Short communications

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INTEGRABILITY OF THE FOURIER TRANSFORMS OF α-MONOTONE FUNCTIONS

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Abstract. We define a class of α-monotone functions. We study of integrability conditions of the Fourier transforms of functions from this class.

In this paper we study integrability conditions for the cosine Fourier transforms

$$\widehat{f}(y) = \int_0^\infty f(x)\cos xy\ dx$$

of α-monotone functions.

Let us give several definitions. Let μ be Lebesgue measure on $\mathbb{R}_+$, f be a μ-measurable function on $\mathbb{R}_+$, then by f^* we denote the non-increasing rearrangement of f,

$$f^*(t) = \inf\{\sigma : \mu\{x \in \mathbb{R}_+ : |f(x)| > \sigma\} \leqslant t\}.$$

Definition 1. Let $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. Then the Lorentz spaces $L_{p,q}(\mathbb{R}_+)$ is the set of μ-measurable functions f for which, the functional

$$\|f\|_{L_{p,q}} := \begin{cases} \left(\int\limits_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{for } 0 < p < \infty \ \textit{and} \ 0 < q < \infty, \\ \sup\limits_{t\geqslant 0} t^{\frac{1}{p}} f^*(t) & \text{for } 0 < p \leqslant \infty \ \textit{and} \ q = \infty, \end{cases}$$

is finite.

In [8], Y. Sagher proved the following theorem.

Theorem A. *Let $f(x)$ be a non-negative on $\mathbb{R}_+$, non-increasing to 0 on $(0,+\infty)$ function. Then*

$$\|f\|_{L_{p,q}} \sim \|\widehat{f}\|_{L_{p',q}}, \quad 1 < p < \infty, 0 < q \leqslant \infty, \tag{1}$$

Throughout this paper, p' denotes the conjugate index of $p : \frac{1}{p}+\frac{1}{p'} = 1$. Also, we denote by C positive constant that may be different on different occasions. In addition, $T \sim S$ means that $\frac{1}{C}S \leqslant T \leqslant CS$.

Also, Y. Sagher in [8] proved analogical result for weighted Lebesgue spaces. There are several results related with Sagher's results. The relation (1) of Theorem A was generalized by E.Nursultanov in [6] (see also [7]) for weak monotone functions in the Lorentz spaces $L_{p,q}$, $1 < p < 2$. Later A. Kopezhanova, E. Nursultanov, and L.-E. Persson in [3] generalized (1) for weak monotone functions in the Lorentz spaces with general weights. The generalizations of Y. Sagher's result for weighted Lebesgue spaces were obtained in [4], [2], [1].

The main goal of this paper is to generalize Theorem A replacing the monotonicity condition by the weaker condition of α-monotonicity.

We will need the following definition of Riemann-Liouville's fractional integrals and derivatives(see [9]).

Definition 2. Let $f(x)$ be a measurable on $(0,\infty)$, $0 < \alpha < 1$. Then the integral

$$I^\alpha f(x) = (I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt \quad \text{for } x > 0.$$

is called fractional integral of the order α. By $\mathcal{D}^\alpha f(x)$ we denote the fractional derivative of the order α $(0 < \alpha < 1)$, i.e.,

$$\mathcal{D}^\alpha f(x) = (\mathcal{D}_-^\alpha f)(x) = -\frac{d}{dx}(I^{1-\alpha} f)(x).$$

Remark 1. If $\alpha = 1$, then the fractional derivative $\mathcal{D}^1$ is understood as follows: $\mathcal{D}^1 f(x) := -f'(x)$. Also, the fractional integral I^0 is understood as: $I^0 f(x) := f(x)$.

Definition 3. Let $0 < \alpha \leqslant 1$. We say that a non-negative function f is α-monotone (or belongs to the class M_α), if $I^{1-\alpha} f(x)$ is non-increasing, local absolutely continuous function on $(0,+\infty)$.

The main result of this paper is the following

Theorem 1. *Let $\alpha \in (0,1]$, $\frac{1}{\alpha} < p < \infty$, $1 \leqslant q < \infty$. Let also $f \in M_\alpha$ and $\widehat{f}(y) = \int_0^\infty f(x) \cos xy \, dx$, $y > 0$. Then*

$$\|\widehat{f}\|_{L_{p,q}} \sim \left(\int_0^\infty \left(t^{\frac{1}{p'}} \overline{f}(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

where

$$\overline{f}(t) = \sup_{y \geqslant t} \frac{1}{y} \left| \int_0^y f(s) ds \right|.$$

Remark 2. From assertion 1 in [5] it follows that there exists $C > 0$ such that

$$\left(\int_0^\infty \left(t^{\frac{1}{p'}}\overline{f}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \leqslant C\|f\|_{L_{p'q}}.$$

Remark 3. If $\alpha = 1$, then the results of Theorem 3 coincide with the results of Theorem A for non-icreasing, absolutely continuous functions.

Indeed, if $f(x)$ is a non-increasing function, then $f^*(t) = f(t)$ a.e. on $[0,\infty)$. Hence,

$$\overline{f}(t) = \sup_{y\geqslant t}\frac{1}{y}\left|\int_0^y f(s)ds\right| \geqslant \frac{1}{t}\int_0^t f(s)ds \geqslant f(t) = f^*(t).$$

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