**EURASIAN MATHEMATICAL JOURNAL**
ISSN 2077-9879
Volume 6, Number 1 (2015), 56 – 64

# THE RELATIONSHIPS BETWEEN POSETS AND INDEPENDENT SETS OF A MATROID OF ARBITRARY CARDINALITY

**H. Mao**

Communicated by J.A. Tussupov

**Key words:** matroid of arbitrary cardinality, independent set, poset, Boolean lattice.

**AMS Mathematics Subject Classification:** 05B35, 06A06, 06A07.

**Abstract.** By constructing the correspondent relationship between matroids of arbitrary cardinality and posets, under isomorphism, this paper characterizes matroids of arbitrary cardinality without loops. Utilizing this characterization, it realizes the translation of some results from posets to matroid of arbitrary cardinality frameworks. At last, we give the conclusion.

## 1 Introduction

Using the family of closed sets of a matroid of arbitrary cardinality, in [6], it presents the relationship between matroids of arbitrary cardinality and geometric lattices. This relationship has been applied to study on the properties of matroids of arbitrary cardinality ([6-9]). However, both [2] and [5] indicate that not every poset is a geometric lattice though a geometric lattice is a poset. This perhaps limits the applied fields of the results found in [6]. We hope to change the status quo. This asks to find out the relationship between matroids of arbitrary cardinality and some new structures which generalize geometric lattices. To do this, first, we observe and add up the following views:

(1.1) From the results in [7], we may infer that a matroid of arbitrary cardinality is uniquely determined by its collection of independent sets.

(1.2) For a given matroid $M$ of arbitrary cardinality, we may associate a poset whose elements are the independent sets of $M$.

The above views (1.1) and (1.2) inform us that we may build up the correspondent relationship between matroids of arbitrary cardinality and posets. This relationship may extend the applied fields of matroids of arbitrary cardinality. This paper will mainly follow the above analysis to complete its work.

We narrate the construction of this paper as follows. Section 2 review some knowledge relative to posets and matroids of arbitrary cardinality.

In Section 3, for a matroid $M = (S, \mathcal{I})$ of arbitrary cardinality with $\mathcal{I}$ as its family of independent sets, we look in some properties at poset $(\mathcal{I}, \subseteq)$. The important in Section 3 is that under isomorphism, we establish the correspondent relationship between posets with some pre-conditions and matroids of arbitrary cardinality with no loops. Using this relationship, it establishes the relationship between Boolean lattices and matroids of arbitrary cardinality in which every member owes a unique base.

In Section 4, after comparing the relationship between matroids of arbitrary cardinality and posets with the famous relationship between matroids of arbitrary cardinality and geometric lattices, we outline our future works.

In what follows, we assume that $E$ is some arbitrary–possibly infinite–set; $2^E$ denotes the family of all the subsets of $E$. For a set $\{A\}, Max\{A\}$ denotes the maximum element in $\{A\}$. $Y \subset\subset X$ represents $Y$ to be a finite subset of a set $X$. $\mathbb{N}_0$ means the set of non-negative integers.

For simplicity, if there is no confusion in the text, then a poset $(P, \leq)$ is said to be $P$. In a poset $P$, $b \prec a$ stands for "$a$ covers $b$"; the interval $\{x \in P : a \leq x \leq b\}$ is in notation $[a, b]$; for $H = \{a, b\} \subseteq P, \bigvee H$ sometimes is in notation $a \vee b$. If two posets $P_1$ and $P_2$ are isomorphic, then it will be denoted by $P_1 \cong P_2$.

## 2 Preliminaries

This section begins by reviewing some knowledge what are needed in the sequel.

All the knowledge relative to poset theory refers to [2,5]. In [2], it points out that a Boolean algebra may be described to be a Boolean lattice, and vice versa. Therefore, we may equivalently say: $L$ is a Boolean algebra if and only if $L$ is a Boolean lattice.

Some notations and properties of infinite matroids are reviewed here and the others are referred to [1,6-8,11]. The knowledge of finite matroids are seen to [10,12].

**Definition 1.** (1)[1] Assume $m \in \mathbb{N}_0$ and $\mathcal{F} \subseteq 2^E$. Then the pair $M := (E, \mathcal{F})$ is called a *matroid of rank $m$ with $\mathcal{F}$ as its closed sets*, if the following axioms hold:
(F1) $E \in \mathcal{F}$;
(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;
(F3) Assume $F_0 \in \mathcal{F}$ and $x_1, x_2 \in E \setminus F_0$. Then one has either
$\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\}$ or
$F_1 \cap F_2 = F_0$ for certain $F_1, F_2 \in \mathcal{F}$ containing $F_0 \cup \{x_1\}$ or $F_0 \cup \{x_2\}$, respectively.
(F4) $m = max\{n \in \mathbb{N}_0 |$ there exist $F_0, F_1, ..., F_n \in \mathcal{F}$ with $F_0 \subset F_1 \subset ... \subset F_n = E\}$.

The *closure operator* $\sigma = \sigma_M : 2^E \to \mathcal{F}$ of $M$ is defined by $\sigma(A) := \bigcap\limits_{\substack{F \in \mathcal{F} \\ A \subseteq F}} F$. The *rank function* $\rho = \rho_M : 2^E \to \{0, 1, ..., m\}$ of $M$ is defined by $\rho(A) := max\{k \in \mathbb{N}_0|$ there exist $F_0, F_1, ..., F_k \in \mathcal{F}$ with $F_0 \subset F_1 \subset ... \subset F_k = \sigma(A)\}$.

$M$ is called *simple*, if any subset $A \subseteq E$ with $|A| \leq 1$ lies in $\mathcal{F}$.

(2)[7] One calls $A \in \mathcal{I} = \{A \subseteq E | x \in A, x \notin \sigma(A \setminus \{x\})\}$ an *independent set* of $M$.

(3)[12,pp.385-387;&11,p.74] An *independence space* $M_p(E)$ is a set $E$ together with a collection $\mathcal{I}_p$ of subsets of $E$ (called *independent sets*) such that
(i1) $\mathcal{I}_p \neq \emptyset$;
(i2) If $A \in \mathcal{I}_p$ and $B \subseteq A$, then $B \in \mathcal{I}_p$;
(i3) If $A, B \in \mathcal{I}_p$ and $|A|, |B| < \infty$ with $|A| = |B| + 1$, then $\exists a \in A \setminus B$ fits $B \cup \{a\} \in \mathcal{I}_p$;
(i4) If $A \subseteq E$ and every finite subset of $A$ is a member of $\mathcal{I}_p$, then $A \in \mathcal{I}_p$.

(4)[6] Two matroids $M_i = (E_i, \mathcal{F}_i)$ of arbitrary cardinality, where $\mathcal{F}_i$ is the system of closed sets of $M_i, (i = 1, 2)$, are *isomorphic* if there is a bijection $\phi : E_1 \to E_2$ satisfying $A \in \mathcal{F}_1 \Leftrightarrow \phi(A) \in \mathcal{F}_2$. We write $M_1 \simeq M_2$ if $M_1$ and $M_2$ are isomorphic.

In this paper, a matroid $M = (E, \mathcal{F})$ defined as in Definition 1 is called a *matroid of arbitrary cardinality*. A *base* of $M$ is a maximal independent set. We define a *loop* of $M$ to be an element $x$ of $E$ such that $\{x\}$ is not an independent set.

The following statements about loops are obvious.

(l1) $x$ is a loop if and only if $x \in \sigma(\emptyset)$.

(l2) $x$ is a loop if and only if $\rho(\{x\}) = 0$.

(l3) $x$ is a loop if and only if it is not contained in any base.

Therefore, if $M$ is simple, then it has no loops, but not vice versa.

**Lemma 2.1.** (1)[7] *A collection $\mathcal{I}$ of subsets of $E$ is the set of independent sets of a matroid of arbitrary cardinality on $E$ if and only if $\mathcal{I}$ satisfies* (i1) − (i4) *and* (i5)*: $max\{k \in \mathbb{N}_0 |$ there exists $I_0, I_1, \ldots, I_k \in \mathcal{I}$ such that $I_0 \subset I_1 \subset \ldots \subset I_k\} < \infty$.*

(2)[7] *$\mathcal{I} \subseteq 2^E$ is the collection of independent sets of a matroid of arbitrary cardinality $M$ on $E$ if and only if $\mathcal{I}$ satisfies* $(i1), (i2), (i4), (i5)$ *and* $(i3)'$*: For $X \subseteq E$, if $I_1, I_2 \in Max\{I \subseteq X | I \in \mathcal{I}\}$, then $|I_1| = |I_2|$.*

(3)[11] *Every independent subset of an independence space is contained in a basis.*

Based on (1) of Lemma 2.1, in what follows, a matroid $M$ of arbitrary cardinality defined on $E$ will be denoted by $(E, \mathcal{I})$, and sometimes $\mathcal{I}$ is notated to be $\mathcal{I}(M)$. (3) of Definition 1 informs us that $M$ is also an independence space. We may easily state the following result: for two matroids of arbitrary cardinality $M_i$, i.e. $(E_i, \mathcal{I}_i)$, $(i = 1, 2)$, $M_1 \simeq M_2$, i.e. $M_1$ is isomorphic to $M_2$, if and only if there is a bijection $\psi : E_1 \to E_2$ satisfying $I \in \mathcal{I}_1 \Leftrightarrow \psi(I) \in \mathcal{I}_2$.

In [3] and [4], an independence space is called a finitary matroid. Thus a matroid of arbitrary cardinality is a finitary matroid with finite rank. That is to say, a matroid of arbitrary cardinality is a class of infinite matroids and much "like" finite matroids. Since, for a long time, the theory of infinite matroids is much more complicated than that of finite matroids. One of the difficulties has been that there are many reasonable and useful definitions, none of which appeared to capture all the important aspects of finite matroid theory. The "best and simplest" definition of an infinite matroid is to require finite rank. Therefore, it is valuable to deal with matroids of arbitrary cardinality.

As good contributions to infinite matroids such as [3,4,10-12], they provide a series of axioms of finitary matroids. None of these axioms is discussed with the relationship between poset theory and finitary matroids. Hence, according to our knowledge, it is necessary to do the essential work on matroids of arbitrary cardinality–a special class of infinite matroids, with poset theory.

## 3 Relations

This section will deal with the relationship between a matroid $M = (E, \mathcal{I})$ of arbitrary cardinality and a poset. First, we find out some properties of $(\mathcal{I}, \subseteq)$. Second, we search that under what conditions, a poset $P$ will correspond to a matroid $M(P)$ of arbitrary cardinality such that $P$ is isomorphic to $(\mathcal{I}(M(P)), \subseteq)$. Third, it gets the correspondence between the category of matroids of arbitrary cardinality with no loops and the category of some posets with some conditions. Fourth, using a concrete consequence, it shows the importance of the correspondent relationship obtained above by "translation" between matroids of arbitrary cardinality and posets.

**Lemma 3.1.** *For a matroid of arbitrary cardinality $M = (E, \mathcal{I})$, the poset $(\mathcal{I}, \subseteq)$, i.e. $\mathcal{P}(M)$, has the following properties.*

(m1) *$\emptyset$ is the least element in $(\mathcal{I}, \subseteq)$.*

(m2) *For any $I \in \mathcal{I}$, the interval $[\emptyset, I]$ in $(\mathcal{I}, \subseteq)$ is isomorphic to the poset $(2^I, \subseteq)$.*

(m3) *All of maximal chains in $(\mathcal{I}, \subseteq)$ have the same length $|B|$, where $B$ is a base in $M$.*

(m4) *For any $X, Y \in (\mathcal{I}, \subseteq)$, if $h(X) = h(Y) + 1$, then there is $a \in X \setminus Y$ such that $Y \cup a$ covers $Y$ in $(\mathcal{I}, \subseteq)$, where $h$ is the height function of $(\mathcal{I}, \subseteq)$.*

(m5) *Let $\mathcal{A}$ be the collection of atoms in $(\mathcal{I}, \subseteq)$. Every $I \in (\mathcal{I}, \subseteq)$ is a join of atoms, i.e. $I = \bigcup_{a \in \mathcal{A}_I} a = \cup \mathcal{A}_I$, and $|I| = |\mathcal{A}_I|$, where $\mathcal{A}_I$ is the family of atoms contained in $I$ in $(\mathcal{I}, \subseteq)$.*

(m6) *Let $X \subseteq \mathcal{A}$. If there is $Y \subset\subset X$ satisfying $Y \notin (\mathcal{I}, \subseteq)$, then $X \notin (\mathcal{I}, \subseteq)$.*

*Proof.* (m1) is obvious by (i1).

(i2) causes the truth of (m2).

Considered (1) and (3) in Lemma 2.1 with Definition 1, for $I \in \mathcal{I}$, we obtain that there exists a base $B_I$ of $M$ satisfying $I \subseteq B_I$. In addition, (2) in Lemma 2.1 and (i5) assure that all the maximal chains in $(\mathcal{I}, \subseteq)$ have finite length $|B_I|$. Hence, (m3) holds.

Recalling back (m2) and (1) in Lemma 2.1, we see $h(X) = |X|$ for any $X \in \mathcal{I}$. So, (m4) is followed by (i3).

Since $|I| < \infty$ holds for any $I \in \mathcal{I}$ according to (i5), and in addition, we know that $(2^I, \subseteq)$ is a Boolean lattice. Considering (m2) and (m3) with (1) in Lemma 2.1, it follows that $I$ is a join of atoms, i.e., $I = \cup \mathcal{A}_I = \mathcal{A}_I$, and so $|I| = |\mathcal{A}_I|$.

(i4) guarantees the correct of (m6). □

The following example will express that $\mathcal{P}(M)$ may not be a lattice.

**Example 1.** Let $E = \{1, 2\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}\}$. Then it is easy to testify $(E, \mathcal{I})$ to be a matroid of arbitrary cardinality. But, obviously, $(\mathcal{I}, \subseteq)$ is not a lattice.

Conversely, for a poset $P$, we try to find out that under what conditions, there is a matroid of arbitrary cardinality $M$ satisfying $\mathcal{P}(M) \cong P$. In light of (m1), $P$ must keep the least element 0. Therefore, we will consider only the poset with the least element.

**Lemma 3.2.** *Let $P$ be a poset with the least element 0, $\mathcal{A}$ be the collection of atoms in $P$ and $\mathcal{A}_x$ be the atoms contained in $x \in P$. If $P$ satisfies the following* (q1)-(q6), *then it exists a matroid $M(P)$ of arbitrary cardinality such that $P \cong (\mathcal{I}, \subseteq)$ and $M(P)$ has no loops, where $\mathcal{I}$ is the set of independent sets of $M(P)$.*

(q1) *Every element in $P$ is a join of atoms, namely, for $x \in P$, there is $x = \bigvee \mathcal{A}_x$.*

(q2) *For any $x \in P$, there is $b \in MaxP$ satisfying $x \leq b$.*

(q3) *If $b \in MaxP$, then $[0, b] \cong (2^{\mathcal{A}_b}, \subseteq)$.*

(q4) *Every maximal chain in $P$ has the same finite length.*

(q5) *For any $x, y \in P$, if $h(x) = h(y) + 1$, then there exists $a_x \in \mathcal{A}_x \setminus \mathcal{A}_y$ satisfying $y \prec y \cup a_x$ in $P$, where $h$ is the height function of $P$.*

(q6) *For $\mathcal{S} \subseteq \mathcal{A}$, if there is $X \subset\subset \mathcal{S}$ satisfying $\bigvee X \notin P$, then $\bigvee \mathcal{S} \notin P$.*

*Proof.* We will carry out the proof step by step.

Step 1. If $x, y \in P$ and $x \neq y$, then $\mathcal{A}_x \neq \mathcal{A}_y$.

Otherwise, there is $\mathcal{A}_x = \mathcal{A}_y$ but $x \neq y$ for some $x, y \in P$. Therefore by (q1), it follows $x = \bigvee \mathcal{A}_x = \bigvee \mathcal{A}_y = y$, a contradiction.

Step 2. Let $\mathcal{I} = \{\mathcal{A}_x | x \in P\}$. We prove that $(\mathcal{A}, \mathcal{I})$ is a matroid of arbitrary cardinality with $\mathcal{I}$ as its family of independent sets, and additionally, $(\mathcal{A}, \mathcal{I})$ has no loops. We denote $(\mathcal{A}, \mathcal{I})$ as $M(P)$.

Step 2.1 Since $0 < x$ holds for any $x \in P \setminus 0$. It causes 0 not a join of atoms. That is, $\mathcal{A}_0 = \emptyset$, and so, $\emptyset \in \mathcal{I}$.

Step 2.2 Let $\mathcal{A}_y \in \mathcal{I}$ and $X \subseteq \mathcal{A}_y$.

$\mathcal{A}_y \in \mathcal{I}$ reveals $y = \bigvee \mathcal{A}_y \in P$. Since (q2) assures that there is $b_y \in MaxP$ satisfying $y \leq b_y$. It follows $[0, y] \subseteq [0, b_y]$. Additionally, $(2^Z, \subseteq)$ is a Boolean lattice for any set $Z$. Thus, (q3) describes that $[0, b_y]$ is a Boolean lattice, and in addition, $[0, y] \cong (2^{\mathcal{A}_y}, \subseteq)$ holds because of $[0, b_y] \cong (2^{\mathcal{A}_{b_y}}, \subseteq)$. So $[0, y]$ is a Boolean lattice. However, $X \subseteq \mathcal{A}_y$ causes $X \in 2^{\mathcal{A}_y}$. Therefore, under isomorphism, $\bigvee X \in [0, y]$ holds, i.e. $\bigvee X \in P$. Hence, it follows $X \in \mathcal{I}$.

Step 2.3 To prove: $|\mathcal{A}_x| < \infty$ for any $x \in P$.

For $x \in P$, (q2) asserts $x \leq b$ for some $b \in MaxP$. (q4) and (q5) together states that there are $a_{1j} \in \mathcal{A}_b$ $(j = 1, 2, \ldots, n < \infty)$ such that $0 \prec a_{11} \prec a_{11} \cup a_{12} \prec \ldots \prec a_{11} \cup a_{12} \cup \ldots \cup a_{1(n-1)} \cup a_{1n} = b$ is a maximal chain in $P$. (q4) indicates that $n$ is the length of any maximal chains in $P$. Using the induction on $n$, we will carry out the proof in the present step. Actually, if $|\mathcal{A}_b| < \infty$, then $|\mathcal{A}_x| < \infty$. Hence, we prove $|\mathcal{A}_b| < \infty$ for any $n$.

If $n \leq 1$. Then the needed result is true obviously.

We may easily prove if $n = 2$, then $|\mathcal{A}_b| < \infty$

Suppose for any $n < k$ and $k \geq 3$, $|\mathcal{A}_b| < \infty$ holds. Let $n = k \geq 3$ and $|\mathcal{A}_b| \not< \infty$.

Then we assert that $(a_{11} \cup a_{12}) \parallel a_{m_0}$ is true for some $a_{m_0} \in \mathcal{A}_b \setminus \{a_{11}, a_{12}, \ldots, a_{1n}\}$.

Otherwise, for any $a_m \in \mathcal{A}_b \setminus \{a_{11}, a_{12}, \ldots, a_{1n}\}$, there is $(a_{11} \cup a_{12}) \nparallel a_m$. In light of $0 \prec a_m, 0 \prec a_{11}, 0 \prec a_{12}, 0 \prec a_{11} \prec a_{11} \cup a_{12}$ and $|a_m| < |a_{11} \cup a_{12}|$, it brings about $a_m < a_{11} \cup a_{12}$. Considered $a_{12}, a_{11} \cup a_{12} \in P, h(a_{12}) = 1$ with (q3) and (q4), we may easily obtain $h(a_{11} \cup a_{12}) = 2$, $\mathcal{A}_{a_{12}} = \{a_{12}\}$ and $\mathcal{A}_{a_{11} \cup a_{12}} = \{a_{11}, a_{12}\}$. Hence (q5) causes $a_{12} \prec a_{11} \cup a_{12}$. In addition, $a_m < a_{11} \cup a_{12}$ and $h(a_{11} \cup a_{12}) = 2$ express $a_m \prec a_{11} \cup a_{12}$. Therefore, we may reveal that $\{0, a_{11}, a_{12}, a_m, a_{11} \cup a_{12}\}$ is a diamond, and additionally, a subposet of $[0, b]$. Because (q3) assures $[0, b]$ to be a Boolean lattice, by (2) of Lemma 2.1, $\{0, a_{11}, a_{12}, a_m, a_{11} \cup a_{12}\}$ is not a sublattice of $[0, b]$, a contradiction.

Let $a_{t_0} \in \mathcal{A}_b \setminus \{a_{11}, a_{12}, \ldots, a_{1n}\}$ satisfy $a_{t_0} \parallel (a_{11} \cup a_{12})$. Since $a_{11}, a_{12} \in \mathcal{A}_b$ guarantees $a_{11} \parallel a_{t_0}$ and $a_{t_0} < b$. Considered with $n \geq 3$, we get $a_{11} \cup a_{12} < b$. So $\{0, a_{11}, a_{t_0}, a_{11} \cup a_{12}, b\}$ is a subposet of $P$ and also a pentagon. Thus, $\{0, a_{11}, a_m, a_{11} \cup a_{12}, b\}$ is a subposet of $[0, b]$ and a pentagon. Since $[0, b]$ is a Boolean lattice according to (q3). Hence, by Lemma 2.1, the pentagon $\{0, a_{11}, a_{t_0}, a_{11} \cup a_{12}, b\}$ is not be a subposet of $[0, b]$, a contradiction.

Therefore, it has $|\mathcal{A}_b| < \infty$.

Step 2.4 First, we prove $h(x) = |\mathcal{A}_x|$ for any $x \in P$.

Let $h(x) = n$. (q4) compels $n < \infty$. Using the induction on $n$, we will finish the proof.

If $n = 1$. This suggests $0 \prec x$, and so $h(x) = |\mathcal{A}_x| = 1$.

If $n = 2$. Then there exist two different elements $a_1, a_2 \in \mathcal{A}_x$ satisfying $a_1 \vee a_2 = x$. In addition, $h(x) = 2$ asks $a < x$ for any $a \in \mathcal{A}_x$.

If $\mathcal{A}_x \setminus \{a_1, a_2\} \neq \emptyset$, then there is $a_3 \in \mathcal{A}_x \setminus \{a_1, a_2\}$ satisfying $a_3 < x$. We may indicate that $\{0, a_1, a_2, a_3, x\}$ constitutes a diamond to be a subposet of $[0, x]$. However, $[0, x]$ is a subposet of $[0, b_x]$ for some $b_x \in MaxP$. In other words, by (q3), up to isomorphism, $\{0, a_1, a_2, a_3, x\}$ is a subposet of the Boolean lattice $[0, b_x]$, a contradiction. Thus, there is $|\mathcal{A}_x| = 2$.

Suppose the needed result is true for any $n < k$. Now, let $h(x) = n = k$ and $3 \leq k$.

Since $[0, x] \subseteq [0, b]$ is correct for some $b \in MaxP$. Considered this result with (q3) and (q4), we may state that all the maximal chains between $x$ and 0 will indeed have the same finite length $k$. No matter to suppose that $0 \prec X_1 \prec X_2 \prec \ldots \prec X_{t-1} \prec X_t = x$ is a maximal chain between 0 and $x$. In virtue of (q5), it follows that the above chain satisfies $t = h(x) = k$ and $X_1 = a_1, X_2 = a_1 \cup a_2, \ldots, X_{t-1} = a_1 \cup a_2 \cup \ldots \cup a_{t-1}, x = X_t = X_{t-1} \cup a$, where $a, a_j \in \mathcal{A}_x, (j = 1, 2, \ldots, t-1)$. Additionally, (q1) indicates $x = \bigvee \mathcal{A}_x$. Thus, we may obtain $\mathcal{A}_x \subseteq \{a_1, a_2, \ldots, a_{t-1}, a\}$. So, $|\mathcal{A}_x| = |\{a_1, a_2, \ldots, a_{t-1}, a\}| = k = h(x)$ is true.

Second, the result above combined with (q5) will assure the hold of (i3) in $\mathcal{I}$.

Step 2.5 Combining (q1) and (q6) with Step 2.2, it yields out the real of (i4) for $\mathcal{I}$.

Step 2.6 (q4) and Step 2.3 cause the correct of (i5) for $\mathcal{I}$.

Taken Step 2.1–Step 2.6, we may state that $(\mathcal{A}, \mathcal{I})$ is a matroid of arbitrary cardinality with $\mathcal{I}$ as its family of independent sets.

By the definitions of $\mathcal{A}$ and $\mathcal{A}_x$ given above, for any $a \in \mathcal{A}$, there is $a \in P$, and further, there exists $\mathcal{A}_a \in \mathcal{I}$ and $\mathcal{A}_a = \{a\}$. This points out that there does not exist loops in $M(P)$.

Step 3. To prove $P \cong (\mathcal{I}, \subseteq)$.

Let $f : P \to (\mathcal{I}, \subseteq)$ be defined as $x \mapsto \mathcal{A}_x$ for any $x \in P$. We may easily find out that $f$ is a bijection.

Let $x, y \in P$ and $x \vee y \in P$. Then we may state clearly $\mathcal{A}_x, \mathcal{A}_y \subseteq \mathcal{A}_{x \vee y}$. In virtue of (q2), it follows the existence $b \in MaxP$ satisfying $x \vee y \leq b$, and so $x, y \leq b$. Further, $\mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_{x \vee y} \subseteq \mathcal{A}_b$ hold because of (q3). Still considering with (q3), it follows $\mathcal{A}_x \cup \mathcal{A}_y \in (2^{\mathcal{A}_b}, \subseteq)$ which corresponds $x \vee y$ in $[0, b]$. Thus, it proves $\mathcal{A}_{x \vee y} = \mathcal{A}_x \cup \mathcal{A}_y$.

Additionally, if $\mathcal{A}_x \cup \mathcal{A}_y \in (\mathcal{I}, \subseteq)$ and $\mathcal{A}_z = \mathcal{A}_x \cup \mathcal{A}_y$, then $z = x \vee y$.

Let $a, b \in P$ and $a \leq b$. By the definition of the notation $\mathcal{A}_x$, it easily follows $\mathcal{A}_a \subseteq \mathcal{A}_b$.

Obviously, we may obtain that if $X, Y \in (\mathcal{I}, \subseteq)$ and $X \subseteq Y$, then $x \leq y$ in $P$, where $x = \bigvee X, y = \bigvee Y$ according to the definition of $\mathcal{I}$. □

For simplicity, if a poset $P$ has the least element and satisfies (q1)-(q6), then we call $P$ a *pseudo-geometric lattice*.

Now unfortunately, the structure of a matroid $M$ of arbitrary cardinality is not completely specified by the poset $\mathcal{P}(M)$.

**Example 2.** Let $a \notin E_1$ and $M_1 = (E_1, \mathcal{I})$ be a matroid of arbitrary cardinality satisfying $E_1 = \{x \mid \{x\} \in \mathcal{I}\}$. Evidently, $M_2 = (E_1 \cup a, \mathcal{I})$ is a matroid of arbitrary cardinality and $\{a\}$ is a loop of $M_2$ though $\mathcal{P}(M_1) = \mathcal{P}(M_2)$.

This indeterminacy of $M$ from $\mathcal{P}(M)$ is due to the existence of loops. The importance of matroids of arbitrary cardinality without loops lies in the following theorem.

**Theorem 3.1.** *There exists a one to one correspondence between pseudo-geometric lattices and matroids of arbitrary cardinality without loops.*

*Proof.* Let $M = (E, \mathcal{I}(M))$ be a matroid of arbitrary cardinality with no loops. We see that the set of atoms in $\mathcal{P}(M)$ is $E$. By (m3) and (2) in Lemma 2.1, we obtain that any maximal chains in $\mathcal{P}(M)$ have the same finite length. Recalling back Lemma 3.1, we may assure that $\mathcal{P}(M)$ has the least element and satisfies (q1)-(q6). Further, by Lemma 3.2, it causes $M(\mathcal{P}(M))$ satisfying $\mathcal{P}(M) \cong (\mathcal{I}(M(\mathcal{P}(M))), \subseteq)$, that is, $(\mathcal{I}(M), \subseteq) = \mathcal{P}(M) \cong (\mathcal{I}(M(\mathcal{P}(M))), \subseteq)$. Meanwhile, $M(\mathcal{P}(M))$ is defined on the set of atoms of $\mathcal{P}(M)$, that is, on the set $\{\{x\}|x \in E\}$. Hence, $M(\mathcal{P}(M))) \simeq M$ holds. □

**Corollary 3.1.** (1) *Let $M$ be a matroid of arbitrary cardinality without loops defined on $E$. Then $\mathcal{P}(M)$ is a Boolean lattice if and only if $|Max\mathcal{P}(M)| = 1$.*

(2) *Let $P$ be a pseudo-geometric lattice. Then $P$ is a Boolean lattice if and only if $P$ is bounded, i.e. $P$ does also have the greatest element.*

*Proof.* (1)($\Longleftarrow$) It is well known that $(2^{Max\mathcal{P}(M)}, \subseteq)$ is a Boolean lattice if $|Max\mathcal{P}(M)| = 1$. By (m2) and $|Max\mathcal{P}(M)| = 1$, we may believe the real of $\mathcal{P}(M) \cong (2^{Max\mathcal{P}(M)}, \subseteq)$.

($\Longrightarrow$) The supposition, (m3), and (2) in Lemma 2.1 will assure $\mathcal{P}(M)$ to be a bounded lattice, and so $|Max\mathcal{P}(M)| = 1$.

(2) It is a straightforward result from Theorem 3.1 and (1). □

Thus, by Theorem 3.1, we may describe that the study of matroids of arbitrary cardinality without loops is just the study of pseudo-geometric lattices. Many of the interesting properties of matroids of arbitrary cardinality are preserved if we just pay attention to matroids of arbitrary cardinality with no loops.

It is useful to "translate" some of the results about posets to a matroid of arbitrary cardinality frameworks. Here, under the matroid of arbitrary cardinality frameworks, for a Boolean lattice with finite length, we will obtain the same result as that in [2,p.18].

**Theorem 3.2.** *Let $P$ be a Boolean lattice with finite length. Then any interval sublattice of $P$ is a Boolean lattice.*

*Proof.* Combined the supposition that $P$ is a lattice with finite length, the definition of lattice with the dual of [2,p.6,ex.8], we may assure that $P$ has the least element and $|MaxP| = 1$. Considered with the supposition that $P$ is a Boolean lattice with finite length, we may easily find out that $P$ satisfies (q1)-(q6). By Theorem 3.1, under isomorphism, it follows that there is a matroid $M(P)$ of arbitrary cardinality with no loops corresponding to $P$.

In $P$, let $\mathcal{A}$ be the collection of atoms and $\mathcal{A}_x$ be the collection of atoms contained in $x \in P$. Let $[0, b]$ be an interval in $P$. Evidently, $\mathcal{A}_b$ is the set of atoms in $[0, b]$ by the definition of interval. Next we may prove that $(\mathcal{A}_b, \mathcal{I}_b)$ is a matroid of arbitrary cardinality where $\mathcal{I}_b = \{\mathcal{A}_x \mid x \in [0, b]\}$.

Theorem 3.1 informs us that $M(P)$ is defined on $\mathcal{A}$ and $\mathcal{I}(M(P)) = \{\mathcal{A}_x \mid x \in P\}$. Additionally, in view of [11], we may yield out that $M(P) \mid \mathcal{A}_b$, i.e. $(\mathcal{A}_b, \mathcal{I}(M(P)) \mid \mathcal{A}_b)$

in which $\mathcal{I}(M(P)) \mid \mathcal{A}_b = \{X \mid X \subseteq \mathcal{A}_b, X \in \mathcal{I}(M(P))\}$, is an independence space. Recalling back (m3) for $M(P)$, we may state clearly that $\mathcal{I}(M(P)) \mid \mathcal{A}_b$ satisfies (i5). Therefore $M(P) \mid \mathcal{A}_b$ is a matroid of arbitrary cardinality in view of (1) of Lemma 2.1 and (3) of Definition 1.

That $M(P)$ has no loops causes that $M(P) \mid \mathcal{A}_b$ has no loops.

In addition, $\mathcal{I}(M(P)) \mid \mathcal{A}_b = \{X \mid X \in \mathcal{I}(M(P)), X \subseteq \mathcal{A}_b\} = \{\mathcal{A}_x \mid \mathcal{A}_x \in \mathcal{I}(M(P)), \mathcal{A}_x \subseteq \mathcal{A}_b\}$ holds evidently. Thus, there is $Max(\mathcal{I}(M(P)) \mid \mathcal{A}_b) = \mathcal{A}_b$. By (1) in Corollary 3.1, we will gain that $\mathcal{P}(\mathcal{I}(M(P)) \mid \mathcal{A}_b)$ is a Boolean lattice. Simultaneously, under isomorphism, $M(P) \mid \mathcal{A}_b$ corresponds to $[0, b]$ in $P$. Therefore, $[0, b]$ is a Boolean lattice. □

## 4 Conclusion

We may state that under isomorphism, Theorem 3.1 characterizes much more matroids of arbitrary cardinality by posets than that found in [6]. This will perhaps generalize the applied fields of matroids of arbitrary cardinality. Utilizing Theorem 3.1 into the study on matroids of arbitrary cardinality is our future research contexts. More study is left rooms.

## Acknowledgments

The author thanks the unknown referee to give valuable suggestions for this paper. This work was supported by NSF of China 61572011, 61073121, 61202178; NSF of Hebei Province A2013201119, A2014201033.

## References

[1] D. Betten, W.Wenzel, *On linear spaces and matroids of arbitrary cardinality*, Algebra Universalis 49 (2003), 259-288.

[2] G. Birkhoff, *Lattice Theory*, 3rd. ed. American Mathematical Society, Providence, 1967.

[3] R.A. Brualdi, J.H. Mason, *Transversal matroids and Hall's theorm*, Pacific J. of Math. 41 (1972), no. 3, 601–613.

[4] H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh, P.Wollan, *Axioms for infinite matroids*, Advances in Math. 239 (2013), 18–46.

[5] G. Grätzer, *Lattice Theory: Foundation*, Springer Basel AG, Belin, 2011.

[6] H. Mao, *On geometric lattices and matroids of arbitrary cardinality*, Ars Combinatoria. 81 (2006), 23–32.

[7] H. Mao, *Paving matroids of arbitrary cardinality*, Ars Combinatoria 90 (2009), 245–256.

[8] H. Mao, *Characterizations of disconnected matroids*, Algebra Colloquium 15 (2008), no. 1, 129–134.

[9] H. Mao, G. Wang, *Some properties of base-matroids of aribitrary cardinality*, Rockey Mountain J. of Math. 40 (2010), no. 1, 291–303.

[10] J. Oxley, *Matroid Theory*, Oxford University Press, New York, 2006.

[11] J. Oxley, *Infinite Matroid*, In: Matroid Application, ed. by N.White. Cambridge Universtiy Press, Cambridge, 1992, 73–90.

[12] D.J. A.Welsh, *Matroid Theory*, Academic Press Inc., London, 1976.

Hua Mao
Department of Mathematics
Hebei University
071002 Baoding, China
E-mails: yushengmao@263.net

Received: 06.03.2012