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MODULUS OF SUPPORTING CONVEXITY AND SUPPORTING SMOOTHNESS

G.M. Ivanov

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Abstract. We introduce the moduli of the supporting convexity and the supporting smoothness of a Banach space, which characterize the deviation of the unit sphere from an arbitrary supporting hyperplane. We show that the modulus of supporting smoothness, the Banaś modulus, and the modulus of smoothness are all equivalent at zero, the modulus of supporting convexity is equivalent at zero to the modulus of convexity. We prove a Day–Nordlander type result for these moduli.

1 Introduction

The properties of a Banach space are completely determined by its unit ball. The geometry of the unit ball of a Banach space X may be described, for instance, using the properties of some moduli attached to X. (For example, the moduli of convexity, of smoothness, Milman's moduli, etc.) The aim of this paper is to introduce and explore some new type of moduli, which characterize the deviation of the unit sphere from an arbitrary supporting hyperplane.

In the sequel we shall need some additional notation. Let X be a real Banach space. For a set $A \subset X$ by ∂A, $\operatorname{int} A$ we denote the boundary and the interior of A. We use $\langle p, x\rangle$ to denote the value of a functional $p \in X^*$ at a vector $x \in X$. For $R > 0$ and $c \in X$ we denote by $\mathfrak{B}_R(c)$ the closed ball with center c and radius R, by $\mathfrak{B}^*_R(c)$ we denote the ball in the conjugate space. By definition, put $J_1(x) = \{p \in \partial\mathfrak{B}^*_1(o) : \langle p, x\rangle = \|x\|\}$. For convenience, the length of segment ab is denoted by $\|ab\|$, i.e., $\|ab\| = \|a - b\|$.

We say that y is *quasiorthogonal* to the vector $x \in X \setminus \{o\}$ and write $y\urcorner x$ if there exists a functional $p \in J_1(x)$ such that $\langle p, y\rangle = 0$. Note that the following conditions are equivalent:

– y is quasiorthogonal to x
– for any $\lambda \in \mathbb{R}$ the vector $x + \lambda y$ lies in the supporting hyperplane to the ball $\mathfrak{B}_{\|x\|}(o)$ at x;
– for any $\lambda \in \mathbb{R}$ the following inequality holds $\|x + \lambda y\| \geqslant \|x\|$;
– x is orthogonal to y in the sense of Birkhoff–James ([6], Ch. 2, §1).

Let

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} :\ x, y \in \mathfrak{B}_1(0),\ \|x-y\| \geq \varepsilon\right\}$$

and

$$\rho_X(\tau) = \sup\left\{\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1 :\ \|x\| = 1, \|y\| = \tau\right\}.$$

The functions $\delta_X(\cdot) : [0,2] \to [0,1]$ and $\rho_X(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ are referred to as the moduli of convexity and smoothness of X respectively.

Let f and g be two non-negative functions, each one defined on a segment $[0,\varepsilon]$. We shall consider f and g as *equivalent at zero,* denoted by $f(t) \asymp g(t)$ as $t \to 0$, if there exist positive constants a, b, c, d, e such that $af(bt) \leqslant g(t) \leqslant cf(dt)$ for $t \in [0, e]$.

The rest of this paper is organized as follows. In Section 2 we prove several simple technical lemmas, in Section 3 we introduce the definitions of the modulus of supporting convexity and the modulus of supporting smoothness and consider their basic properties, in Section 4 we show these modulus are equivalent at zero to the modulus of convexity and smoothness respectively, in Section 5 we prove that the moduli of smoothness, of supporting smoothness and the modulus of Banaś are all equivalent at zero, and, finally, in Section 6 we prove some estimates for these moduli concerning the maximal value of the Lipschitz constant for the metric projection operator onto a hyperplane.

2 Technical results

In this section we prove several simple technical results.

The proof of the next lemma is trivial.

Lemma 2.1. *Suppose the set* $\mathfrak{B}_1(o) \setminus \operatorname{int} \mathfrak{B}_r(o_1)$ *is nonempty. Then it is arcwise connected.*

Lemma 2.2. *Let* X_2 *be a two-dimensional Banach space. Suppose* $a, b, c, d \in \partial\mathfrak{B}_1(o)$ *and the segments* ab, cd *intersect in point* x. *Then the following inequality holds*

$$\min\{\|cx\|, \|xd\|\} \leqslant \max\{\|ax\|, \|xb\|\}.$$

Proof. Assume the converse. Then for some $\varepsilon > 0$ we get $\min\{\|cx\|, \|xd\|\} > \max\{\|ax\|, \|xb\|\} + \varepsilon = r$. Since the segment ab belongs to $\operatorname{int}\mathfrak{B}_r(x)$ and separates it into two parts, then we cannot connect points c, d in $\mathfrak{B}_1(o) \setminus \operatorname{int}\mathfrak{B}_r(x)$. This contradicts Lemma 2.1. The lemma is proved. □

Lemma 2.3. *Let* $x, y \in X,\ x \neq o,\ p \in \partial\mathfrak{B}^*_1(o)$ *such that* $\langle p, x\rangle = \|x\|$. *Then*

$$\|x+y\| \leqslant \|x\| + \langle p, y\rangle + 2\|x\|\,\rho_X\!\left(\frac{\|y\|}{\|x\|}\right). \tag{2.1}$$

Proof. By definition of the modulus of smoothness, we get

$$\frac{1}{2}\left(\frac{\|x+y\|}{\|x\|} + \frac{\|x-y\|}{\|x\|}\right) - 1 \leqslant \rho_X\!\left(\frac{\|y\|}{\|x\|}\right).$$

Multiplying both sides by $2\|x\|$, after some transformations we obtain:

$$\|x+y\| \leqslant 2\|x\| - \|x-y\| + 2\|x\|\,\rho_X\!\left(\frac{\|y\|}{\|x\|}\right) \leqslant$$
$$2\|x\| + \langle p, y-x\rangle + 2\|x\|\,\rho_X\!\left(\frac{\|y\|}{\|x\|}\right) = \|x\| + \langle p, y\rangle + 2\|x\|\,\rho_X\!\left(\frac{\|y\|}{\|x\|}\right).$$

□

Lemma 2.4. *For any vectors $x, y \in X \setminus \{o\}$ the following inequality is true*

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leqslant \frac{2\|x-y\|}{\|x\|}.$$

Proof. Using the triangle inequality, we get

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| = \left\|\left(\frac{x}{\|x\|} - \frac{y}{\|x\|}\right) + \left(\frac{y}{\|x\|} - \frac{y}{\|y\|}\right)\right\| \leqslant$$
$$\leqslant \left\|\left(\frac{x}{\|x\|} - \frac{y}{\|x\|}\right)\right\| + \left\|\left(\frac{y}{\|x\|} - \frac{y}{\|y\|}\right)\right\| \leqslant$$
$$\leqslant \frac{1}{\|x\|}\|x-y\| + \|y\|\left|\frac{1}{\|x\|} - \frac{1}{\|y\|}\right| \leqslant \frac{2\|x-y\|}{\|x\|}.$$

□

3 Definitions and basic properties

Let $x, y \in \partial\mathfrak{B}_1(o)$ be such that $y \urcorner x$. By definition, put

$$\lambda_X(x,y,r) = \min\{\lambda \in \mathbb{R} : \|x + ry - \lambda x\| = 1\}$$

for any $r \in [0,1]$. Denote

$$\lambda_X^-(x,y,r) = \min\{\lambda_X(x,y,r),\ \lambda_X(x,-y,r)\};$$
$$\lambda_X^+(x,y,r) = \max\{\lambda_X(x,y,r),\ \lambda_X(x,-y,r)\}.$$

Definition 1. For any $r \in [0,1]$ and $x \in \partial\mathfrak{B}_1(o)$ we define the *modulus of local supporting convexity* as

$$\lambda_X^-(x,r) = \inf \lambda_X^-(x,y,t),$$

and respectively, the *modulus of local supporting smoothness* as

$$\lambda_X^+(x,r) = \sup \lambda_X^+(x,y,t),$$

where we choose (y,t) such that $\|y\| = 1,\ y \urcorner x,\ 0 \leqslant t \leqslant r$ to minimize (maximize) $\lambda_X^-(x,r)$ $(\lambda_X^+(x,r))$.

It is clear that $\lambda_X^-(x,r) \leqslant \lambda_X^+(x,r) \leqslant 1$.

Definition 2. For any $r \in [0,1]$ we define the *modulus of supporting convexity* as

$$\lambda_X^-(r) = \inf \lambda_X^-(x,t),$$

and respectively, the *modulus of supporting smoothness* as

$$\lambda_X^+(r) = \sup \lambda_X^+(x,t),$$

where we choose (x,t) such that $x \in \mathfrak{B}_1(o)$, $0 \leqslant t \leqslant r$ to minimize (maximize) $\lambda_X^-(r)$ $(\lambda_X^+(r))$.

Let us explain the geometrical meaning of the moduli of supporting convexity and of supporting smoothness. Fix $y, x \in \partial\mathfrak{B}_1(o)$ such that $y \urcorner x$. Consider the plane $L = \mathrm{Lin}\{y, x\}$. We use (a_1, a_2) to denote the vector $a = a_1 y + a_2 x$ in this plane. The coordinate line $\ell = \{(a_1, a_2) | a_1 \in \mathbb{R}, a_2 = 0\}$ is a tangent to the unit "circle" $S = L \cap \partial\mathfrak{B}_1(x)$. By the convexity of the ball, there is a convex function $f : [-1,1] \to \mathbb{R}$ such that for $a_1 \in [-1,1]$ the point $(a_1, f(a_1))$ belongs to the lower semicircle of S (see Fig. 1). Hence for $a_1 \in [-1,1]$ the functions $\lambda_X^-(|a_1|)$ and $\lambda_X^+(|a_1|)$ are the lower and upper bounds to the $f(a_1)$ respectively, i.e. the following inequalities hold $\lambda_X^-(|a_1|) \leqslant f(a_1) \leqslant \lambda_X^+(|a_1|)$.

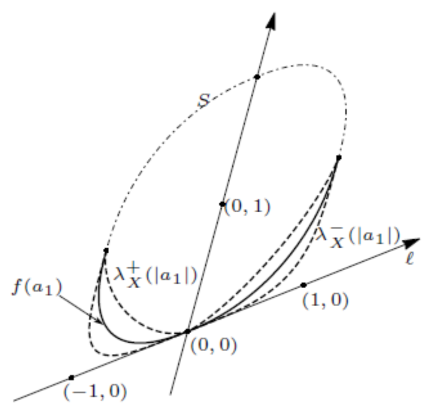

Figure 1: Geometrical meaning of the $\lambda_X^+(r)\,, \lambda_X^-(r)\,.$

Lemma 3.1. *Let X be an arbitrary Banach space, then:*

(i) $\lambda_X^+(0) = \lambda_X^-(0) = 0$;

(ii) for any $r \in [0,1]$ the following inequality holds: $0 \leqslant \lambda_X^-(r) \leqslant \lambda_X^+(r) \leqslant r$;

(iii) for any $0 < r_1 < r_2 < 1$ we have

$$\frac{r_2}{r_1}\lambda_X^-(r_1) \leqslant \lambda_X^-(r_2)\,, \tag{3.1}$$

$$\lambda_X^-(r_2) - \lambda_X^-(r_1) \leqslant \frac{r_2 - r_1}{1 - r_1}; \tag{3.2}$$

(iv) the modulus of supporting convexity is an increasing, continuous function on $[0,1)$ and moreover it is a strictly increasing function on the set $\{r \in [0,1] : \lambda_X^-(r) > 0\}$;

(v) *the modulus of supporting smoothness is a strictly increasing, convex and continuous function on* $[0,1]$ *and furthermore* $\lambda_X^+(1) = 1$.

Proof. Let us introduce some notation. Fix $x, y \in \partial\mathfrak{B}_1(o)$ such that $y \urcorner x$, and real numbers r_1, r_2 such that $0 < r_1 < r_2 < 1$. Let $z = x + y$, $z_i = x + r_i y$ where $i = 1,2$. Let $y_1, y_2 \in \partial\mathfrak{B}_1(o)$ such that $y_i z_i \parallel ox$ and the intersection of the segment $y_i z_i$ and the ball $\mathfrak{B}_1(o)$ is the point y_i where $i = 1,2$. (see Fig. 2). By construction $\|y_i z_i\| = \lambda_X(x, y, r_i)$ where $i = 1,2$. The reader will have no difficulty in showing that it is enough to prove all the assertions of this Lemma for $\lambda_X(x, y, r)$. Now, let us prove the Lemma.

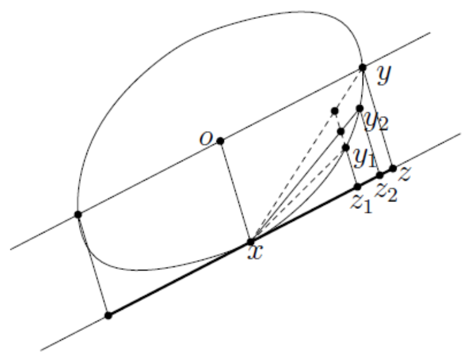

Figure 2: Illustration for Lemma 3.1.

1. By the definitions, we have $\lambda_X^+(0) = \lambda_X^-(0) = 0$.

2. The first two inequalities of assertion (ii) are trivial. By similarity, we have $\lambda_X(x, y, r) \leqslant r$. Indeed, $y_1 z_1 \parallel zy$ and $y_1 z_1 \subset \triangle xyz$. Taking the supremum we get assertion (ii).

3. Taking into account that $\mathfrak{B}_1(o)$ is convex, we get $y_1 z_1 \subset x y_2 z_2$. By construction we have that $y_1 z_1 \parallel z_2 y_2$. By the similarity, we get $\|y_2 z_2\| \geqslant \frac{r_2}{r_1}\|y_1 z_1\|$, i.e. $\frac{r_2}{r_1}\lambda_X(x, y, r_1) \leqslant \lambda_X(x, y, r_2)$. Taking the infimum in $\lambda_X(x, y, r_2)$, we complete the proof of inequality (3.1).
   By the convexity of the unit ball, we obtain that segment $y_2 z_2$ lies in trapezoid $y_1 z_1 z y$. By construction $y_2 z_2 \parallel y_1 z_1 \parallel yz$. By similarity, we get
   $$\|y_2 z_2\| - \|y_1 z_1\| \leqslant (1 - \|y_1 z_1\|)\frac{r_2 - r_1}{1 - r_1} \leqslant \frac{r_2 - r_1}{1 - r_1}.$$
   Taking the infimum in $\|y_1 z_1\| \to \lambda_X^-(r_1)$, we have $\|y_2 z_2\| - \lambda_X^-(r_1) \leqslant \frac{r_2 - r_1}{1 - r_1}$. This yields (3.2).

4. Assertion (iv) is the direct consequence of assertion (iii).

5. The function $\lambda_X^+(\cdot)$ is the supremum of the convex functions, therefore it's convex. Since $\lambda_X^+(\cdot)$ is a convex bounded function and $\lambda_X(x, y, r)$ is continuous in r, we obtain that $\lambda_X^+(\cdot)$ is continuous on $[0,1]$. We will prove that $\lambda_X^+(r) > 0$ on $(0,1]$ in Lemma 4.2 below. By this and the equality $\lambda_X^+(0) = 0$ and convexity of the modulus of supporting smoothness, we get that it is a strictly increasing function.

The inequality $\lambda^+_X(r) \leqslant r$ was proved in assertion (ii). The equality $\lambda^+_X(1) = 1$ is the consequence of inequality (5.1) at $r = 1$, which will be proved below.

□

From Lemma 3.1 we have that in the definitions of the moduli of the supporting smoothness and supporting convexity one may choose $t = r$.

Remark 1. Since any two plane central sections of the unit ball in a Hilbert space H are equal, we have

$$\lambda^+_H(r) = \lambda^-_H(r) = \delta_H(2r) = 1 - \sqrt{1 - r^2}.$$

4 Comparison of supporting moduli with the moduli of convexity and smoothness

Theorem 4.1. *Let X be an arbitrary Banach space. Then $\lambda^-_X(\varepsilon) \asymp \delta_X(\varepsilon)$ as $\varepsilon \to 0$ and for any $r \in [0; 1]$:*

$$\delta_X(r) \leqslant \lambda^-_X(r) \leqslant \delta_X(2r)\,. \tag{4.1}$$

Proof. 1) By the definition of the modulus of supporting convexity for any $\varepsilon > 0$ there exists a parallelogram $xyzd$ such that $x, z \in \partial\mathfrak{B}_1(o)$, the point d lies in the segment xo and $\|xy\| = r$, $xy \urcorner ox$, $\|yz\| \leqslant \lambda^-_X(r) + \varepsilon$. Therefore $\|od\| = 1 - \|yz\|$, consequently $\delta_X(r) = \delta_X(\|zd\|) \leqslant \|yz\| \leqslant \lambda^-_X(r) + \varepsilon$. Passing to the limit as $\varepsilon \to 0$, we obtain the left-hand side of chain (4.1).
2) Let us prove the right-hand side of chain (4.1).
Fix $r \in (0, 1)$ (if $r = 0$ or $r = 1$ the inequality is trivial). By the definition of the modulus of supporting convexity for any $\varepsilon > 0$ there exist points $a_\varepsilon, b_\varepsilon$ on the unit sphere such that $\|a_\varepsilon b_\varepsilon\| \geqslant 2r$ and for the point $c_\varepsilon = \frac{a_\varepsilon + b_\varepsilon}{2}$ the following inequality holds:

$$1 - \|oc_\varepsilon\| \leqslant \delta_X(2r) + \varepsilon. \tag{4.2}$$

Let the ray oc_ε intersect the unit sphere in a point x. Denote by l_1 the supporting line to the unit sphere such that l_1 lies in the plane $oa_\varepsilon b_\varepsilon$ and $x \in l_1$. Let l_2 be a line such that $l_1 \parallel l_2$ and $c_\varepsilon \in l_2$. Denote by f, g the points of intersections of $\partial\mathfrak{B}_1(o)$ and the line l_2. From Lemma 2.2 it follows that $\|f - c_\varepsilon\| \geqslant r$ or $\|g - c_\varepsilon\| \geqslant r$. Without loss of generality, put $\|g - c_\varepsilon\| \geqslant r$. Let l_3 be a line such that $l_3 \parallel oc_\varepsilon$ and $g \in l_\varepsilon$. By definition, we put $y = l_3 \cap l_1$ (see Fig. 3). Then

$$\delta_X(2r) + \varepsilon \geqslant \|c_\varepsilon x\| \geqslant \lambda^-_X\left(x, \frac{y-x}{\|y-x\|}, \|y-x\|\right) \geqslant \lambda^-_X\left(x, \frac{y-x}{\|y-x\|}, r\right) \geqslant \lambda^-_X(r)\,,$$

i.e., $\delta_X(2r) + \varepsilon \geqslant \lambda^-_X(r)$. Passing to the limit as $\varepsilon \to 0$, we complete the proof. □

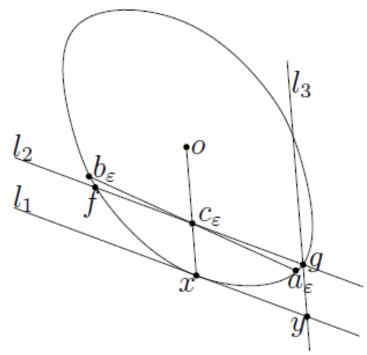

Figure 3: Illustration for Theorem 4.1.

Lemma 4.1. *Let $r \in [0, \frac{1}{2}]$. Then*

$$\lambda_X^+(r) \leqslant \rho_X(2r)\,. \tag{4.3}$$

Proof. Denote $\lambda = \lambda_X^+(r)$. Since $\lambda_X^+(r) \leqslant r$ for any $r \in [0,1]$, then $\lambda \leqslant \frac{1}{2}$. Let $\mu \in (0,\lambda)$. By the Definitions 1, 2 there exist $x, y \in \partial\mathfrak{B}_1(o)$ such that $y\urcorner x$ and $\lambda_X(x,y,r) = \mu' \in (\mu,\lambda)$, and consequently $\|x + ry - \mu' x\| = 1$. Since $y\urcorner x$ there exists $p \in J_1(x) = J_1(x - \mu' x)$ such that $\langle p, y\rangle = 0$.

Using Lemma 2.3, we get

$$1 = \|x + ry - \mu' x\| \leqslant \|x - \mu' x\| + \langle p, ry\rangle + 2(1-\mu')\rho_X\left(\frac{r}{1-\mu'}\right) =$$
$$= 1 - \mu' + 2(1-\mu')\rho_X\left(\frac{r}{1-\mu'}\right).$$

To complete the proof, it suffices to note that $\mu' < \frac{1}{2}$, $\rho_X(0) = 0$ and the modulus of smoothness is a convex function. □

Lemma 4.2. *Let $r \in [0,1]$. Then*

$$\rho_X\left(\frac{r}{2}\right) \leqslant \lambda_X^+(r)\,. \tag{4.4}$$

Proof. Taking into account the definition of the modulus of smoothness, it follows that for any $\tau \in \left[0, \frac{1}{2}\right]$ and $\varepsilon \in [0, \rho_X(\tau))$ there exist x and y such that the following inequality is true

$$\|x + \tau y\| + \|x - \tau y\| - 2 \geqslant 2(\rho_X(\tau) - \varepsilon). \tag{4.5}$$

Without loss of generality, we can assume that $\|x + \tau y\| \geqslant \|x - \tau y\|$ (therefore $\|x + \tau y\| \geqslant 1$). Denote $u = \frac{x+\tau y}{\|x+\tau y\|}$, $v = \frac{x-\tau y}{\|x-\tau y\|}$. By Lemma 2.4, we obtain

$$\|u - v\| \leqslant \frac{4\tau}{\|x + \tau y\|}; \tag{4.6}$$

By the triangle inequality, we get

$$\|u+v\| \leqslant \frac{2\,\|x\|}{\|x+\tau y\|} + \|x-\tau y\| \left| \frac{1}{\|x+\tau y\|} - \frac{1}{\|x-\tau y\|} \right|$$
$$= 2 - \frac{1}{\|x+\tau y\|}(\|x+\tau y\| + \|x-\tau y\| - 2).$$

Now, by inequality (4.5), we have that

$$\|u+v\| \leqslant 2 - \frac{2(\rho_X(\tau)-\varepsilon)}{\|x+\tau y\|}. \tag{4.7}$$

Let us consider the plane ouv. By ω denote a point lying on the smallest arc uv of the unit circle such that the supporting line to the unit ball at ω is parallel to uv. Obviously, either $\lambda_X\left(\omega, \frac{u-v}{\|u-v\|}, \frac{\|u-v\|}{2}\right) \geqslant 1 - \frac{\|u+v\|}{2}$ or $\lambda_X\left(\omega, -\frac{u-v}{\|u-v\|}, \frac{\|u-v\|}{2}\right) \geqslant 1 - \frac{\|u+v\|}{2}$, i.e. $\lambda_X^+\left(\frac{\|u-v\|}{2}\right) \geqslant 1 - \frac{\|u+v\|}{2}$. Combining this with inequality (4.7), we get

$$\frac{2(\rho_X(\tau)-\varepsilon)}{\|x+\tau y\|} \leqslant 2\lambda_X^+\left(\frac{\|u-v\|}{2}\right).$$

Now, by inequality (4.6), we obtain

$$\frac{2}{\|x+\tau y\|}(\rho_X(\tau)-\varepsilon) \leqslant 2\lambda_X^+\left(\frac{2\tau}{\|x+\tau y\|}\right) \leqslant \frac{2}{\|x+\tau y\|}\lambda_X^+(2\tau)\,.$$

Multiplying both sides by $\frac{\|x+\tau y\|}{2}$ and passing to the limit as $\varepsilon \to 0$, we obtain (4.4). □

Remark 2. By Lemma 4.2 and the properties of the modulus of smoothness, it follows that $\lambda_X^+(r) > 0$ for all $r > 0$.

By Lemmas 4.1, 4.2 and the properties of the modulus of smoothness we have the following result.

Theorem 4.2. *Let X be an arbitrary Banach space. Then $\lambda_X^+(\tau) \asymp \rho_X(\tau)$ as $\tau \to 0$ and for any $r \in [0, \frac{1}{2}]$:*

$$\rho_X\Big(\frac{r}{2}\Big) \leqslant \lambda_X^+(r) \leqslant \rho_X(2r)\,.$$

5 Comparison with the Banaś modulus

In the paper [1] J. Banaś defined and studied some new modulus of smoothness. Namely, he defined

$$\delta_X^+(\varepsilon) = \sup\left\{1 - \frac{\|x+y\|}{2} \;:\; x, y \in \mathfrak{B}_1(o),\ \|x-y\| \leqslant \varepsilon\right\}, \quad \varepsilon \in [0,2].$$

The function $\delta_X^+(\cdot)$ is called the *Banaś modulus*. In the papers [1, 2, 3, 4] several interesting results concerning this modulus were obtained. Particulary, in [1], J. Banaś proved that a space is uniformly smooth iff $\frac{\delta_X^+(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. However, from the definition a space is uniformly smooth if and only if $\frac{\rho_X(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. This leads to the question: are the modulus of smoothness and the modulus of Banaś equivalent at zero? It is easy to check that there exist positive constant a, b such that $\delta_X^+(t) \leqslant a\rho_X(bt)$, but the lower estimate of the modulus of Banaś in terms of the modulus of smoothness is unknown. In the next theorem we prove that the modulus of Banaś and the modulus of supporting smoothness are equivalent at zero, so Theorem 4.2 answers the above question.

Theorem 5.1. *Let X be an arbitrary Banach space. Then $\delta_X^+(\varepsilon) \asymp \delta_X(\varepsilon)$ as $\varepsilon \to 0$ and the following inequalities hold:*

$$\delta_X^+(2r) \le \lambda_X^+(r) \qquad \forall r \in [0,1]; \tag{5.1}$$

$$\lambda_X^+(r) \le 2\delta_X^+(3r) \qquad \forall r \in \left[0, \frac{2}{3}\right]. \tag{5.2}$$

Proof. 1) First we shall prove inequality (5.1) for $r \in [0,1)$.
Let a, b be points of the unit sphere such that $\|a-b\| \leqslant 2r$. By X_2 denote the plane aob.

There exists a point y_2 of the unit sphere of the plane X_2 such that the supporting line l_2 to the unit ball at this point is parallel to ab. By definition, put $y_1 = oy_2 \cap ab$. There exists a point a_2 in the projection of the point a on l_2 such that the segments y_1y_2, aa_2 are equal in length and parallel. The point b_2 is defined in the same way, such that y_1y_2 and bb_2 are parallel (see Fig. 6). Without loss of generality we assume that $\|y_2a_2\| \leqslant r < 1$. Since the modulus of supporting smoothness is an increasing function, we have $\|y_1y_2\| = \|aa_2\| \leqslant \lambda_X^+(y_2, \|y_2a_2\|) \leqslant \lambda_X^+(y_2, r)$. Taking the supremum, we obtain inequality (5.1).

Taking into account that the modulus of Banaś is a continuous and increasing function, we obtain inequality (5.1) for $r = 1$.

2) Let us prove inequality (5.2).
By the definition of modulus of supporting smoothness for any $\varepsilon \in (0, \lambda_X^+(r))$ there exist
– a point $x \in \partial\mathfrak{B}_1(o)$;
– a line ℓ_1 supporting to the unit ball at point x;
– a point y on ℓ_1 and a point $z \in \partial\mathfrak{B}_1(o)$ such that
$\|xy\| = r$, $\|yz\| > 0$, $zy \parallel ox$ and $\lambda^+\left(x, \frac{xy}{\|xy\|}, r\right) = \|yz\| > \lambda_X^+(r) - \varepsilon > 0$.

Let ℓ_2 be a line parallel to ℓ_1 such that $z \in \ell_2$. Let z, z_1 be points of the intersections of line ℓ_2 and $\partial\mathfrak{B}_1(o)$. By y_1 denote the projections of z_1 on ℓ_1 such that $z_1y_1 \parallel ox$ (see Fig. 4).

We shall prove that $\|zz_1\| \geqslant 2r$. In the converse case, $\|xy_1\| < r$. Note that if we fix $x, y \in \partial\mathfrak{B}_1(o)$ such that $y \urcorner x$, then the function $\lambda^+(x, y, \cdot)$ is strictly increasing on the set of its positive values. Since xy_1 and xy lie on the same line and by to the definition of λ^+, we obtain

$$\lambda^+\left(x, \frac{xy}{\|xy\|}, r\right) = \|yz\| = \|y_1z_1\| \leqslant \lambda^+\left(x, \frac{xy_1}{\|xy_1\|}, \|xy_1\|\right) <$$
$$< \lambda^+\left(x, \frac{xy_1}{\|xy_1\|}, r\right) = \lambda^+\left(x, \frac{xy}{\|xy\|}, r\right).$$

This is a contradiction, consequently $\|zz_1\| \geqslant 2r$.

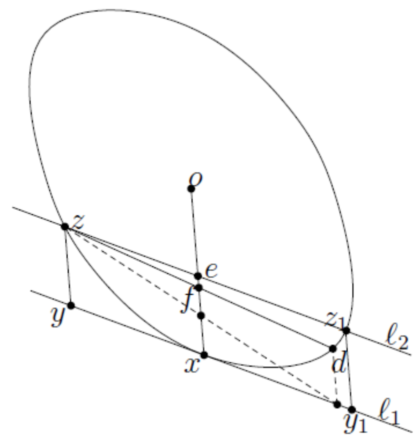

Figure 4: Illustration for the second part of Theorem 5.1.

By definition, put $e = ox \cap zz_1$. By the continuity reasons there exists a point d on the arc z_1x of the unit sphere such that for the point $f = zd \cap ox$ the following equality holds $\|d - f\| = \|f - z\|$. Since ℓ_1 is a supporting line to the unit sphere, we have $\|xf\| \geqslant \frac{\|yz\|}{2}$. Note that $\|dz\| \leqslant 2(\|ze\| + \|ef\|) \leqslant 3r$. Combining the last two inequalities, we get

$$\delta^+_X(3r) \geqslant \|xf\| \geqslant \frac{\lambda^+_X(r) - \varepsilon}{2}.$$

Passing to the limit as $\varepsilon \to 0$, we obtain inequality (5.2). □

From Theorems 4.2 and 5.1 we have the following corollary.

Corollary 5.1. *Let X be an arbitrary Banach space, then $\delta^+_X(\varepsilon) \asymp \rho_X(\varepsilon)$ as $\varepsilon \to 0$ and the following inequalities hold:*

$$\frac{1}{2}\rho_X\left(\frac{r}{6}\right) \leqslant \delta^+_X(r) \leqslant \rho_X(r)\,, \quad r \in \left[0, \frac{1}{2}\right].$$

The Day-Nordlander theorem (see [7]) asserts that $\delta_X(\varepsilon) \leqslant \delta_H(\varepsilon)$ for $\varepsilon \in [0, 2]$, where H denotes an arbitrary Hilbert space. On the other hand, repeating the arguments from the paper [7] we can show that for any Banach space the following estimate is true $\delta^+_H(\varepsilon) \leqslant \delta^+_X(r)$ for $\varepsilon \in [0, 2]$. From this and Theorems 4.2, 5.1 we obtain a Day–Nordlander type result for the moduli of supporting convexity and supporting smoothness:

Corollary 5.2. *Let X be an arbitrary Banach space. Then*

$$\lambda^-_X(r) \leqslant \lambda^-_H(r) = 1 - \sqrt{1-r^2} = \lambda^+_H(r) \leqslant \lambda^+_X(r) \qquad \forall r \in [0,1].$$

If at least one of these inequalities turns into equality, then X is a Hilbert space.

6 Estimates for Lipschitz constant for the metric projection onto a hyperplane

Let us introduce the following characteristic of a space:

$$\xi_X = \sup_{\substack{\|x\|=1,\\ \|y\|=1}} \sup_{p\in J_1(y)} \|x - \langle p, x\rangle y\|.$$

Note that if $y \in \partial\mathfrak{B}_1(0)$, $p \in J_1(y)$, then the vector $(x - \langle p, x\rangle y)$ is a metric projection of x onto the hyperplane $H_p = \{x \in X : \langle p, x\rangle = 0\}$. So, $\xi_X = \sup_{y\in\mathfrak{B}_1(o)} \sup_{p\in J_1(x)} \xi^p_X$, where ξ^p_X is half of diameter of a unit ball's projection onto the hyperplane H_p. Therefore, ξ_X is the maximal value of the Lipschitz constant for the metric projection operator onto a hyperplane. Obviously, $\xi_X \leq 2$ and $\xi_H = 1$ for a Hilbert space H.

Theorem 6.1. *For any Banach space X the following inequality is true:*

$$\frac{1}{1-\lambda^-_X\left(\frac{1-\lambda^-_X(1)}{2}\right)} \leqslant \xi_X \leqslant \frac{1}{1-\lambda^+_X\left(\frac{1-\lambda^-_X(1)}{2}\right)}. \tag{6.1}$$

Proof. First let us introduce some notation. Let x_0 be an arbitrary point on the unit sphere. Let l be a supporting line to the unit ball at the point x_0. Define l_2 as the line such that the following conditions hold:
a) $l_2 \parallel ox_0$;
b) $l_2 \cap l \neq \emptyset$, by definition, put $x_2 = l_2 \cap l$;
c) l a is supporting line to the unit ball at some point y_2;
d) $\|y_2x_2\| \leqslant 1$.
Let x_1 be a point on segment x_0x_2 such that $\|x_0x_1\| = 1$, let l_1 be a line such that $x_1 \in l_1$ and $l_1 \parallel ox_0$. By definition, put y_1 as the intersection point of line l_1 and the segment oy_2. Let b be a point on $\partial\mathfrak{B}_1(o)$ such that the segment ob is parallel to x_0x_1. By construction, we have that x_0x_1bo is a parallelogram, therefore $b \in l_1$ and $y_1 \in x_1b$. Let a be the intersection point of the line l_1 and the unit sphere such that $a \in x_1y_1$. From the intercept theorem, we have $\frac{\|x_0x_2\|}{\|oy_2\|} = \frac{\|x_0x_1\|}{\|oy_1\|}$. Therefore

$$\|x_0x_2\| = \frac{1}{\|oy_1\|} = \frac{1}{1-\|y_1y_2\|}. \tag{6.2}$$

Since x_0x_1bo is a parallelogram, we get $\|x_1b\| = \|ox_0\| = 1$. By construction we have that $\|x_0x_1\| = 1$. Therefore,

$$\|ab\| \leqslant 1 - \lambda^-_X(1)\,. \tag{6.3}$$

Define a_2 as the projection of the point a on l_2 such that $aa_2 \parallel oy_2$. In the same way we

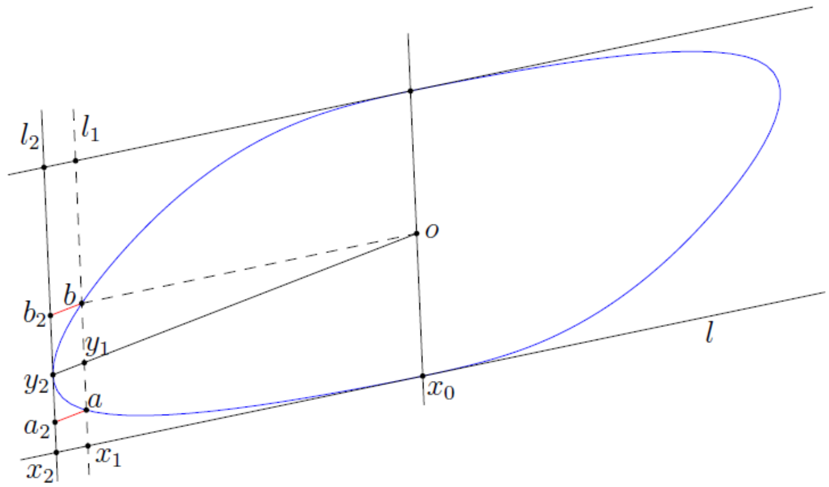

Figure 5: Illustration for Theorem 6.1.

define the point b_2. Then the segments y_1y_2, aa_2 and bb_2 are parallel and equal in length (as parallel segments between two parallel lines). By the definition of the modulus of supporting convexity and by inequality (6.3), we obtain

$$\|y_1y_2\| \leqslant \lambda^+_X(\min\{\|a_2y_2\|, \|y_2b_2\|\}) \leqslant \lambda^+_X\left(\frac{\|ab\|}{2}\right) \leqslant \lambda^+_X\left(\frac{1-\lambda^-_X(1)}{2}\right). \tag{6.4}$$

Combining this and equality (6.2), we finally prove the right-hand side of inequality (6.1).

Let ε be an arbitrary positive real number. Note that we could choose a point x_0 such that $\|x_1a\| \leqslant \lambda^-_X(1)+\varepsilon$, i.e. $\|ab\| \geqslant 1-\lambda^-_X(1)-\varepsilon$. Like in (6.4), we obtain

$$\|y_1y_2\| \geqslant \lambda^-_X(\max\{\|a_2y_2\|, \|y_2b_2\|\}) \geqslant \lambda^-_X\left(\frac{\|ab\|}{2}\right) \geqslant \lambda^-_X\left(\frac{1-\lambda^-_X(1)-\varepsilon}{2}\right).$$

Passing to limit as $\varepsilon \to 0$ and using inequality (6.2), we prove the left-hand side of inequality (6.1). □

Remark 3. The estimate (6.1) is reached in case of a Hilbert space. The right-hand side of inequality (6.1) does not exceed 2, i.e. this estimate is not trivial.

Conjecture 6.1. *The right-hand side of inequality (6.1) becomes an equality in case of L_p, $p \in (1; +\infty)$.*

In the following lemma we obtain a lower estimate of the modulus of supporting smoothness by the inverse function to the modulus of convexity.

Lemma 6.1. *For any $r \in [0,1]$ the following inequalities hold:*

$$1-\frac{1}{2}\delta^{-1}_X\left(1-\frac{r}{2}\right) \leqslant 1-\frac{1}{2}\delta^{-1}_X\left(1-\frac{r}{\xi_X}\right) \leqslant \lambda^+_X(r). \tag{6.5}$$

Proof. The left-hand side of inequality (6.5) is a straightforward consequence of the inequality $\xi_X \leq 2$. Let us prove the right-hand side of inequality (6.5). In case of $r = 0$ it is trivial. Let x_0 be an arbitrary point on the unit sphere. Define H_x as a supporting hyperplane to the unit ball at the point x_0. Let x_1 be a point of the supporting hyperplane H_x such that $\|x_0x_1\| = r$. Denote the ray $\{ox_0 + \alpha x_0x_1 : \alpha \geqslant 0\}$ as ℓ. Let l_1, l_2 be the lines parallel to ox_0 such that
a) l_2 is a supporting line to the unit ball at the point y_2 and $l_2 \cap \ell = x_2$;
b) l_1 intersects the ray ℓ at x_1 and intersects the unit sphere at points a, b.
Let $y_1 = oy_2 \cap ab$ (see Fig. 6).

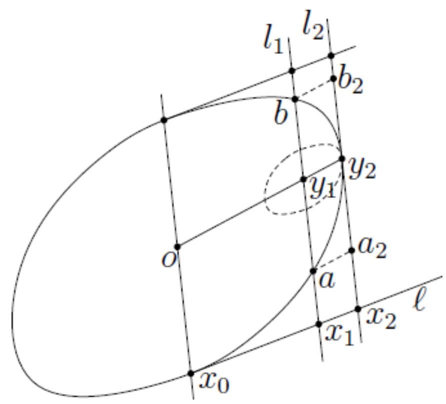

Figure 6: Illustration for Lemma 6.1 and for the first part of Theorem 5.1.

By the definition of $\lambda_X^+(r)$ and since the unit ball is centrally symmetric, we get $\|ab\| \geqslant 2(1 - \lambda_X^+(r))$. Obviously, $\|y_1y_2\| \geqslant \delta_X(\|ab\|)$. Consequently, we have

$$\delta_X\big(2(1 - \lambda_X^+(r))\big) \leqslant \delta_X(\|ab\|) \leqslant \|y_1y_2\|\,. \tag{6.6}$$

Using the intercept theorem, we obtain

$$\|y_1y_2\| = \frac{\|y_1y_2\|}{\|oy_2\|} = \frac{\|x_1x_2\|}{\|x_0x_2\|} = \frac{\|x_0x_2\| - \|x_0x_1\|}{\|x_0x_2\|} = 1 - \frac{r}{\|x_0x_2\|} \leqslant 1 - \frac{r}{\xi_X}. \tag{6.7}$$

By inequalities (6.6) and (6.7), we have

$$\delta_X\big(2(1 - \lambda_X^+(r))\big) \leqslant 1 - \frac{r}{\xi_X}$$

□

It is easy to check that in a Hilbert space H the following equality holds

$$\delta_H^{-1}(\tau) = 2\sqrt{1 - (1 - \tau)^2}.$$

Substituting this in inequality (6.5) and since $\xi_H = 1$, we obtain

$$\delta_H(2r) = 1 - \frac{1}{2}\delta_H^{-1}(1 - r) \leqslant \lambda_H^+(r).$$

According to (1), we have that if X is a Hilbert space, then the right hand estimate in inequality (6.5) is reached.

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Grigory Ivanov
Department of Higher Mathematics
Moscow Institute of Physics and Technology
Institutskii pereulok 9
141700 Dolgoprudny, Russia
and
School of Applied Mathematics and Information Science
National Research University Higher School of Economics
Bolshoi Trekhsvyatitelskiy 3
109028 Moscow, Russia
E-mails: grimivanov@gmail.com

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