# Short communications

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## DESCRIPTION OF THE DOMAIN OF DEFINITION OF THE ELECTROMAGNETIC SCHRÖDINGER OPERATOR IN DIVERGENCE FORM

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**Abstract.** In this work, conditions are found on the coefficients of the electromagnetic Schrödinger operator in divergence form that provide the coincidence of the domain of definition of the closure of the given operator of the second order in the Sobolev space in $n$-dimensional case.

## 1 Introduction

In this work, we study the electromagnetic Schrödinger operator in $L_2(\mathbb{R}_n)$ $(n \geq 3)$ in divergence form

$$H(a;b;c) = \sum_{j,k=1}^{n} \left( \frac{1}{i} \frac{\partial}{\partial x_j} + b_j(x) \right) a_{jk}(x) \left( \frac{1}{i} \frac{\partial}{\partial x_k} + b_k(x) \right) + c(x), \tag{1.1}$$

where $i = \sqrt{-1}$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}_n$, $a(x) = (a_{jk}(x))_{1 \leq j,k \leq n}$ is a real symmetric matrix function, $b(x) = (b_1(x), b_2(x), ..., b_n(x))$ is a real magnetic potential, and $c(x)$ is a real electric potential.

Denote by $C_0^\infty(\mathbb{R}_n)$ the set of all complex-valued functions in $C^\infty(\mathbb{R}_n)$ with compact support in $\mathbb{R}_n$. Under some conditions on $a_{jk}(x)$, $b_j(x)$ and $c(x)$, $H(a;b;c)$ is defined correctly in $C_0^\infty(\mathbb{R}_n)$ and is an unbounded symmetric operator in $L_2(\mathbb{R}_n)$. Note that the concept of essential self-adjointness of differential operators is an initial point for any research in the field of quantum system where these operators are used as Hamiltonians (see, e.g., [1], [2]).

The essential self-adjointness of the symmetric operator $H_0(a;b;c)$ generated by the elliptic differential expression (1.1), with the domain of definition $D(H_0\ (a;b;c)) =$

$C_0^\infty(\mathbb{R}_n)$ and rather general conditions on the coefficients $a_{jk}(x)$, $b_j(x)$ and $c(x)$, has been studied in [3].

Denote the closure of the operator $H_0(a;b;c)$ by $H$, and the one of the operator $-\Delta$ in $C_0^\infty(\mathbb{R}_n)$ by $H_0$. It is known that the coincidence of domains of these operators plays an important role in solving problems of scattering theory. Of course, to guarantee the validity of equality $D(H) = D(H_0) := W_2^2(\mathbb{R}_n)$ (the latter is the Sobolev space) the conditions imposed on the coefficients $a_{jk}(x)$, $b_j(x)$ and $c(x)$ must be stronger than for the self-adjointness of the operator $H$. For $n = 3$, such conditions on the coefficients have been found in [4].

The purpose of this work is to establish the coincidence of the domain of definition of the operator $H$ in the space $W_2^2(\mathbb{R}_n)$ for $n \geq 3$.

## 2 Main results

Suppose that the coefficients $a_{jk}(x)$, $b_j(x)$ $(j,k = 1,2,...,n)$ and $c(x)$ satisfy the following conditions:

Condition A)

a$_1$) $a(x) = (a_{jk}(x))_{1\leq j,k\leq n}$ is a real symmetric matrix function,

a$_2$) for every $x \in \mathbb{R}_n$, $a(x)$ is a positive definite matrix (the ellipticity condition),

a$_3$) $a_{jk}(x) \in C^2(\mathbb{R}_n)$, $j,k = 1,2,...,n$,

a$_4$) $\frac{\partial a_{jk}(x)}{\partial x_j} \in L_\infty(\mathbb{R}_n)$, $j,k = 1,2,...,n$,

a$_5$) $\lim\limits_{|x|\to+\infty} \left\{ \sup\limits_{\omega\in S_{n-1}} |a_{jk}(x) - \delta_{jk}| \right\} = 0$, $j,k = 1,2,...,n$,

where $S_{n-1}$ is the unit sphere in $\mathbb{R}_n$, $x = |x|\,\omega$, and $\delta_{jk}$ is the Kronecker symbol;

Condition B)

b$_1$) $b_j(x)$ are the real-valued functions on $\mathbb{R}_n$, $j = 1,2,...,n$,

b$_2$) $b_j(x) \in C^1(\mathbb{R}_n)$, $j = 1,2,...,n$,

b$_3$) $\frac{\partial b_k(x)}{\partial x_j} \in L_\infty(\mathbb{R}_n)$, $j,k = 1,2,...,n$;

Condition C)

$$c(x) \in \begin{cases} L_2(\mathbb{R}_n) + L_\infty(\mathbb{R}_n), & if \quad n = 3, \\ L_p(\mathbb{R}_n) + L_\infty(\mathbb{R}_n) \ (p > \frac{n}{2}), & if \quad n \geq 4. \end{cases}$$

Let us recall (see [8; Chapter X.2, p. 185]) that if the relative bound of the operator $B$ with respect to $A$ is equal to zero, then $B$ is said to be infinitely small with respect to $A$.

**Lemma 2.1.** *The operator $D_j := \frac{\partial}{\partial x_j}$ $(j = 1,2,...,n)$ is infinitely small with respect to $H_0$.*

*Proof.* A brief scheme of the proof is as follows. As is known (see [7, p. 200, Proposition 3.5]), $\forall\varepsilon > 0$, $\exists\delta > 0$, such that

$$\|u\|_{W_2^1(\mathbb{R}_n)} \leq \varepsilon \|u\|_{W_2^2(\mathbb{R}_n)} + \delta \|u\|_{L_2(\mathbb{R}_n)} \tag{2.1}$$

for every $u(x) \in W_2^2(\mathbb{R}_n)$. Taking into account the estimate

$$\|D_j u\|_{L_2(\mathbb{R}_n)} \leq \|u\|_{W_2^1(\mathbb{R}_n)},$$

the equivalence of the norms

$$\|u\|_{W_2^2(\mathbb{R}_n)} \text{ and } \|-\Delta u\|_{L_2(\mathbb{R}_n)} + \|u\|_{L_2(\mathbb{R}_n)},$$

and the arbitrariness of a positive number $\varepsilon$, from (2.1) we get the validity of Lemma 2.1. □

Consider the symmetric differential operator $L_0$ in $L_2(\mathbb{R}_n)$ generated by the elliptic differential expression

$$H(a;0;0) = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial}{\partial x_k}\right)$$

with $C_0^\infty(\mathbb{R}_n)$ as the domain of definition. The closure of the operator $L_0$ is denoted by $L$.

The following lemma is true.

**Lemma 2.2.** *Let condition A) be satisfied. Then the following assertions are true:*

$l_1$) $L_0$ *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}_n)$*;*

$l_2$) $D(L) = D(H_0) := W_2^2(\mathbb{R}_n)$*;*

$l_3$) *the norms*

$$\|u\|_{W_2^2(\mathbb{R}_n)} \text{ and } \|Lu\|_{L_2(\mathbb{R}_n)} + \|u\|_{L_2(\mathbb{R}_n)} \equiv \|u\|_L$$

*are equivalent.*

*Proof.* A brief scheme of the proof is as follows. Following [3], we denote by $a^+(x)$ the greatest eigenvalue of the matrix $a(x) = (a_{jk}(x))_{1\le j,k\le n}$. Let

$$a^*(r) = \max_{|x|\le r} a^+(x)$$

and

$$a^{**}(r) = \begin{cases} a^*(r), & if \quad 0 \le r \le 1, \\ \max\left\{a^*(1), \max a^+(x)/\left|x\right|^2\right\}, & if \quad r > 1. \end{cases}$$

By condition A) we have

$$\int^{+\infty} \frac{dr}{\sqrt{a^*(r)a^{**}(r)}} = +\infty.$$

Consequently, assertion $l_1$ follows by the Ikebe-Kato theorem (see [3, Theorem 1]).

Positive definiteness of the smooth matrix-function $a(x) = (a_{jk}(x))_{1\le j,k\le n}$ implies that the inequality

$$\|u\|_L \le const\,\|u\|_{W_2^2(\mathbb{R}_n)} \tag{2.2}$$

is true for every $u(x)$ in $C_0^\infty(\mathbb{R}_n)$, where *const* does not depend on $u(x)$. Consequently,

$$W_2^2(\mathbb{R}_n) \subset D(L). \tag{2.3}$$

Now let us show that

$$D(L) \subset W_2^2(\mathbb{R}_n). \tag{2.4}$$

Let $u(x) \in D(L)$, and $\varphi(x) \in C^\infty(\mathbb{R}_n)$ be a function with $0 \leq \varphi(x) \leq 1$ and

$$\varphi(x) = \begin{cases} 1, & if \quad |x| \leq R, \\ 0, & if \quad |x| \geq R+1, \end{cases}$$

where $R$ is a positive number. Let

$$u(x) = u(x)\varphi(x) + u(x)(1 - \varphi(x)). \tag{2.5}$$

It follows by the results of Ikebe and Kato (see [3, Lemma 3]) that if $u(x) \in D(L)$, then $u(x)\varphi(x) \in W_2^2(\mathbb{R}_n)$ and $Lu \in L_2(\mathbb{R}_n)$. Let us prove that for sufficiently large values of $R$ the function $u(x)(1 - \varphi(x))$ also belongs to $W_2^2(\mathbb{R}_n)$. By Lemma 2.1, the equivalence of the norms

$$\|u\|_{W_2^2(\mathbb{R}_n)} \text{ and } \|-\Delta u\|_{L_2(\mathbb{R}_n)} + \|u\|_{L_2(\mathbb{R}_n)}$$

and condition $\mathrm{a}_5$ it follows that for sufficiently large $R$ there exist numbers $0 < \alpha < 1$ and $\beta > 0$ such that

$$\|L(1-\varphi)u(x) - H_0(1-\varphi)u(x)\|_{L_2(\mathbb{R}_n)} \leq$$
$$\leq \alpha \|H_0(1-\varphi)u(x)\|_{L_2(\mathbb{R}_n)} + \beta \|u(x)\|_{L_2(\mathbb{R}_n)} \tag{2.6}$$

for every $u(x) \in D(L)$. As the operators $L(1-\varphi)$ and $H_0(1-\varphi)$ are closed ones, inequality (2.6) and the Kato-Rellich theorem imply (see Theorem X.12, Section X.2 in [8]) that

$$D(L(1-\varphi)) = D(H_0(1-\varphi)).$$

From the obtained equality we conclude that if $u(x) \in D(L)$, then $u(x)(1 - \varphi(x)) \in W_2^2(\mathbb{R}_n)$. Thus, by representation (2.5) we obtain $u(x) \in W_2^2(\mathbb{R}_n)$, which means that inclusion (2.4) is true. By (2.3) and (2.4) we get the validity of assertion $l_2$.

Assertion $l_3$ follows from inequality (2.2), assertion $l_2$ and the closed graph theorem. □

The main result of this work is the following

**Theorem 2.1.** *Let conditions A)-C) be satisfied. Then the following assertions are true:*

$t_1$) $H_0(a; b; c)$ *is essentially self-adjoint on* $L_2(\mathbb{R}_n)$*;*

$t_2$) $H$ *is semi-bounded from below;*

$t_3$) $D(H) = D(H_0) := W_2^2(\mathbb{R}_n)$.

The proof of this theorem is based on Lemmas 2.1, 2.2 and the Kato-Rellich theorem.

**Remark 7.** Note that condition $\mathrm{a}_5$ imposes some restrictions on the growth of coefficients $a_{jk}(x)$, $j, k = 1, 2, ..., n$, at infinity. Uraltseva [9] and Laptev [5] showed that due to the fast growth of functions $a_{jk}(x)$, $j, k = 1, 2, ..., n$, the operator $L_0$ may not be essentially self-adjoint on $C_0^\infty(\mathbb{R}_n)$ when $n \geq 3$. Maz'ya [6] proved that the cases $n = 1, 2$ are exceptions.

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