EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 6, Number 1 (2015), 123 – 131

ON THE COMPLETENESS AND MINIMALITY OF SETS OF BESSEL FUNCTIONS IN WEIGHTED L^2-SPACES

B.V. Vynnyts'kyi, R.V. Khats'

Communicated by E.D. Nursultanov

Key words: Bessel function, entire function, complete system, minimal system, biorthogonal system.

AMS Mathematics Subject Classification: Primary 30B60, 33C10, 34B30, 42A65; Secondary 30D10, 30D20, 44A15, 46E30.

Abstract. We establish a criterion for the completeness and minimality of the system $(x^{-p-1}\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N})$ in the space $L^2((0;1);x^{2p}dx)$ where J_ν is the Bessel function of the first kind of index $\nu \geqslant 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of distinct nonzero complex numbers.

1 Introduction

Let $\alpha \in \mathbb{R}$ and $L^2((0;1);x^\alpha dx)$ be the space of measurable functions $f : (0;1) \to \mathbb{C}$ such that the function $t \to t^{\alpha/2}f(t)$ belongs to $L^2(0;1)$; the inner product and the norm in $L^2((0;1);x^\alpha dx)$ are given respectively by $\langle f_1; f_2\rangle = \int_0^1 t^\alpha f_1(t)\overline{f_2(t)}\,dt$ and $\|f\|^2 := \int_0^1 t^\alpha |f(t)|^2\,dt$. Let $J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$ be Bessel's function of the first kind of index ν. It is well known that the function J_ν for $\nu > -1$ has an infinite number of real zeros, among them positive zeros ρ_k, $k \in \mathbb{N}$, and negative zeros $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$ (see [8, p. 94], [16, p. 350], [19, p. 483]). All zeros are simple except, perhaps, the zero $\rho_0 = 0$. A system $(e_k : k \in \mathbb{N}_0)$ of the Hilbert space is said to be minimal (see [9, p. 131], [10, p. 4258], [12]) if for each $n \in \mathbb{N}_0$ the element e_n does not belong to the closure of the linear span of the system $(e_k : k \in \mathbb{N}_0 \setminus \{n\})$. A system is minimal if and only if it has a biorthogonal system (see [10, p. 4258], [12]).

Various approximation properties of the system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ were studied in a number of papers (see, for instance, [2], [6, p. 40], [7], [8, p. 97], [13], [16, p. 357], [17]–[19]). In particular, it is well known that the system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ is an orthogonal basis for the space $L^2(0;1)$ if $\nu > -1$ (see [7], [16, p. 356], [19]). From this it follows the following statement (see [7], [16, p. 357], [19]).

Theorem A. *Let $\nu > -1$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of positive zeros of J_ν. Then the system $(\Theta_{k,\nu} : k \in \mathbb{N})$, $\Theta_{k,\nu}(x) := x^{-\nu}J_\nu(x\rho_k)$, is complete and minimal in $L^2((0;1);x^{2\nu+1}dx)$.*

Using $J_\nu(z) = O(z^\nu)$ as $z \to 0+$ (see [8, p. 127], [19, p. 43]) we have that the function $t \to t^{-p-1}\sqrt{t\rho}J_\nu(t\rho)$ belongs to $L^2((0;1);x^{2p}dx)$ for each $\nu \in (0;+\infty)$, $p \in \mathbb{R}$ and $\rho \in \mathbb{C}$. We obtain the following result.

Theorem 1.1. *Let $\nu \geqslant 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of positive zeros of J_ν. Then the system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$, $\Theta_{k,\nu,p}(x) := x^{-p-1}\sqrt{x\rho_k}J_\nu(x\rho_k)$, is complete and minimal in $L^2((0;1);x^{2p}dx)$.*

If the system $(\Theta_{k,\nu} : k \in \mathbb{N})$ is complete and minimal in $L^2((0;1);x^{2\nu-1}dx)$ then it is complete and minimal in $L^2((0;1);x^{2\nu+1}dx)$. Therefore Theorem A follows from Theorem 1.1 for $\nu \geqslant 1/2$. Moreover, we establish a criterion for the completeness and minimality of the system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ in $L^2((0;1);x^{2p}dx)$ if $\nu \geqslant 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ is an arbitrary sequence of distinct nonzero complex numbers (see Theorem 3.4).

2 Preliminaries

To prove our main results we need the following auxiliary lemmas.

Lemma 2.1. (see [1], [4], [15]) *Let $\nu \geqslant -1/2$. A function F has the representation*

$$F(z) = \int_0^1 z^{-\nu}\sqrt{t}J_\nu(zt)\gamma(t)\,dt$$

with $\gamma \in L^2(0;1)$ if and only if it is an even entire function of exponential type $\sigma \leqslant 1$ such that $z^{\nu+1/2}F(z) \in L^2(0;+\infty)$.

Lemma 2.2. (see [3, p. 67], [14, p. 212]) *Let $\nu > -1$. Then every function $f \in L^2(0;+\infty)$ can be represented in the form*

$$f(z) = \int_0^{+\infty} \sqrt{zt}J_\nu(zt)h(t)\,dt$$

with some function $h \in L^2(0;+\infty)$. In this case,

$$h(t) = \int_0^{+\infty} \sqrt{zt}J_\nu(zt)f(z)\,dz,$$

and $\|f\| = \|h\|$.

3 Main results

Theorem 3.1. *Let $\nu \geqslant 1/2$. An entire function Ω has the representation*

$$\Omega(z) = \int_0^1 z^{-\nu}\sqrt{t}J_\nu(tz)t^{-1}q(t)\,dt \tag{3.1}$$

with $q \in L^2(0;1)$ if and only if it is an even entire function of exponential type $\sigma \leqslant 1$ such that $z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \in L^2(0;+\infty)$. In this case,

$$q(t) = \int_0^{+\infty} \sqrt{tz}J_{\nu-1}(tz)z^{-\nu+1/2}(z^{2\nu}\Omega(z))'\,dz. \tag{3.2}$$

Proof. *Necessity.* Let Ω have the representation (3.1). Using Schwartz's inequality and a well-known formula [9, p. 6 (7)] for finding the type of an entire function in terms of its Taylor coefficients, by direct calculation we get that Ω is an even entire function of exponential type $\sigma \leqslant 1$. Since $(z^\nu J_\nu(z))' = z^\nu J_{\nu-1}(z)$ (see [8, p. 14], [19, p. 45]), we obtain

$$z^{-2\nu+1}(z^{2\nu}\Omega(z))' = \int_0^1 z^{-(\nu-1)}\sqrt{t}J_{\nu-1}(tz)q(t)\,dt.$$

Put $\Psi(z) := z^{\nu-1/2}z^{-2\nu+1}(z^{2\nu}\Omega(z))' = z^{-\nu+1/2}(z^{2\nu}\Omega(z))'$. Since $\nu - 1 \geqslant -1/2$ then by Lemma 2.1 the function Ψ belongs to $L^2(0;+\infty)$. *Sufficiency.* Since Ω is an even function, then Ω' is an odd function. Therefore $z^{-2\nu+1}(z^{2\nu}\Omega(z))'$ is an even entire function of exponential type $\sigma \leqslant 1$. Hence, according to Lemma 2.1, there exists the function $q \in L^2(0;1)$ such that

$$z^{-2\nu+1}Q'(z) = \int_0^1 z^{-(\nu-1)}\sqrt{t}J_{\nu-1}(tz)q(t)\,dt, \quad Q'(z) = \int_0^1 z^{\nu}\sqrt{t}J_{\nu-1}(tz)q(t)\,dt,$$

where $Q(z) := z^{2\nu}\Omega(z)$. Since $Q(0) = 0$ then using Fubini's theorem, we get

$$Q(z) = \int_0^z dw \int_0^1 w^\nu \sqrt{t}J_{\nu-1}(wt)q(t)\,dt = \int_0^1 \sqrt{t}t^{-\nu-1}q(t)\,dt \int_0^z t(wt)^\nu J_{\nu-1}(wt)\,dw.$$

Further, using $((tw)^\nu J_\nu(tw))'_w = t(tw)^\nu J_{\nu-1}(tw)$, we have $\int_0^z t(wt)^\nu J_{\nu-1}(wt)\,dw = (tz)^\nu J_\nu(tz)$. Thus

$$Q(z) = \int_0^1 z^\nu \sqrt{t}J_\nu(tz)t^{-1}q(t)\,dt,$$

whence it follows (3.1). Furthermore, if the function Ω is representable in the form (3.1), then

$$z^{-\nu+1/2}(z^{2\nu}\Omega(z))' = \int_0^1 \sqrt{tz}J_{\nu-1}(tz)q(t)\,dt.$$

Therefore by Lemma 2.2 we get (3.2). The theorem is proved. □

Let $\widetilde{E}_{0,2}$ be the class of the entire functions Ω that can be represented in the form (3.1), and let $E_{0,2}$ be the class of even entire functions Ω of exponential type $\sigma \leqslant 1$ such that $z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \in L^2(0;+\infty)$.

Corollary 3.1. $\widetilde{E}_{0,2} = E_{0,2}$.

Lemma 3.1. *Let $\nu \geqslant 1/2$ and $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ for $k \neq n$. For a system $(\Theta_{k,\nu,0} : k \in \mathbb{N})$ to be incomplete in the space $L^2(0;1)$ it is necessary and sufficient that a sequence $(\rho_k : k \in \mathbb{Z}\setminus\{0\})$, $\rho_{-k} := -\rho_k$, is a subsequence of zeros of some nonzero function $\Omega \in E_{0,2}$.*

Proof. According to the well-known completeness criterion [9, p. 131], the considered system is incomplete in $L^2(0;1)$ if and only if there exists a nonzero function $q \in L^2(0;1)$ such that

$$\int_0^1 \rho_k^{-\nu}\sqrt{x}J_\nu(x\rho_k)x^{-1}q(x)\,dx = 0$$

for all $k \in \mathbb{N}$. Therefore, taking into account Theorem 3.1, we obtain the required proposition. The lemma is proved. □

Lemma 3.2. *Let $\nu \geqslant 1/2$ and an entire function $\Omega \in E_{0,2}$ be defined by the formula (3.1). Then (here and so on by C_1, C_2, … we denote arbitrary positive constants) for all $z \in \mathbb{C}$, we have*

$$|\Omega(z)| \leqslant C_3(1+|z|)^{-\nu}\exp(|\Im z|).$$

Proof. Since [11, p. 125 (A.4)]

$$|\sqrt{z}J_\nu(z)| \leqslant C_1 e^{|\Im z|}\left(\frac{|z|}{1+|z|}\right)^{\nu+1/2}, \quad z \in \mathbb{C},$$

then applying Schwartz's inequality, for all $z \in \mathbb{C}$ we get

$$|\Omega(z)| \leqslant \frac{\|q\|}{|z|^{\nu+1/2}}\left(\int_0^1 \left|\sqrt{tz}J_\nu(tz)\right|^2 t^{-2}\,dt\right)^{1/2} \leqslant C_2\left(\int_0^1 e^{2t|\Im z|}\frac{t^{2\nu-1}}{(1+t|z|)^{2\nu+1}}\,dt\right)^{1/2}$$

$$\leqslant C_2\frac{e^{|\Im z|}}{|z|^\nu}\left(\int_0^{|z|}\frac{u^{2\nu-1}}{(1+u)^{2\nu+1}}\,du\right)^{1/2} = C_2\frac{e^{|\Im z|}}{|z|^\nu}\left(\frac{|z|^{2\nu}}{2\nu(1+|z|)^{2\nu}}\right)^{1/2} = \frac{C_3 e^{|\Im z|}}{(1+|z|)^\nu}.$$

The lemma is proved. □

Lemma 3.3. *Let $\nu \geqslant 1/2$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_m^2$ for $k \neq m$. Let a sequence $(\rho_k : k \in \mathbb{Z}\setminus\{0\})$, $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even entire function $G \notin E_{0,2}$ of exponential type $\sigma \leqslant 1$ for which on the rays $\{z : \arg z = \varphi_j\}$, $j \in \{1;2;3;4\}$, $\varphi_1 \in [0;\pi/2)$, $\varphi_2 \in [\pi/2;\pi)$, $\varphi_3 \in (\pi;3\pi/2]$, $\varphi_4 \in (3\pi/2;2\pi)$, we have*

$$|G(z)| \geqslant C_4(1+|z|)^{-\nu-1/2}\exp(|\Im z|). \tag{3.3}$$

Then the system $(\Theta_{k,\nu,0} : k \in \mathbb{N})$ is complete in $L^2(0;1)$.

Proof. Assume the converse. Then according to Lemma 3.1 there exists a nonzero entire function $\Omega \in E_{0,2}$ for which the sequence $(\rho_k : k \in \mathbb{Z}\setminus\{0\})$ is a subsequence of zeros. Let $V(z) = \Omega(z)/G(z)$. Then V is an even entire function of order $\tau \leqslant 1$ for which (see Lemma 3.2)

$$|V(z)| \leqslant C_5\sqrt{1+|z|}, \quad \arg z = \varphi_j, \quad j \in \{1;2;3;4\}.$$

Therefore according to the Phragmén-Lindelöf theorem (see [9, p. 39], [10, p. 4263]) the function V is a constant. Hence $\Omega(z) = C_6 G(z)$ and $\Omega \notin E_{0,2}$. Thus we have a contradiction and the proof of the lemma is completed. □

Lemma 3.4. *Let $\nu \geqslant 1/2$. If an even entire function L belongs to $E_{0,2}$ and has a root at a point $\rho \neq 0$, then the function $\widetilde{L}(z) = L(z)/(z^2-\rho^2)$ also belongs to $E_{0,2}$.*

Proof. Let $\Phi(z) := z^{-\nu+1/2}(z^{2\nu}L(z))'$. Then $\Phi \in L^2(0;+\infty)$. Further, the function $\widetilde{L}$ is an even entire function of exponential type $\sigma \leqslant 1$. Furthermore,

$$\widetilde{\Phi}(z) := z^{-\nu+1/2}(z^{2\nu}\widetilde{L}(z))' = \frac{\Phi(z)}{z^2-\rho^2} - 2z^{\nu+3/2}\frac{L(z)}{(z^2-\rho^2)^2},$$

$$\int_{1+|\rho|}^{+\infty}\left|\frac{\Phi(x)}{x^2-\rho^2}\right|^2 dx \leqslant C_7\int_{1+|\rho|}^{+\infty}|\Phi(x)|^2\,dx < +\infty,$$

and according to Lemma 3.2

$$\int_{1+|\rho|}^{+\infty}\left|x^{\nu+3/2}\frac{L(x)}{(x^2-\rho^2)^2}\right|^2 dx = \int_{1+|\rho|}^{+\infty}\frac{x^{2\nu+3}}{(x^2-\rho^2)^4}|L(x)|^2\,dx < +\infty.$$

Hence the function $\widetilde{\Phi}$ belongs to $L^2(0;+\infty)$. This concludes the proof of the lemma. □

Lemma 3.5. *Let $\nu \geqslant 1/2$. If an even entire function L has zeros at points $\rho_k \neq 0$, $k\in\mathbb{N}$, and the function $L(z)/(z^2-\rho_1^2)$ belongs to $E_{0,2}$, then the functions $L_k(z) := L(z)/(z^2-\rho_k^2)$ also belong to $E_{0,2}$ for every $k\in\mathbb{N}\setminus\{1\}$.*

Proof. Let $Q_k(z) = (\rho_k^2-\rho_1^2)\frac{L(z)}{(z^2-\rho_k^2)(z^2-\rho_1^2)}$. Then $Q_k(z) = (\rho_k^2-\rho_1^2)\frac{L_1(z)}{z^2-\rho_k^2}$ and $L_k = Q_k + L_1$. Therefore, taking into account the previous lemma, we get the required proposition. □

Lemma 3.6. *Let $\nu\geqslant 1/2$ and $(\rho_k : k\in\mathbb{N})$ be an arbitrary sequence of distinct nonzero complex numbers such that $\rho_k^2\neq\rho_m^2$ for $k\neq m$. If the sequence $(\rho_k : k\in\mathbb{N})$ is a subsequence of zeros of some even entire function G which has simple roots at all points ρ_k and the function $G(z)/(z^2-\rho_1^2)$ belongs to $E_{0,2}$, then the system $(\Theta_{k,\nu,0} : k\in\mathbb{N})$ has a biorthogonal system $(\gamma_k : k\in\mathbb{N})$ in $L^2(0;1)$. The biorthogonal system $(\gamma_k : k\in\mathbb{N})$ is formed, in particular, by the functions γ_k, defined by the equality*

$$\overline{\gamma_k(t)} = \int_0^{+\infty}\sqrt{tz}J_{\nu-1}(tz)z^{-\nu+1/2}(z^{2\nu}G_k(z))'\,dz, \tag{3.4}$$

where

$$G_k(z) := \frac{2G(z)}{\rho_k^{\nu-1/2}G'(\rho_k)(z^2-\rho_k^2)}.$$

Proof. According to Lemma 3.5 the functions G_k belong to $E_{0,2}$ for every $k\in\mathbb{N}$. Therefore there exist nonzero elements γ_k of the space $L^2(0;1)$ such that

$$G_k(z) = \int_0^1 z^{-\nu}\sqrt{t}J_\nu(tz)t^{-1}\gamma_k(t)\,dt,$$

and by Theorem 3.1 the functions γ_k can be found by (3.4). Moreover,

$$\rho_n^{\nu+1/2}G_k(\rho_n) = \begin{cases}1, & k=n,\\ 0, & k\neq n.\end{cases}$$

This completes the proof of the lemma. □

Theorem 3.2. *Let $\nu\geqslant 1/2$ and $(\rho_k : k\in\mathbb{N})$ be an arbitrary sequence of nonzero complex numbers such that $\rho_k^2\neq\rho_n^2$ for $k\neq n$. The system $(\Theta_{k,\nu,0} : k\in\mathbb{N})$ is complete and minimal in $L^2(0;1)$ if and only if the sequence $(\rho_k : k\in\mathbb{Z}\setminus\{0\})$, $\rho_{-k} := -\rho_k$, is a sequence of zeros of some even entire function $G\notin E_{0,2}$ such that the function $G(z)/(z^2-\rho_1^2)$ belongs to $E_{0,2}$.*

Proof. If the considered system is minimal then there exists a nonzero function $\gamma_1 \in L^2(0;1)$ such that

$$\int_0^1 \rho_k^{-\nu} \sqrt{t} J_\nu(t\rho_k) t^{-1} \gamma_1(t)\, dt = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1. \end{cases}$$

Let $T(z) = \int_0^1 z^{-\nu} \sqrt{t} J_\nu(tz) t^{-1} \gamma_1(t)\, dt$. The function $G(z) = (z^2 - \rho_1^2) T(z)$ is the required, because the function $T(z) = G(z)/(z^2 - \rho_1^2)$ belongs to $E_{0,2}$ and has zeros at all points ρ_k, all its zeros are simple and it has no other zeros. Indeed, if ρ is another root of the function G then the function $D(z) = G(z)/(z^2 - \rho^2)$ which has roots at all points ρ_k, would belongs to $E_{0,2}$ that, according to Lemma 3.1, contradicts the completeness of the considered system. Besides, the function G does not belongs to $E_{0,2}$, because otherwise the system would be incomplete. Conversely, if all the conditions of the theorem hold then, basing on Lemma 3.6, we obtain the required proposition. The proof of theorem is thus completed. □

Theorem 3.3. *Let $\nu \geqslant 1/2$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of positive zeros of J_ν. Then the system $(\Theta_{k,\nu,0} : k \in \mathbb{N})$ is complete and minimal in the space $L^2(0;1)$. There exists a biorthogonal system $(\gamma_k : k \in \mathbb{N})$ in this space which formed by the functions γ_k, defined by the formula*

$$\overline{\gamma_k(t)} = \frac{2}{\rho_k J_{\nu+1}^2(\rho_k)} t \sqrt{t\rho_k} J_\nu(t\rho_k). \tag{3.5}$$

Proof. The sequence $(\rho_k : k \in \mathbb{Z} \setminus \{0\})$, $\rho_{-k} := -\rho_k$, is a sequence of zeros of the function $G(z) := z^{-\nu} J_\nu(z)$ which is an even entire function of exponential type $\sigma \leqslant 1$ [5, p. 48]. Using $J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(|z|^{-3/2} e^{|\Im z|}\right)$ as $z \to \infty$ and $|\arg z| < \pi$ (see [8, p. 127], [16, p. 352], [19, p. 199]), we obtain (3.3). Further $z^{-\nu+1/2}(z^{2\nu} G(z))' = z^{-\nu+1/2}(z^\nu J_\nu(z))' = \sqrt{z} J_{\nu-1}(z)$. Furthermore, by [13, Lemma 2.1, p. 226] we have $\int_0^n x|J_{\nu-1}(x)|^2\, dx \geqslant C_8 n$ for $n \in \mathbb{N}$. Therefore $G \notin E_{0,2}$. Hence, by Lemma 3.3 the system $(\Theta_{k,\nu,0} : k \in \mathbb{N})$ is complete in $L^2(0;1)$. Moreover, since (see [8, pp. 96-97], [16, p. 347])

$$\int_0^1 t J_\nu(t\rho_k) J_\nu(t\rho_n)\, dt = \begin{cases} \frac{1}{2} J_{\nu+1}^2(\rho_n), & k = n, \\ 0, & k \neq n, \end{cases}$$

then

$$\int_0^1 t^{-1} \sqrt{t\rho_k} J_\nu(t\rho_k) \overline{\gamma_n(t)}\, dt = \frac{2\sqrt{\rho_k \rho_n}}{\rho_n J_{\nu+1}^2(\rho_n)} \int_0^1 t J_\nu(t\rho_k) J_\nu(t\rho_n)\, dt = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

Thus the system $(\Theta_{k,\nu,0} : k \in \mathbb{N})$ is also minimal in $L^2(0;1)$. This completes the proof of the theorem. □

Remark that by using Hankel's integral [19, p. 429 (4)] we can obtain formula (3.5) by Lemma 3.6.

Theorem 3.4. *Let $\nu \geqslant 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ for $k \neq n$. The system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ is complete and minimal in $L^2((0;1);x^{2p}dx)$ if and only if the sequence $(\rho_k : k \in \mathbb{Z}\setminus\{0\})$, $\rho_{-k} := -\rho_k$, is a sequence of zeros of some even entire function $G \notin E_{0,2}$ such that the function $G(z)/(z^2 - \rho_1^2)$ belongs to $E_{0,2}$.*

Proof. A system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ is complete and minimal in $L^2((0;1);x^{2p}dx)$ if and only if the system $(\Theta_{k,\nu,0} : k \in \mathbb{N})$ is complete and minimal in $L^2(0;1)$. Therefore by Theorem 3.2 we obtain the required proposition. □

Theorem 1.1 is an immediate consequence of the Theorem 3.3. Theorem 1.1 implies the following assertion.

Corollary 3.2. *Let $\nu \geqslant 1/2$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of positive zeros of J_ν. Then the system $(\Theta_{k,\nu} : k \in \mathbb{N})$ is complete and minimal in $L^2((0;1);x^{2\nu-1}dx)$.*

References

[1] N.I. Akhiezer, *To the theory of coupled integral equations,* Kharkov State University Press. 25 (1957), 5–31 (in Russian).

[2] R.P. Boas, H. Pollard, *Complete sets of Bessel and Legendre functions,* Ann. of Math. 48 (1947), no. 2, 366–384.

[3] M.M. Dzhrbashyan, *Integral transforms and representations of functions in the complex domain,* Nauka, Moscow, 1966 (in Russian).

[4] J.L. Griffith, *Hankel transforms of functions zero outside a finite interval,* J. Proc. Roy. Soc. New South Wales. 89 (1955), 109–115.

[5] G.R. Grozev, Q.I. Rahman, *Lagrange interpolation in the zeros of Bessel functions by entire functions of exponential type and mean convergence,* Methods Appl. Anal. 3 (1996), no. 1, 46–79.

[6] J.R. Higgins, *Completeness and basis properties of sets of special functions,* London-New York-Melbourne, Cambridge University Press, Cambridge, 1977.

[7] H. Hochstadt, *The mean convergence of Fourier-Bessel series,* SIAM Rev. 9 (1967), 211–218.

[8] B.G. Korenev, *Bessel functions and their applications,* Taylor Francis, Inc., London-New York, 2002.

[9] B.Ya. Levin, *Lectures on entire functions,* Transl. Math. Monogr. Amer. Math. Soc., Providence, R.I., 150 (1996).

[10] A.M. Sedletskii, *Analytic Fourier transforms and exponential approximations, I.* J. Math. Sci. 129 (2005), no. 6, 4251–4408.

[11] F. Serier, *The inverse spectral problem for radial Schrödinger operators on* $[0,1]$*,* J. Differential Equations. 235 (2007), no. 1, 101–126.

[12] I. Singer, *Bases in Banach spaces,* Vol. 1. Springer-Verlag, Berlin, 1970.

[13] K. Stempak, *On convergence and divergence of Fourier-Bessel series,* Electron. Trans. Numer. Anal. 14 (2002), 223–235.

[14] E.C. Titchmarsh, *Introduction to the theory of Fourier integrals,* 2nd ed. Clarendon Press, Oxford, 1948.

[15] K.R. Unni, *Hankel transforms and entire functions,* Bull. Amer. Math. Soc. 71 (1965), no. 3, 511–513.

[16] V.S. Vladimirov, *Equations of mathematical physics,* Nauka, Moscow, 1981 (in Russian).

[17] B.V. Vynnyts'kyi, R.V. Khats', *Completeness and minimality of systems of Bessel functions,* Ufa Math. J. 5 (2013), no. 2, 132–141.

[18] B.V. Vynnyts'kyi, R.V. Khats', *Some approximation properties of the systems of Bessel functions of index* $-3/2$*,* Mat. Stud. 34 (2010), no. 2, 152–159.

[19] G.N. Watson, *A treatise on the theory of Bessel functions,* Cambridge University Press, Cambridge, 1944.

Bohdan V. Vynnyts'kyi, Ruslan V. Khats'
Institute of Physics, Mathematics, Economics and Innovation Technologies
Drohobych Ivan Franko State Pedagogical University
3 Stryis'ka St.
82100 Drohobych, Ukraine
E-mails: vynnytskyi@ukr.net, khats@ukr.net

Received: 02.07.2014