of elliptic nonlinear problems,* Nonlinear Differ. Equ. Appl. 7 (2000), 309–321.

[4] W. Allegretto, Y.X. Huang, *A Picones identitiy for the p-Laplacian and applications,* Nonlinear Analysis T. M. A. 32 (1998), 819–830.

[5] W. Allegretto, *Sturm theorems for degenerate elliptic equations,* Proc. Amer. Math. Soc. 129 (2001), 3031–3035.

[6] O. Doslý, *The Picone identity for a class of partial differential equations,* Mathematica Bohemica. 127 (2002), 581–589.

[7] O. Doslý, P. Řehák, *Half-linear differential equations,* North-Holland Mathematics Studies, 202, Elsevier Science B. V., Amsterdam, 2005.

[8] S. Fisnarová, R. Marik, *Generalized Picone and Riccati inequalites for half-linear differential operators with arbitrary elliptic matrices,* Electronic J. Diff. Equations. (2010), no. 111, 1–13.

[9] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities,* Cambridge Univ. Press, 1988.

[10] J. Jaroš, T. Kusano, *A Picone type identity for second order half-linear differential equations,* Acta Math. Univ. Comenianae. Vol. LXVIII, (1999), no. 1, 137–151.

[11] J. Jaroš, T. Kusano, N. Yoshida, *Forced superlinear oscillations via Picone's identity,* Acta Math. Univ. Comenianae. LXIX (2000), 107–113.

[12] J. Jaroš, T. Kusano, N. Yoshida, *Picone type inequalies for half linear elliptic equations and their applications,* Adv. Math. Sci. Appl. 12 (2002), no. 2, 709–724.

[13] K. Kreith, *Oscillation theory,* Lecture Notes in Mathematics, no. 324, Springer-Verlag, 1973.

[14] W. Leighton, *Comparison theorems for linear differential equations of second order,* Proc. Amer. Math. Soc. 13 (1962), 603–610.

[15] H.J. Li, C.C. Yeh, *Sturmian comparison theorem for half-linear second order differential equations,* Proc. Roy. Soc. Edinburgh 125A (1995), 1193–1204.

[16] E. Muller-Pfeiffer, *Sturm comparison theorems for non-selfadjoint differential equations on non-compact intervals,* Math. Nachr. 159 (1992), 291–298.

[17] M. Picone, *Sui valori eccezionali di un parametro da cui dipende un equazione differenziale lineare ordinaria del second ordine,* Ann. Scuola Norm. Pisa 11 (1909), 1–141.

[18] C. Sturm, *Sur les equations differentielles linearies du second ordere,* J. Math. Pures. Appl., 1 (1836), 106–186.

[19] C.A. Swanson, *Comparison and oscillation theory of linear differential equations,* Academic Press, New York, 1968.

[20] T. Tadie, *Oscillation criteria for semilinear elliptic equations with a damping term in R^n,* Electronic J. Diff. Equations (2010), no. 51, 1–5.

[21] A. Tiryaki, *Sturm-Picone type theorems for second-order nonlinear differential equations,* Electronic J. Diff. Equations 2014 (2014), no. 146, 1–11.

[22] A. Tiryaki, *Sturm-Picone type theorems for second-order nonlinear elliptic differential equations,* Electronic J. Diff. Equations 2014 (2014), no. 214, 1–10.

[23] J. Tyagi, V. Raghavendra, *A note on generalization of Sturm's comparison theorem,* Nonlinear Dyn. Syst. Theory 8 (2008), no. 2, 213–216.

[24] J. Tyagi, *Generalizations of Sturm-Picone theorem for second order nonlinear differential equations,* Taiwanese Journal of Mathematics 17 (2013), no. 1, 361–378.

[25] N. Yoshida, *Oscillation theory of partial differential equations.* World Scientific, 2008.

[26] N. Yoshida, *A Picone identity for half-linear elliptic equations and its applications to oscillation theory,* Nonlinear Anal. 71 (2009), 4935–4951.

[27] C. Zhang, S. Sun, *Sturm-Picone comparison theorem of second-order linear equationson time scales,* Adv. Diff. Eqs. (2009),Article ID 496135, 12pages doi:10.1155/2009/496135.

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