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A SURVEY OF THE RECENT RESULTS ON CHARACTERIZATIONS OF EXPONENTIAL STABILITY AND DICHOTOMY OVER FINITE DIMENSIONAL SPACES

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Abstract. The main purpose of this article is the investigation of the recent advances on the exponential stability and dichotomy of autonomous and nonautonomous linear differential systems, in both continuous and discrete cases i.e. $\dot{x}(t) = Ax(t)$, $\dot{x}(t) = A(t)x(t)$, $x_{n+1} = Ax_n$ and $x_{n+1} = A_n x_n$ in terms of the boundedness of solutions of some Cauchy problems, where A, A_n, and $A(t)$ are square matrices, for any $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$.

1 Introduction

The natural tendency for abstraction and for generalization in the study of differential systems has led to the theory of linear operator groups and linear operator semi-groups. In 1888, Giuseppe Peano [44, 45], took the first step for writing a system of scalar differential equations, briefly as one single matrix differential equation. Moreover, Peano wrote the variation of constant formula with the help of the exponential of a matrix with respect to the operatorial norm as:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{t!} A^k.$$

The concept of asymptotic stability is fundamental in the theory of ordinary and partial differential equations. In this way the stability theory leads to real applications. The recent advances of the stability theory deeply interact with spectral theory, harmonic analysis, modern topics of complex functions theory and also with control theory. Finding of necessary and sufficient conditions for a system to be asymptotically stable, is justified by the existence of a vast field of applications, especially in the domain of equations of the mathematical physics.

In 1892, Alexander Lyapunov [36] proved that if A is a square matrix with complex entries then the group of operators $\{e^{tA}\}_{t\in\mathbb{R}}$ is asymptotically stable (or equivalently exponentially stable), i.e. $\lim_{t\to\infty} ||e^{tA}|| = 0$, if and only if the spectrum of the matrix

A is included in the open left half-plane of the complex plane. This classical result has already entered in the present mathematical folklore. In essence, the proof of this result is based on the *spectral mapping theorem* which states that

$$\sigma(e^{tA}) = e^{t\sigma(A)} \quad \text{for all } t \in \mathbb{R}$$

and operates under the assumptions exposed before.

In 1930, Oscar Perron [46] introduced the concept of exponential dichotomy for linear differential systems. Perron also established a connection between the exponential dichotomy and the conditional stability of the system. Extensions of the Perron problem to the general framework of the infinite dimensional Banach spaces were obtained by M.G. Krein, R. Bellman, J.L. Massera and J.J. Schäffer in the period 1948-1966. This vast domain of research is far from exhausted, as proved by the existence in the mathematical literature of the last four decades of an impressive number of papers and monographs dedicated to this interesting topic. We mention here only some of the authors: W. Arendt, A.V. Balakrishnan, V. Barbu, B. Basit, A.G. Baskakov, C.J.K. Batty, C. Buşe, C. Corduneanu, R. Datko, K. Engel, H.O. Fattorini, C. Foiaş, I. Gohberg, J.A. Goldstein, A. Halanay, E. Hille, F.L. Huang, A. Ichikawa, Yu. Latushkin, Yu.I. Lyubich, M. Megan, N.V. Minh, V. Müller, R. Nagel, F. Neubrander, Jan van Neerven, A. Pazy, A. Pogan, G. Da Prato, C.I. Preda, P. Preda, J. Prüss. M. Reghiş, A.L. Sasu, B. Sasu, R. Schnaubelt, Vu Quoc Phong, G. Weiss [1, 4, 6, 8, 9, 10, 20, 21, 22, 24, 25, 23, 28, 27, 29, 30, 31, 32, 33, 34, 37, 40, 39, 38, 41, 42, 43, 50, 51, 49, 48, 47, 53, 54, 52, 55, 57].

In a particular case and using the most simple terms, Perron's result may be formulated as follows:

The system

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R} \tag{$A(t)$}$$

is exponentially stable if and only if it is admissible relative to the space of all continuous and bounded functions, i.e. for each input f, continuous and bounded function defined on the semi-axis of all nonnegative real numbers ($\mathbb{R}_+$), *the output, i.e. the solution of the Cauchy problem*

$$\dot{y}(t) = A(t)y(t) + f(t), \quad t \geq 0, \quad y(0) = 0, \tag{$A(t), f, 0$}$$

is bounded.

A more general concept of admissibility can be defined as follows: Let us denote the solution of $(A(t), f, 0)$ by $y_f(\cdot, 0)$ and let $\mathcal{X}_+$ and $\mathcal{Y}_+$ be two nonempty sets of functions defined on $\mathbb{R}_+$. The system $(A(t))$ is called $(\mathcal{X}_+, \mathcal{Y}_+)$-*admissible* if for each input $f \in \mathcal{X}_+$, the output $y_f(\cdot, 0)$ belongs to $\mathcal{Y}_+$.

The enunciation: *the system* $(A(t))$ *is uniformly exponentially stable if and only if it is* $(\mathcal{X}_+, \mathcal{Y}_+)$-*admissible*, will be called (ad-hoc) *theorem of Perron's type.*

In particular, when $\mathcal{X}_+ = \mathcal{Y}_+$ is a certain normed space of functions, the corresponding theorem of Perron leads to a spectral mapping theorem for the so-called evolution semigroup associated to the system $(A(t))$. For more details on this topic we refer readers to the monograph [20], by Carmen Chicone and Yuri Latushkin.

It is well known that if a nonzero solution of the scalar differential equation $\dot{x}(t) = ax(t),\ t \in \mathbb{R}$ is asymptotically stable then each other solution has the same property and this happens if and only if for each real number μ and each complex number b the solution of the Cauchy problem:

$$\dot{z}(t) = az(t) + e^{i\mu t}b, \quad t \geq 0, \quad z(0) = 0,$$

is bounded.

This result can be extended with approximatively the same formulation for the case of bounded linear operators acting on a Banach space X, [5]. The result can also be extended for strongly continuous bounded semigroups, [14, 17, 16, 42]. For discrete systems, see [15, 56, 61]. Under a slightly different assumption the result on stability is also preserved for all strongly continuous semigroups acting on complex Hilbert spaces, see for example [41, 47] and references therein. For counter-examples, see [13].

In 2008, A. Zada [58] extended the above result to the case of dichotomic matrices. His proofs uses the Spectral Decomposition Theorem. Since then many papers have been devoted to the study of the above subject in the continuous case as well as in the discrete case.

The main aim of this article is to present recent results on the exponential stability and exponential dichotomy of the autonomous systems $\dot{x}(t) = Ax(t)$, $x_{n+1} = Ax_n$ and the nonautonomous systems $\dot{x}(t) = A(t)x(t)$ and $x_{n+1} = A_n x_n$, where A, A_n and $A(t)$ are square matrices, in terms of the boundedness of the solutions of the corresponding Cauchy problems.

This article is organized as follows: In the first section we give results on the decomposition of the solutions of autonomous systems, in the second section we consider results on the exponential stability and dichotomy of autonomous systems, in the third section we present results on discrete characterization of the exponential stability and dichotomy for nonautonomous systems and in the last section we discuss the exponential stability and dichotomy of nonautonomous systems in the continuous case.

2 Decomposition of the solutions of autonomous systems

Consider the following linear differential Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t), & t \in \mathbb{R} \\ x(0) = x_0, \end{cases} \qquad (A, 0, x_0)_c$$

where A is a square matrix of order m and x_0 is a fixed vector in $\mathbb{C}^m$. It is clear that the Cauchy problem $(A, 0, x_0)_c$ has a unique solution given by

$$\phi(t) = e^{tA}x_0, \quad t \in \mathbb{R}.$$

Problem: How can we decompose $\phi(t)$ so that the stability of the system $\dot{x}(t) = Ax(t)$ can be easily discussed?

In [58], $\phi(t)$ was decomposed with the help of the spectral decomposition theorem. Here we are recalling some background and lemmas, without proof, for the spectral decomposition theorem.

Let $p(\lambda) = a_0 + a_1\lambda + \cdots + a_k\lambda^k \in \mathbb{C}[\lambda]$. Then by $p(A)$ we mean the matrix

$$p(A) = a_0 I + a_1 A + \cdots + a_k A^k.$$

Clearly if $p(\lambda) = 1$, then $p(A) = I$, and if $p(\lambda) = \lambda$ then $p(A) = A$. Also if $p, q \in \mathbb{C}[\lambda]$ then $(pq)(A) = p(A)q(A)$.

Lemma 2.1. [59] *Let A be a square matrix of order m. The polynomial $p(A)$ and the exponential e^{tA} commutes i.e.*

$$e^{tA}p(A) = p(A)e^{tA}.$$

Lemma 2.2. [59] *Let p_1 and p_2 be complex-valued polynomials such that p_1 and p_2 are relatively prime. Then*

$$\ker[p_1p_2(B)] = \ker[p_1(B)] \oplus \ker[p_2(B)].$$

The set of all eigenvalues of a matrix A is called the spectrum of A and is denoted by $\sigma(A)$. The polynomial of degree m defined by

$$P_A(\lambda) = \det(\lambda I - A) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m$$

is called the characteristic polynomial associated with A. The spectrum of A is the set of all roots of the polynomial P_A. Let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $k \leq m$ be the spectrum of A. Then

$$P_A(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_k)^{m_k},$$

$$m_1 + m_2 + \cdots + m_k = m.$$

Note that the polynomials $p = (\lambda - \lambda_i)^{m_i}$ and $q = (\lambda - \lambda_j)^{m_j}$ are relatively prime because $\lambda_i \neq \lambda_j$ when $i \neq j$. From the Hamilton-Cayley Theorem it follows that

$$P_A(A) = 0 = (A - \lambda_1 I)^{m_1}(A - \lambda_2 I)^{m_2} \ldots (A - \lambda_k I)^{m_k}.$$

Taking kernel of both sides, we get

$$\ker(0) = \ker[(A - \lambda_1 I)^{m_1}(A - \lambda_2 I)^{m_2} \ldots (A - \lambda_k I)^{m_k}].$$

As $\ker(0) = \mathbb{C}^m$, we have

$$\mathbb{C}^m = \ker[(A - \lambda_1 I)^{m_1}(A - \lambda_2 I)^{m_2} \ldots (A - \lambda_k I)^{m_k}].$$

Applying Lemma 2.2, we obtain

$$\mathbb{C}^m = \ker(A - \lambda_1 I)^{m_1} \oplus \ker(A - \lambda_2 I)^{m_2} \oplus \cdots \oplus \ker(A - \lambda_k I)^{m_k}.$$

For each $j \in \{1, 2, \ldots, k\}$ let us denote $W_j := \ker(A - \lambda_j I)^{m_j}$. Then

$$\mathbb{C}^m = W_1 \oplus W_2 \oplus \cdots \oplus W_k. \tag{2.1}$$

Lemma 2.3. [59] *The subspace $\ker(A - \lambda_j I)^{m_j}$ is e^{tA}-invariant.*

The following result shows that any solution of the Cauchy problem $(A, 0, x_0)_c$ can be split in the sum of k solutions of the system $\dot{x}(t) = Ax(t)$. Moreover, each of such summands has a relatively simple structure described as follows.

Theorem 2.1. [59] *Let A be a square matrix of order m. For each $x \in \mathbb{C}^m$ there exist $w_j \in W_j$ $(j \in \{1, 2, \ldots, k\})$ such that*

$$e^{tA}x = e^{tA}w_1 + e^{tA}w_2 + \cdots + e^{tA}w_k, \quad t \in \mathbb{R}.$$

Moreover, if $w_j(t) := e^{tA}w_j$ then $w_j(t) \in W_j$ for all $t \in \mathbb{R}$ and there exist $\mathbb{C}^m$-valued polynomials $p_j(t)$ with $\deg(p_j) \leq m_j - 1$ such that

$$w_j(t) = e^{\lambda_j t}p_j(t), \quad t \in \mathbb{R},\ j \in \{1, 2, \ldots, k\}.$$

In the discrete case the Cauchy problem associated with matrix A is

$$\begin{cases} z_{n+1} = Az_n, & z_n \in \mathbb{C}^m, \quad n = 0, 1, 2, \ldots \\ z(0) = z_0. \end{cases} \qquad (A, 0, z_0)_d$$

Clearly the solution of $(A, 0, z_0)_d$ is $z_n = A^n z_0$.

Problem: *How can we decompose the above solution $A^n z_0$ so that the exponential stability of the system $(A, 0, z_0)_d$ can be easily discussed?*

In [18], the solution of $(A, 0, z_0)_d$ was decomposed with the help of the Spectral Decomposition Theorem (in discrete form).

Let us denote $z(n+1) - z(n)$ by $\Delta z(n)$. Then concerning $\Delta z(n)$ we have the following lemma.

Lemma 2.4. [59] *If $\Delta^N q(n) = 0$ for all $n = 0, 1, 2 \ldots$ and $N \geq 1$ is a natural number, then q is a $\mathbb{C}^m$-valued polynomial of degree less than or equal to $N - 1$.*

The following result shows that any solution of the Cauchy problem $(A, 0, z_0)_d$ can be split in the sum of k solutions of the system $x_{n+1} = Ax_n$. Moreover, each of such summands has a relatively simple structure described as follows.

Theorem 2.2. [18] *Let A be a square invertible matrix of order m. For each $y \in \mathbb{C}^m$ there exist $w_i \in W_i$ where $W_i = \ker(A - \lambda_i I)^{n_i}$, $(i \in \{1, 2, \ldots, k\})$ such that*

$$A^n y = A^n w_1 + A^n w_2 + \cdots + A^n w_k.$$

Moreover, if $w_i(n) = A^n w_i$ then $w_i(n) \in W_i$ for all $n = 0, 1, 2, \ldots$ and there also exist $\mathbb{C}^m$-valued polynomials $t_i(n)$ with $\deg(t_i) \leq n_i - 1$ such that

$$w_i(n) = \lambda_i^n t_i(n), \quad n = 0, 1, 2, \ldots\ , \ i \in \{1, 2, \ldots, k\}.$$

3 Stability and dichotomy of autonomous systems

In [58], the stability and dichotomy of the systems $\dot{x}(t) = Ax(t)$ was discussed. The authors decompose the complex plane as follows. Suppose that $\mathbb{C}_+ = \{z \in \mathbb{C} : Re(z) > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} : Re(z) < 0\}$ and $i\mathcal{R} = \{i\eta : \eta \in \mathbb{R}\}$. Clearly $\mathbb{C} = \mathbb{C}_+ \cup \mathbb{C}_- \cup i\mathcal{R}$. With the help of above decomposition of $\mathbb{C}$, we can now state the following definition.

Definition 3.1. [58] Consider the system

$$\dot{x}(t) = Ax(t) \tag{$(A)_c$}$$

The system $(A)_c$ is called

(i) **stable** if $\sigma(A)$ belongs to $\mathbb{C}_-$ or, equivalently, if there exist two positive constants N and ν such that $\|e^{tA}\| \leq Ne^{-\nu t}$ for all $t \geq 0$,

(ii) **expansive** if $\sigma(A)$ belongs to $\mathbb{C}_+$ and

(iii) **dichotomic** if $\sigma(A)$ does not intersect the set $i\mathcal{R}$.

With the help of the decomposition of $\mathbb{C}$ and the decomposition of the solution of $\dot{x}(t) = Ax(t)$ the following result was obtained.

Theorem 3.1. [58] *The system $(A)_c$ is stable if and only if for each real number μ and each non-zero vector b in finite dimensional space $\mathbb{C}^m$ the solution of the following Cauchy problem*

$$\begin{cases} \dot{w}(t) = Aw(t) + e^{i\mu t}b, \quad t \geq 0, \\ w(0) = 0 \end{cases} \tag{$(A,\mu,b,0)_c$}$$

is bounded.

As the system $(A)_c$ is expansive if and only if $(-A)_c$ is stable, the following corollary was also stated.

Corollary 3.1. [58] *The system $(A)_c$ is expansive if and only if for each real number μ and each non-zero vector b in finite dimensional space $\mathbb{C}^m$ the solution of the Cauchy problem $(-A, \mu, b, 0)_c$ is bounded.*

In the same paper this result was also extended to dichotomy.

Theorem 3.2. [58] *The system $(A)_c$ is dichotomic if and only if there exists a projection P with $e^{tA}P = Pe^{tA}$ for all $t \geq 0$ such that for each real number μ and each non-zero vector b in finite dimensional space $\mathbb{C}^m$ the following two Cauchy problems:*

$$\begin{cases} \dot{u}(t) = Au(t) + e^{i\mu t}Pb, \quad t \geq 0, \\ u(0) = 0 \end{cases} \tag{$(A,\mu,Pb,0)_c$}$$

and

$$\begin{cases} \dot{w}(t) = -Aw(t) + e^{i\mu t}(I-P)b, \quad t \geq 0, \\ w(0) = 0 \end{cases} \tag{$(-A,\mu,(I-P)b,0)_c$}$$

are bounded.

The next problem was as follows.

Problem. *Can we extend the results of [58] to any arbitrary value of the initial vector in the Cauchy problem $(A, \mu, b, 0)_c$ i.e. can we replace the Cauchy problem $(A, \mu, b, 0)_c$ by $(A, \mu, b, x_0)_c$?*

Results concerning this problem were obtained in [64]. Thus the results of [58] were extended as follows.

Theorem 3.3. [64] *The system $(A)_c$ is asymptotically stable if and only if for each real number μ and each non-zero vector b in $\mathbb{C}^m$ the solution of the following Cauchy problem:*

$$\begin{cases} \dot{w}(t) = Aw(t) + e^{i\mu t}b, \quad t \geq 0, \\ w(0) = w_0 \end{cases} \tag{$(A, \mu, b, w_0)_c$}$$

is bounded.

Corollary 3.2. [64] *The system $(A)_c$ is expansive if and only if for each real number μ and each non-zero vector b in $\mathbb{C}^m$ the solution of $(-A, \mu, b, w_0)_c$ is bounded.*

By using projection P, the above result was also extended to dichotomy as follows.

Theorem 3.4. [64] *The system $(A)_c$ is dichotomic if and only if there exist a projection P having the property $e^{tA}P = Pe^{tA}$ for all $t \geq 0$ such that for each real number μ and each non-zero vector b in $\mathbb{C}^m$ the solutions of the following Cauchy problems*

$$\begin{cases} \dot{u}(t) = Au(t) + e^{i\mu t}Pb, \quad t \geq 0, \\ u(0) = Pu_0 \end{cases}$$

and

$$\begin{cases} \dot{w}(t) = -Aw(t) + e^{i\mu t}(I - P)b, \quad t \geq 0, \\ w(0) = (I - P)w_0 \end{cases}$$

are bounded.

After the publishing of [58], the following question arose.

Problem. *Can we extend the results of [58] to discrete autonomous systems* ?

Answer to the above problem was given in [18].

In [18], the stability and dichotomy of the system $x_{n+1} = Ax_n$ were discussed. The authors decomposed $\mathbb{C}$ in the following way. Suppose that $\Omega_1 = \{z \in \mathbb{C} : |z| = 1\}$, $\Omega_1^+ = \{z \in \mathbb{C} : |\,z| > 1\}$ and $\Omega_1^- = \{z \in \mathbb{C} : |\,z| < 1\}$. Clearly $\mathbb{C} = \Omega_1 \cup \Omega_1^+ \cup \Omega_1^-$.

Definition 3.2. Consider a system

$$w_{n+1} = Aw_n. \tag{$(A)_d$}$$

The system $(A)_d$ is said to be

(i) **stable** if $\sigma(A)$ is contained in Ω_1^- or, equivalently, if there exist two constants $N > 0$ and $\nu > 0$ such that $\|A^n\| \leq Ne^{-\nu n}$ for all $n = 0, 1, 2 \ldots,$

(ii) **expansive** if $\sigma(A)$ is contained in Ω_1^+ and

(iii) **dichotomic** if $\sigma(A)$ does not intersect set Ω_1.

Since any expansive matrix A with spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is invertible, its inverse is stable, because

$$\sigma\left(A^{-1}\right) = \left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_k}\right\} \subset \Omega_1^-.$$

In [18], with the help of the above decomposition of $\mathbb{C}$ and from the decomposition of the solution $A^n z_0$ of the system $z_{n+1} = Az_n$ the following results were obtained.

Theorem 3.5. [18] *The matrix A is stable if and only if for each $\mu \in \mathbb{R}$ and each $b \in \mathbb{C}^m$ the solution of the discrete Cauchy problem*

$$\begin{cases} u_{n+1} = Au_n + e^{i\mu n}b, & n \in \mathbb{Z}_+ \\ u_0 = 0, \end{cases} \qquad (A, \mu, b, 0)_d$$

is bounded.

Corollary 3.3. [18] *A matrix A is expansive if and only if it is invertible and for all $\mu \in \mathbb{R}$ and all $b \in \mathbb{C}^m$ the solution of the discrete Cauchy problem*

$$\begin{cases} w_{n+1} = A^{-1}w_n + e^{i\mu n}b, & n \in \mathbb{Z}_+ \\ w_0 = 0, \end{cases}$$

is bounded.

Theorem 3.6. [18] *The system $(A)_d$ is dichotomic if and only if there exists a projection P having the property $AP = PA$ such that for each $\mu \in \mathbb{R}$ and each vector $b \in \mathbb{C}^m$ the solutions of the following two discrete Cauchy problems*

$$\begin{cases} u_{n+1} = Au_n + e^{i\mu n}Pb, & n \in \mathbb{Z}_+ \\ u_0 = 0 \end{cases}$$

and

$$\begin{cases} w_{n+1} = A^{-1}w_n + e^{i\mu n}(I - P)b, & n \in \mathbb{Z}_+ \\ w_0 = 0 \end{cases}$$

are bounded.

If in the above problem we replace a matrix A by an operator A, then still the same theorem holds.

Theorem 3.7. [18] *A bounded linear operator A acting on the complex Banach space X is dichotomic if and only if there exists a projection P on X that commutes with A and such that for each real number μ and each vector $b \in X$ the solutions of the following two Cauchy problems*

$$\begin{cases} u_{n+1} = Au_n + e^{i\mu n}Pb, & n \in \mathbb{Z}_+ \\ u_0 = 0 \end{cases}$$

and

$$\begin{cases} w_{n+1} = A^{-1}w_n + e^{i\mu n}(I - P)b, & n \in \mathbb{Z}_+ \\ w_0 = 0 \end{cases}$$

are bounded.

The next problem was as follows.

Problem. *Can we extend the results of [18] to any arbitrary value of the initial vector in the Cauchy problem* $(A,\mu,b,0)_d$ *i.e. can we replace the Cauchy problem* $(A,\mu,b,0)_d$ *by* $(A,\mu,b,x_0)_d$?

Results concerning this problem were obtained in [60]. Thus the results of [18] were extended as follows:

Theorem 3.8. [60] *The system* $(A)_d$ *is stable if and only if for each real number* μ *and any arbitrary vector* b *in* $\mathbb{C}^m$ *the solution of the discrete Cauchy problem*

$$\begin{cases} u_{n+1} = Au_n + e^{i\mu n}b, & n \in \mathbb{Z}_+ \\ u(0) = u_0, \end{cases} \qquad (A,\mu,b,0)_d$$

is bounded.

Corollary 3.4. [60] *The system* $(A)_d$ *is expansive if and only if* A *is invertible and for each real number* μ *and any arbitrary vector* b *in* $\mathbb{C}^m$ *the solution of the discrete Cauchy problem*

$$\begin{cases} u_{n+1} = A^{-1}u_n + e^{i\mu n}b, & n \in \mathbb{Z}_+ \\ u(0) = u_0, \end{cases}$$

is bounded.

Theorem 3.9. [60] *The system* $(A)_d$ *is dichotomic if and only if there exists a projection* P *with* $AP = PA$ *such that for each real number* μ *and each arbitrary vector* b *in* $\mathbb{C}^m$ *the solutions of the following two discrete Cauchy problems*

$$\begin{cases} u_{n+1} = Au_n + e^{i\mu n}Pb, & n \in \mathbb{Z}_+ \\ u(0) = Pu_0 \end{cases}$$

and

$$\begin{cases} w_{n+1} = A^{-1}w_n + e^{i\mu n}(I-P)b, & n \in \mathbb{Z}_+ \\ w(0) = (I-P)w_0 \end{cases}$$

are bounded.

4 Discrete characterization of stability and dichotomy for nonautonomous systems

After the results of [18], one of the natural question is as follows.

Problem. *By using the idea of [18], can we extend the stability result of the system* $x_{n+1} = Ax_n$ *to nonautonomous systems* $x_{n+1} = A_n x_n$ *i.e. can we replace the square matrix* A *by a sequence of matrices* A_n, where $n \in \mathbb{Z}_+$?

Working with nonautonomous systems is always more complicated than with autonomous systems. In [2], a partial answer to the above problem was given for periodic systems.

Let $\mathbb{Z}_+$ be the set of all nonnegative integer numbers. A family $\mathcal{U} = \{U(p,q) : (p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+\}$ of square matrices having complex scalars as entries is called an N-periodic discrete evolution family if it satisfies the following conditions:

(1) $U(p,q)U(q,r) = U(p,r)$ for all nonnegative integers $p \geq q \geq r$.

(2) $U(p,p) = I$ for all $p \in \mathbb{Z}_+$.

(3) $U(p+N, q+N) = U(p,q)$ for all nonnegative integers $p \geq q$.

We will use such families to solve the following discrete Cauchy problem.

$$\begin{cases} y_{n+1} = A_n y_n + e^{i\mu n} b, & n \in \mathbb{Z}_+ \\ \quad y_0 = 0\,, \end{cases} \qquad (A_n, \mu, b, 0)$$

in the case when (A_n) are square N-periodic matrices, i.e. $A_{n+N} = A_n$ for all $n \in \mathbb{Z}_+$. Define

$$U(n,j) := \begin{cases} A_{n-1}A_{n-2}\ldots A_j, & j \leq n-1 \\ I, & j = n, \end{cases}$$

then the family $\{U(n,j)\}_{n\geq j\geq 0}$ is a discrete N-periodic evolution family and the solution $(y_n(\mu,b))$ of the Cauchy problem $(A_n,\mu,b)_0$ is given by:

$$y_n(\mu,b) = \sum_{j=1}^{n} U(n,j) e^{i\mu(j-1)} b. \tag{4.1}$$

For further details related to the general theory of difference equations we refer to [26]. The first result of [2] is stated in the following form.

Theorem 4.1. [2] *The sequence $(y_n(\mu,b))$ given in (3.1) is bounded for any real number μ and any m-vector b if the matrix $U(N,0)$ is stable.*

Now the following natural question arises.

Problem. *Is the converse of the above theorem is also true?*

In [2], a partial converse of Theorem 4.1 was given as follows:

Theorem 4.2. [2] *If for each $\mu \in \mathbb{R}$ and each non zero $b \in \mathbb{C}^m$ the sequence $(y_{Nk}(\mu,b))_k$ is bounded and the matrix $V_\mu = \sum\limits_{\nu=1}^{N} U(N,\nu)e^{i\mu\nu}$ is invertible then the matrix $U(N,0)$ is stable.*

In [2], an example was also given which shows that the assumption on invertibility of V_μ, for each real number μ, cannot be removed.

Open Problem. *Can we find a strong version of Theorems* 4.1 *and* 4.2 *that represents the results of both theorems as a single theorem?*

In the same paper [2], a strong version of a Barbashin's type theorem is also obtained which states the following result.

Theorem 4.3. [2] *Let $(U(n,k))_{n\geq k}$ be an N-periodic evolution family. If for each vector $b \in \mathbb{C}^m$ the inequality*

$$\sup_{k\geq 1} \sum_{j=1}^{Nk} \|U(Nk,k)b\| = M(b) < \infty$$

holds then the matrix $U(N,0)$ is stable.

An interesting problem can be stated as follows.

Open Problem. *If the following uniform inequality*

$$\sup_{\mu\in\mathbb{R}}\sup_{k\geq 1}||y_{2k}(\mu,b)||=K(b)<\infty \tag{4.2}$$

holds for all $b\in\mathbb{C}^m$ then is the matrix $U(N,0)$ stable?

Under an assumption similar to (3.2), Jan van Neerven proved in [41] that a strongly continuous semigroup acting on a complex Banach space is exponentially stable. Moreover, when the semigroup acts in a complex Hilbert space it is uniformly exponentially stable. A transparent proof of this later result can be found in [47]. In connection with (3.2) we also mention that in [11] it is proved that if a vector valued function has a bounded holomorphic extension to the open right half plane then its primitive grows like $M(1+t)$ *for* $t\geq 0$.

Working further with equation $(A_n,\mu,b,0)$, the following stronger results were obtained in [7].

Let $\mathbb{Z}^+$ be the set of all nonnegative integers and $m,q\in\mathbb{Z}^+$, $m\geq 1, q>1$, be fixed. Denote by $S_{q,0}(\mathbb{Z}^+,\mathbb{C}^m)$ the set of all $\mathbb{C}^m$-valued, q-periodic sequences (z_n), with $z_0=0$. Let us consider the following difference equation and discrete Cauchy problems:

$$x_{n+1}=A_nx_n,\qquad n\in\mathbb{Z}^+ \tag{A_n}$$

$$\begin{cases} y_{n+1}=A_ny_n+e^{i\mu n}b, & n\in\mathbb{Z}^+,\\ y_0=0 \end{cases} \tag{$A_n,\mu,b,0$}$$

and

$$\begin{cases} w_{n+1}=A_nw_n+e^{i\mu n}z_n, & n\in\mathbb{Z}^+,\\ w_0=0 \end{cases} \tag{$A_n,\mu,z_n,0$}$$

where A_n is a q-periodic, $\mathcal{L}(\mathbb{C}^m)$-valued sequence, $z_n\in S_{q,0}(\mathbb{Z}^+,\mathbb{C}^m)$ and μ is a real parameter. We know that the solution of $(A_n,\mu,b,0)$ is given by (3.1) and similarly the solution of $(A_n,\mu,z,0)$ can be obtained from (3.1) by replacing only b by z_n. In [7] the following theorem was obtained which is stronger than Theorems 4.1 and 4.2.

Theorem 4.4. *The following four statements are equivalent.*

1. *The system (A_n) is uniformly asymptotically stable, i.e. there exist two positive constants N and ν such that*

$$\|U(n,j)\|\leq Ne^{-\nu(n-k)},\quad\forall\ \ n\geq k\geq 0$$

2. *For each $\mu\in\mathbb{R}$ and each $(z_n)\in S_{q,0}(\mathbb{Z}^+,\mathbb{C}^m)$ the solution of $(A_n,\mu,z_n,0)$ is bounded, i.e.*

$$\sup_{n\geq 1}\left\|\sum_{j=1}^{n}U(n,j)e^{i\mu(j-1)}z_{j-1}\right\|=M(\mu,(z_n))<\infty;$$

3. *For each* $b \in \mathbb{C}^m$ *the solution of* $(A_n, \mu, b, 0)$ *is uniformly bounded, i.e.*

$$\sup_{\mu \in \mathbb{R}} \sup_{n \geq 1} \|y_n(\mu, b)\| = M(b) < \infty;$$

4. *For each* $\mu \in \mathbb{R}$ *and each* $b \in \mathbb{C}^m$ *Íthe solution of* $(A_n, \mu, b, 0)$ *is bounded, i. e.*

$$\sup_{n \geq 1} \left\| \sum_{j=1}^{n} U(n, j) e^{i\mu(j-1)} b \right\| = M(\mu, b) < \infty;$$

and for each $\mu \in \mathbb{R}$ *the operator* $V_\mu = \sum_{\nu=1}^{q} e^{i\mu\nu} U(q, \nu)$ *is invertible.*

In [58], the stability results were also extended to the case of dichotomy. So a natural question can be stated as follows:

Problem. *Is it possible to extend Theorems 4.1 and 4.2 for dichotomy?*

In [62], Theorems 4.1 and 4.2 were extended for dichotomy in the following way.

Theorem 4.5. [62] *Let* $N \geq 2$ *be a fixed natural number. The Poincaré map* $U(N, 0)$ *is dichotomic if and only if for each* $\mu \in \mathbb{R}$ *the matrix* V_μ *is invertible and there exists a projection* P *with* $PU(N, 0) = U(N, 0)P$ *and* $PV_\mu = V_\mu P$ *such that for each* $\mu \in \mathbb{R}$ *and each non-zero vector* b *in* $\mathbb{C}^m$*, the solutions of discrete Cauchy problems*

$$\begin{cases} u_{n+1} = A_n u_n + e^{i\mu n} Pb, \quad n \in \mathbb{Z}^+ \\ u_0 = 0 \end{cases} \tag{$A_n, \mu, b, 0$}$$

and

$$\begin{cases} v_{n+1} = A_n^{-1} v_n + e^{i\mu n} (I - P) b, \quad n \in \mathbb{Z}^+ \\ v_0 = 0 \end{cases} \tag{$A_n^{-1}, \mu, (I-P)b, 0$}$$

are bounded.

5 Characterization of exponential stability and dichotomy for nonautonomous systems

The case of exponential stability for nonautonomous system is more complicated than for autonomous systems. In [19], the following approach for the exponential stability of such system was developed.

Consider the homogenous time-dependent differential system

$$\dot{x} = A(t)x, \tag{$A(t)$}$$

where $A(t)$ is a 2-periodic continuous function, i.e. $A(t + 2) = A(t)$ for all $t \in \mathbb{R}$. It is well known that the system $(A(t))$ is *uniformly exponentially stable*, i.e. there exist two positive constants N and ν such that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq N e^{-\nu(t-s)} \quad \text{for all} \quad t \geq s,$$

if and only if the spectrum of the matrix $V := \Phi(2)$ lies inside of the circle of radius one, where $\Phi(t)$ is the resolvent matrix of $(A(t))$. See e.g. [16], where even the infinite dimensional version of this result is stated. The following natural question arises:

Problem. *Is the negativeness of all the eigenvalues of $A(t)$ yields the exponential stability of the system $(A(t))$?*

Answer to this question is NO, for counter example, see [55].

So motivation for the paper [19] was to search some other tools to investigate the exponential stability of the system $\dot{x}(t) = A(t)x(t)$.

Consider two arbitrary functions $h_1, h_2 : [0, 2] \to \mathbb{C}$ given by

$$h_1(u) = \begin{cases} u, & u \in [0,1) \\ 2-u, & u \in [1,2] \end{cases} \qquad \text{and} \qquad h_2(u) = u(2-u).$$

Now let us consider the vectorial non-homogenous Cauchy problem

$$\begin{cases} \dot{y}(t) = A(t)y(t) + e^{i\mu t}f(t), & t \in \mathbb{R}_+ \\ y(0) = 0, \end{cases} \qquad (A(t), \mu, f(t), 0)$$

where f is some continuous function. Denote by $P_{2,0}(\mathbb{R}_+, \mathbb{C}^n)$, the space of all continuous and 2-periodic functions g with the property that $g(0) = 0$. We endow this space with the norm "sup". For each $k \in \{1, 2\}$ let us consider the set $\mathcal{A}_k$ consisting of all functions $f \in P_{2,0}(\mathbb{R}_+, \mathbb{C}^n)$ given for $t \in [0, 2]$ by $f(t) = \Phi(t)h_k(t)$, where $\Phi(t)$ is the resolvent matrix of $(A(t))$.

Theorem 5.1. [19] *The following two statements hold true.*

(i) If the system $(A(t))$ is uniformly exponentially stable then for each continuous and bounded function f and each real number μ the solution of $(A(t), \mu, f, 0)$ is bounded.

(ii) Let $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. If for each $f \in \mathcal{A}$ and for each real number μ the solution of the Cauchy Problem $(A, \mu, f, 0)$ is bounded then the system $(A(t))$ is uniformly exponentially stable.

From Theorem 5.1, the following corollary was obtained.

Corollary 5.1. [19] *The system $(A(t))$ is uniformly exponentially stable if and only if for each real number μ and each function f belonging to $P_{2,0}(\mathbb{R}_+, \mathbb{C}^n)$ the solution of $(A(t), \mu, f, 0)$ is bounded.*

Also the following corollary of Datko type was stated.

Corollary 5.2. [19] *The system $(A(t))$ is uniformly exponentially stable if and only if for each vector b*

$$\sum_{j=1}^{\infty} ||\Phi(2j)b|| < \infty. \tag{5.1}$$

It is not difficult to see that the requirement (4.1) may be replaced by the apparently weaker one, namely by the inequality

$$\sum_{j=1}^{\infty} |\langle \Phi(2j)b, b \rangle| < \infty, \quad \forall \ b \in \mathbb{C}^n.$$

Now one more natural question arises here.

Problem. *Can we extend the results of Theorem 5.1 from 2-periodic systems to any q-periodic systems?*

In [3], the following answer to the above problem was given.

Consider the q-periodic system

$$\dot{x}(t) = A(t)x(t). \tag{$A(t)$}$$

We know that the Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t \in \mathbb{R} \\ x(0) = I, \end{cases}$$

has a unique solution denoted by $\Phi(t)$. Also It is well known that $\Phi(t)$ is an invertible matrix and that its inverse is the unique solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = -x(t)A(t), & t \in \mathbb{R} \\ x(0) = I. \end{cases}$$

The family $\mathcal{U} = \{U(t,s), t, s \in \mathbb{R}\}$, where $U(t,s) := \Phi(t)\Phi^{-1}(s)$, is called evolution family and has the following properties:

(i) $U(t,t) = I$, for all $t \in \mathbb{R}$;

(ii) $U(t,s) = U(t,r)U(r,s)$ for all $t, s, r \in \mathbb{R}$;

(iii) $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$ for all $t, s \in \mathbb{R}$;

(iv) $\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s)$ for all $t, s \in \mathbb{R}$;

(v) The map $(t,s) \mapsto U(t,s) : \mathbb{R}^2 \to \mathcal{M}(n,\mathbb{C})$ is continuous, where $\mathcal{M}(n,\mathbb{C})$ is the space of all square matrices of order n. If, in addition, the map $A(\cdot)$ is q- periodic, for some positive number q, then:

(vi) $U(t+q, s+q) = U(t,s)$ for all $t, s \in \mathbb{R}$;

(vii) There exist $\omega \in \mathbb{R}$ and $M_\omega \geq 1$ such that

$$\|U(t,s)\| \leq M_\omega e^{\omega(t-s)}, \quad t \geq s.$$

Concerning the evolution family $\mathcal{U} = \{U(t,s), t, s \in \mathbb{R}\}$ in [3], the following proposition was presented.

Proposition 5.1. [3] *Let $\mathcal{U} = \{U(t,s), t, s \in \mathbb{R}\}$ be a strongly continuous and q-periodic evolution family acting on the Banach space X. Then the following four statements are equivalent:*

1. The family $\mathcal{U}$ is uniformly exponentially stable, i.e. there exist two positive constants N and ν such that

$$\|U(t,s)\| \leq Ne^{-\nu(t-s)}, \text{ for all } t \geq s;$$

2. There exist two positive constants N and ν such that

$$\|U(t,0)\| \leq Ne^{-\nu t}, \text{ for all } t \geq 0;$$

3. The spectral radius of $U(q,0)$ is less than one, i.e.

$$r(U(q,0)) := \sup\{|\lambda| \quad \lambda \in \sigma(U(q,0))\} = \inf_{n\geq 1} \|U(q,0)^n\|^{\frac{1}{n}} < 1.$$

4. For each real number μ, one has

$$\sup_{\nu\geq 1} \left\| \sum_{k=0}^{\nu-1} e^{i\mu k} U(q,0)^{\nu-k} \right\| := L(\mu) < \infty.$$

Let $P_q^0(\mathbb{R}^+, X)$ be the set of all continuous X-valued functions f defined on $\mathbb{R}^+$, with $f(0) = 0$ and $f(t+q) = f(t)$ for $t \in \mathbb{R}^+$. The following theorem is stated in [3].

Theorem 5.2. [3] *Let $\mathcal{U} = \{U(t,s), t, s \in \mathbb{R}\}$ be a strongly continuous and q-periodic evolution family on the Banach space X. If for each $\mu \in \mathbb{R}$ and each $f \in P_q^0(\mathbb{R}^+, X)$ one has*

$$\sup_{t>0} \| \int_0^t e^{i\mu s} U(t,s) f(s) ds \| = K(\mu, f) < \infty$$

then $\mathcal{U}$ is uniformly exponentially stable.

For a given real number μ and a given family $(A(t))$ the following Cauchy problem was considered.

$$\begin{cases} \dot{x}(t) = A(t)x(t) + e^{i\mu t} I \quad t \geq 0 \\ x(0) = 0. \end{cases} \qquad (A(t), \mu, I, 0)$$

Obviously, the solution of $(A(t), \mu, I, 0)$ is given by

$$\Phi_\mu(t) = \int_0^t U(t,s) e^{i\mu s} ds. \qquad (*)$$

The next result is stated in [3].

Theorem 5.3. [3] *The following two statements hold true:*

1. For each real number μ and each vector $b \in \mathbb{C}^n$, the solution of the Cauchy Problem $(A(t), \mu, b, 0)$ is bounded on $\mathbb{R}_+$, if the spectral radius of $(U(q,0)$ is less than one.

2. Conversely, if for each real μ and each vector $b \in \mathbb{C}^n$, the solution of the Cauchy Problem $(A(t), \mu, b, 0)$ is bounded on $\mathbb{R}_+$ and in addition for each real number μ the matrix $\Phi_\mu(q)$ is an invertible one, where $\Phi_\mu(q)$ is given in $(*)$, then the system $(A(t))$ is uniformly exponentially stable.

One more result is stated in [3].

Theorem 5.4. [3] *The system $(A(t))$ is uniformly exponentially stable if and only if for each $b \in \mathbb{C}^n$ the solution of $(A(t), \mu, b, 0)$ is bounded on $\mathbb{R}_+$, uniform with respect to the parameter μ on $\mathbb{R}$, i.e.*

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \left\| \int_0^t U(t,s) e^{i\mu s} b ds \right\| = K(b) < \infty,$$

for all $b \in \mathbb{C}^n$.

In [3], from Theorem 5.4 the following weak version of the Barbashin theorem is stated.

Corollary 5.3. [3] *The time varying q-periodic system* $(A(t))$ *is uniformly exponentially stable if and only if for each* $x \in \mathbb{C}^n$ *one has*

$$\sup_{t\geq 0} \int_0^t | < \Phi(t)\Phi^{-1}(s)x, x > |ds < \infty.$$

Open problem. *Can we extend the result of Corollary 5.3 to infinite dimensional space?*

Theorem 4.4 also yields the following corollary for strong version of the Barbashin Theorem.

Corollary 5.4. [3] *The time varying q-periodic system* $(A(t))$ *is uniformly exponentially stable if and only if for each* $b \in \mathbb{C}^n$ *one has*

$$\sup_{t\geq 0} \int_0^t \|U(t,s)b\|ds = K(b) < \infty.$$

Now one more natural question arises in this context.

Problem. *Can we extend the results and ideas of [18] and [3] to Dichotomy, by using the idea of [58] in autonomous case*?

In this regards article [35] presents the following result.

The evolution family $\mathcal{U}$ is said to have a uniform exponential dichotomy with respect to the projector P (i.e. $P \in \mathcal{L}(\mathbb{C}^n)$ and $P^2 = P$) if there exist positive constants N_1, N_2, ν_1 and ν_2 such that

(1) $U(t,s)P = PU(t,s)$, for all $t \geq s \in \mathbb{R}$

(2) $\|U(t,s)P\| \leq N_1 e^{-\nu_1(t-s)}$, for all $t \geq s \in \mathbb{R}$

(3) $\|QU(t,s)\| \leq N_2 e^{-\nu_2(t-s)}$, for all $t \geq s \in \mathbb{R}$.

Here, $Q := I - P$ and $U(s,t)$ is the inverse of $U(t,s)$. It is clear that $Q^2 = Q$ and $PQ = QP = 0$.

Then by using Proposition 5.1, the following result on dichotomy was obtained in [35].

Theorem 5.5. [35] *The following statements are equivalent:*

i. The evolution family $\mathcal{U}$ *has an exponential dichotomy with respect to the projector* P.

ii. The following holds:

1. $\sup_{\mu\in\mathbb{R}} \sup_{t\in\mathbb{R}} \left\| \int_0^t e^{i\mu s} U(t,s)P ds \right\| < \infty.$

2. *the solution of the equation*

$$\begin{cases} \dot{Y}(t) = -Y(t)A(t) + e^{i\mu t}Q, \quad Y(t) \in \mathcal{L}(\mathbb{C}^n),\ t \geq s \\ Y(s) = 0 \end{cases}$$

has a limit in $\mathcal{L}(\mathbb{C}^n)$ *as* s *tends to* $-\infty$ *(i.e.* $\int_{-\infty}^{t} e^{i\mu s} QU(s,t)ds$ *exists) and*

$$\sup_{\mu\in\mathbb{R}} \sup_{t\in\mathbb{R}} \Big\| \int_{-\infty}^{t} e^{i\mu s} QU(s,t)ds \Big\| < \infty,$$

In [63], a similar result of dichotomy, as in Theorem 5.5, was obtained. The main theorem of [63] is stated as follows.

Theorem 5.6. [63] *Let* $q > 0$. *If the matrix* $L := U(q,0)$ *is dichotomic and there exists a projection* P *commuting with* L, $\Phi_\mu(q)$ *and* $\Psi_\mu(q)$ *then for each* $\mu \in \mathbb{R}$ *and each non-zero vector* $b \in \mathbb{C}^m$ *the solutions of the following Cauchy problems*

$$\begin{cases} \dot{X}(t) = A(t)X(t) + e^{i\mu t}Pb, \quad t \geq 0 \\ X(0) = 0, \end{cases} \qquad (A(t), \mu, Pb, 0)$$

and

$$\begin{cases} \dot{X}(t) = -X(t)A(t) + e^{i\mu t}(I-P)b, \quad t \geq 0 \\ X(0) = 0, \end{cases} \qquad (-A(t), \mu, (I-P)b, 0)$$

are bounded, where $\Phi_\mu(q)$ *and* $\Psi_\mu(q)$ *are the solutions of* $(A(t), \mu, I, 0)$ *and* $(-A(t), \mu, I, 0)$ *respectively.*

Conclusions. The main objective of this article is to discuss the recent results on exponential stability and dichotomy of autonomous and nonautonomous first order linear differential systems on finite dimensional spaces. This is very helpful for the people working in this area of research.

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