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# NORMABILITY AND DUALITY IN THE TWO-DIMENSIONAL LORENTZ SPACES

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**Abstract.** The two-dimensional Lorentz space $L_2^{p,q}$ is defined as a special case from the two-dimensional space $\Lambda_2^p(w)$ just as was done in the classical dimension one. The normability and duality of the space $L_2^{p,q}$ are discussed.

## 1 Introduction

In [7] (see also [3]), Lorentz introduced the classical Lorentz space, denoted by $\Lambda^p(w)$ which consists of all measurable functions for which

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty (f^*(t))^p w(t)dt\right)^{1/p} < \infty,$$

where $0 < p < \infty$, $w$ is a weight function (a non-negative, locally integrable function on $(0,\infty)$, not equivalent to 0) and

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \qquad t \geq 0$$

is the non-increasing rearrangement of $f$, $\mu_f$ being its distribution function.

In [2] (see also [1]), the authors defined a two-dimensional analogue of the space $\Lambda^p(w)$, denoted by $\Lambda_2^p(w)$, as follows

Let $E \subset \mathbf{R}^2$ and for $x_1 \in \mathbf{R}$, $\varphi_E(x_1) = |\{x_2 \in \mathbf{R} : (x_1, x_2) \in E\}|$. The decreasing rearrangement of $\varphi_E$ is given by

$$\varphi_E^*(t_1) = \inf\{\lambda : |\{x_1 \in \mathbf{R} : \varphi_E(x_1) > \lambda\}| \leq t_1\}, \qquad t_1 \geq 0.$$

Then, the two-dimensional decreasing rearrangement of the set $E$ is defined by

$$E^* = \{(t_1, t_2) \in \mathbf{R}_+^2 : 0 < t_2 < \varphi_E^*(t_1)\}.$$

Now, with these notations, the two-dimensional Lorentz space $\Lambda_2^p(w)$ is the space of all measurable functions $f$ on $\mathbf{R}^2$ for which

$$\|f\|_{\Lambda_2^p(w)} = \left(\int_{\mathbf{R}_+^2} (f^*(t_1, t_2))^p w(t_1, t_2)dt_1dt_2\right)^{1/p} < \infty, \tag{1.1}$$

where $w$ is a non-negative, locally integrable function on $\mathbf{R}_+^2$, not equivalent to 0 and $f^*$ is the two-dimensional decreasing rearrangement of $f$ given by the so called Layer-cake formula

$$f^*(t_1, t_2) = \int_0^\infty \chi_{\{|f|>t\}^*}(t_1, t_2) dt.$$

In a rather surprising result ([2], Theorem 3.7), it was proved that for $1 \le p < \infty$, the expression (1.1) is a norm if and only if $w(t_1, t_2) = v(t_2)$ for some decreasing weight $v$ on $\mathbf{R}^+$. This motivates us to define a variant of $\Lambda_2^p(w)$, to be denoted by $L_2^{p,q}$, which consists of all measurable functions on $\mathbf{R}^2$ for which

$$\|f\|_{L_2^{p,q}} = \left( \int_{\mathbf{R}_+^2} (t_2)^{q/p-1} f^*(t_1, t_2)^q dt_1 dt_2 \right)^{1/q} < \infty. \tag{1.2}$$

It is clear from the above discussion that for $1 \le q < p < \infty$, the space $L_2^{p,q}$ is a normed space. We prove, in this paper that, in fact, the restriction $1 \le q < p < \infty$ is necessary for $L_2^{p,q}$ to be normable. A similar situation exists in dimension one, see, e.g., [6].

Next, our aim is to overcome the restriction $q < p$ in (1.2) to make it a norm. A natural idea, similar to the one-dimensional case, is to replace $f^*$ by $f^{**}$ defined by

$$f^{**}(t_1, t_2) = \frac{1}{t_1 t_2} \int_0^{t_1} \int_0^{t_2} f^*(s_1, s_2) ds_1 ds_2.$$

However, this does not suffice the purpose. In fact, it was shown in [1] that $f^{**}$ is not sublinear. Those authors considered, instead, the function $f_{2,1}^{**}$ defined by

$$f_{2,1}^{**}(t_1, t_2) = \frac{1}{t_1} \int_0^{t_1} \left( \frac{1}{t_2} \int_0^{t_2} f_2^*(., \tau) d\tau \right)_1^* (\sigma) d\sigma$$

and proved that this function is sublinear. Here $f_2^*$ denotes the one-dimensional rearrangement of the function $f$ with respect to the second variable keeping the first variable fixed and in the similar sense, the expression $(...)_1^*$ should be understood. Taking advantage of this fact, we replace $f^*$ in (1.2) by $f_{2,1}^{**}$ and prove that the corresponding expression

$$\|f\|'_{L_2^{p,q}} := \left( \int_{\mathbf{R}_+^2} t_2^{q/p-1} f_{2,1}^{**}(t_1, t_2)^q dt_1 dt_2 \right)^{1/q} \tag{1.3}$$

is a norm for $1 < p, q < \infty$. It is noted, in view of ([1], Theorem 2.4) (see also [4]), that for $1 < p, q < \infty$, the quasi-norm (1.2) and the norm (1.3) are equivalent.

Finally, from the one-dimensional case, it is known that the dual space of $L^{p,q}$, that is, $(L^{p,q})^*$ can be identified as $L^{p',q'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. We shall prove that the similar duality holds also in dimension two.

## 2 Normability

We prove the following normability result

**Theorem 2.1.** *The Lorentz space $L_2^{p,q}$ is not normable for the following cases:*

(i) $0 < p < \infty, \quad 0 < q < 1$; (ii) $0 < p < 1, \quad 1 < q < \infty$.

*Proof.* Our strategy would be to construct a sequence of functions $\{f_k\}, k = 1, 2, ...$ such that in both cases (i) and (ii), the ratio

$$A_k := \frac{\left\| \sum_{k=1}^{n} f_k \right\|_{L_2^{p,q}}}{\sum_{k=1}^{n} \|f_k\|_{L_2^{p,q}}} \to \infty \quad \text{as} \quad n \to \infty$$

(i) For all $x = (x_1, x_2) \in [0,1] \times [0,1]$ and $k = 1, 2, ...$, define the functions $f_k$ as

$$f_k(x) = \begin{cases} \left(\frac{q}{p}\right)^{1/q} 2^{(1+p/q)k}, & 0 < x_1 < 2^{-kp}, 0 < x_2 < 2^{-kp} \\ 0, & \text{otherwise} . \end{cases}$$

Then

$$\begin{aligned} \|f_k\|_{L_2^{p,q}} &= \left( \int_0^\infty \int_0^\infty t_2^{q/p-1} ((f_k)^*(t_1, t_2))^q dt_1 dt_2 \right)^{1/q} \\ &= \left( \int_0^{2^{-kp}} \int_0^{2^{-kp}} t_2^{q/p-1} \left( \left(\frac{q}{p}\right)^{1/q} 2^{(1+p/q)k} \right)^q dt_1 dt_2 \right)^{1/q} \\ &= \left( \left(\frac{q}{p}\right) 2^{(q+p)k} \int_0^{2^{-kp}} dt_1 \int_0^{2^{-kp}} t_2^{q/p-1} dt_2 \right)^{1/q} = 1 \end{aligned}$$

so that

$$\sum_{k=1}^{n} \|f_k\|_{L_2^{p,q}} = \sum_{k=1}^{n} 1 = n.$$

Now, let

$$g_n(x) = \frac{1}{n} \sum_{k=1}^{n} f_k(x),$$

i.e.,

$$g_n(x) = \begin{cases} \frac{1}{n} \left(\frac{q}{p}\right)^{1/q} \left( \sum_{i=1}^{n} 2^{i(1+p/q)} \right), & 0 < x_1 < 2^{-np}, 0 < x_2 < 2^{-np} \\ \frac{1}{n} \left(\frac{q}{p}\right)^{1/q} \left( \sum_{i=1}^{n-1} 2^{i(1+p/q)} \right), & 2^{-np} < x_1 < 2^{-(n-1)p}, 2^{-np} < x_2 < 2^{-(n-1)p} \\ \vdots & \\ \frac{1}{n} \left(\frac{q}{p}\right)^{1/q} \left( 2^{(1+p/q)} + 2^{2(1+p/q)} \right), & 2^{-3p} < x_1 < 2^{-2p}, 2^{-3p} < x_2 < 2^{-2p} \\ \frac{1}{n} \left(\frac{q}{p}\right)^{1/q} \left( 2^{(1+p/q)} \right), & 2^{-2p} < x_1 < 2^{-p}, 2^{-2p} < x_2 < 2^{-p} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $g_n(x)$ is a decreasing function for each $n = 1, 2, ....$ Also for $p \neq q$

$$\begin{aligned}
\|g_n\|^q_{L_2^{p,q}} &= \int_0^\infty \int_0^\infty t_2^{q/p-1}((g_n)^*(t_1,t_2))^q dt_1 dt_2 \\
&= \frac{q}{p}\int_0^{2^{-np}} \int_0^{2^{-np}} t_2^{q/p-1} \left(\frac{1}{n}\sum_{k=1}^n 2^{(1+p/q)k}\right)^q dt_1 dt_2 \\
&\quad + \frac{q}{p}\sum_{m=1}^{n-1} \int_{2^{-(n-m+1)p}}^{2^{-(n-m)p}} \int_{2^{-(n-m+1)p}}^{2^{-(n-m)p}} t_2^{q/p-1} \left(\frac{1}{n}\sum_{k=1}^{n-m} 2^{(1+p/q)k}\right)^q dt_1 dt_2 \\
&= I_0 + \sum_{m=1}^{n-1} I_m. \qquad (2.1)
\end{aligned}$$

Now,

$$\begin{aligned}
I_0 &= \frac{q}{p}\int_0^{2^{-np}} \int_0^{2^{-np}} t_2^{q/p-1} \left(\frac{1}{n}\sum_{k=1}^n 2^{(1+p/q)k}\right)^q dt_1 dt_2 \\
&= \frac{1}{n^q}\left(\sum_{k=1}^n 2^{(1+p/q)k}\right)^q 2^{-np}2^{-nq} \\
&= \frac{1}{n^q}\left(\frac{2^{(1+p/q)}\left(2^{(1+p/q)n}-1\right)}{2^{(1+p/q)}-1}\right)^q 2^{-n(p+q)} \\
&= \frac{1}{n^q}\left(1-2^{-(1+p/q)}\right)^{-q}\left(\left(2^{(1+p/q)n}-1\right)2^{-n(p+q)1/q}\right)^q \\
&= \frac{1}{n^q}\left(1-2^{-(1+p/q)}\right)^{-q}\left(1-2^{-(1+p/q)n}\right)^q \qquad (2.2)
\end{aligned}$$

and

$$\begin{aligned}
I_m &= \frac{q}{p}\int_{2^{-(n-m+1)p}}^{2^{-(n-m)p}} \int_{2^{-(n-m+1)p}}^{2^{-(n-m)p}} t_2^{q/p-1} \left(\frac{1}{n}\sum_{k=1}^{n-m} 2^{(1+p/q)k}\right)^q dt_1 dt_2 \\
&= \left(\frac{q}{p}\right)\left(\frac{1}{n^q}\right)\left(\sum_{k=1}^{n-m} 2^{(1+p/q)k}\right)^q \int_{2^{-(n-m+1)p}}^{2^{-(n-m)p}} dt_1 \int_{2^{-(n-m+1)p}}^{2^{-(n-m)p}} t_2^{q/p-1} dt_2 \\
&= \frac{1}{n^q}\left(\frac{2^{(1+p/q)}\left(2^{(1+p/q)(n-m)}-1\right)}{2^{(1+p/q)}-1}\right)^q \left(2^{-(n-m)p}-2^{-(n-m+1)p}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(2^{-q(n-m)} - 2^{-q(n-m+1)}\right) \\
&= \frac{1}{n^q} \left(1 - 2^{-(1+p/q)}\right)^{-q} \left(2^{(1+p/q)(n-m)} - 1\right)^q 2^{-(n-m)(p+q)} \left(1 - 2^{-p}\right) \left(1 - 2^{-q}\right) \\
&= \frac{1}{n^q} \left(1 - 2^{-(1+p/q)}\right)^{-q} \left(1 - 2^{-(1+p/q)(n-m)}\right)^q \left(1 - 2^{-p}\right) \left(1 - 2^{-q}\right). \qquad (2.3)
\end{aligned}
$$

By using (2.3) and (2.4) in (2.2), we get

$$
\begin{aligned}
\|g_n\|^q_{L_2^{p,q}} &= \frac{1}{n^q} \left(1 - 2^{-(1+p/q)}\right)^{-q} \left(1 - 2^{-(1+p/q)n}\right)^q \\
&\quad + \sum_{m=1}^{n-1} \frac{1}{n^q} \left(1 - 2^{-(1+p/q)}\right)^{-q} \left(1 - 2^{-(1+p/q)(n-m)}\right)^q \left(1 - 2^{-p}\right) \left(1 - 2^{-q}\right) \\
&\geq \left(1 - 2^{-(1+p/q)}\right)^{-q} \Bigg( \frac{1}{n^q} \left(1 - 2^{-(1+p/q)n}\right)^q \\
&\quad + \frac{1}{n^q} \left(1 - 2^{-p}\right) \left(1 - 2^{-q}\right) \left( \sum_{m=1}^{n-1} \left(1 - 2^{-(1+p/q)(n-m)}\right) \right)^q \Bigg) \\
&= \left(1 - 2^{-(1+p/q)}\right)^{-q} \Bigg( \left( \frac{\left(1 - 2^{-(1+p/q)n}\right)}{n} \right)^q \\
&\quad + \left(1 - 2^{-p}\right) \left(1 - 2^{-q}\right) \frac{1}{n^q} \left( \frac{(n-1)n}{2} + \frac{2^{-(1+p/q)} \left(1 - 2^{-(n-1)(1+p/q)}\right)}{2^{-(1+p/q)} - 1} \right)^q \Bigg) \\
&= \left(1 - 2^{-(1+p/q)}\right)^{-q} \Bigg( \left( \frac{\left(1 - 2^{-(1+p/q)n}\right)}{n} \right)^q \\
&\quad + \left(1 - 2^{-p}\right) \left(1 - 2^{-q}\right) \left( \frac{(n-1)}{2} + \frac{1 - 2^{-(n-1)(1+p/q)}}{(1 - 2^{(1+p/q)})n} \right)^q \Bigg) \\
&\to \infty \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Consequently

$$
A_k = \frac{\left\| \sum_{k=1}^{n} f_k \right\|_{L_2^{p,q}}}{\sum_{k=1}^{n} \|f_k\|_{L_2^{p,q}}} = \frac{n\|g_n\|_{L_2^{p,q}}}{n} \to \infty \quad \text{as} \quad n \to \infty
$$

and we are done in this case.

(ii) We define, in this case, for all $x = (x_1, x_2) \in [0,1] \times [0,1]$ and $k = 1, 2, ...,$ the functions $f_k$ as

$$
f_k(x) = \sum_{i=1}^{n} \frac{n}{(1 + (i+k) \mod n)^{1/p}} \chi_{[0,\frac{1}{n}) \times [\frac{i-1}{n} \frac{i}{n})}(x). \qquad (2.4)
$$

Then

$$\sum_{k=1}^{n} f_k(x) = \sum_{i=1}^{n} \frac{n}{i^{1/p}} \chi_{[0,\frac{1}{n})\times[0,\frac{1}{n})}(x) + \sum_{i=1}^{n} \frac{n}{i^{1/p}} \chi_{[0,\frac{1}{n})\times[\frac{1}{n},\frac{2}{n})}(x) + \ldots + \sum_{i=1}^{n} \frac{n}{i^{1/p}} \chi_{[0,\frac{1}{n})\times[\frac{n-1}{n},1)}(x)$$
$$= \sum_{i=1}^{n} \frac{n}{i^{1/p}} \chi_{[0,\frac{1}{n})\times[0,1)}(x),$$

so that

$$\begin{aligned}
\left\|\sum_{k=1}^{n} f_k\right\|^q_{L_2^{p,q}} &= \int_0^\infty \int_0^\infty t_2^{q/p-1} \left(\sum_{k=1}^{n} f_k(t_1,t_2)\right)^{*q} dt_1 dt_2 \\
&= \int_0^\infty \int_0^\infty t_2^{q/p-1} \left(\sum_{i=1}^{n} \frac{n}{i^{1/p}} \chi_{[0,\frac{1}{n})\times[0,1)}(t_1,t_2)\right)^{*q} dt_1 dt_2 \\
&= \int_0^\infty \int_0^\infty t_2^{q/p-1} \left(\sum_{i=1}^{n} \frac{n}{i^{1/p}}\right)^q \chi_{([0,\frac{1}{n})\times[0,1))^*}(t_1,t_2) dt_1 dt_2 \\
&= \int_0^\infty \int_0^\infty t_2^{q/p-1} \left(\sum_{i=1}^{n} \frac{n}{i^{1/p}}\right)^q \chi_{[0,\frac{1}{n})\times[0,1)}(t_1,t_2) dt_1 dt_2 \\
&= \left(\sum_{i=1}^{n} \frac{n}{i^{1/p}}\right)^q \int_0^{1/n} \int_0^1 t_2^{q/p-1} dt_1 dt_2 \\
&= \left(\sum_{i=1}^{n} \frac{n}{i^{1/p}}\right)^q \left(\frac{1}{n}\right)\left(\frac{p}{q}\right). \qquad (2.5)
\end{aligned}$$

We rearrange the terms of $f_1(x)$ in (2.4) as follows:

$$f_1(x) = \sum_{j=1}^{n+1} a_j \chi_{E_j}(x),$$

where

$$\begin{aligned}
a_1 &= \frac{n}{1^{1/p}}, \quad E_1 = \left[0,\frac{1}{n}\right) \times \left[\frac{n-2}{n},\frac{n-1}{n}\right) \\
a_2 &= \frac{n}{2^{1/p}}, \quad E_2 = \left[0,\frac{1}{n}\right) \times \left[\frac{n-1}{n},1\right) \\
a_3 &= \frac{n}{3^{1/p}}, \quad E_3 = \left[0,\frac{1}{n}\right) \times \left[0,\frac{1}{n}\right)
\end{aligned}$$

and so on. It can be seen that $a_1 > a_2 > ... > a_n > 0$ and $a_{n+1} = 0$. Now,

$$\begin{aligned}
(f_1)^*(t_1,t_2) &= \int_0^\infty \chi_{\{f_1>t\}^*}(t_1,t_2)dt \\
&= \int_{a_2}^{a_1} \chi_{E_1^*}(t_1,t_2)dt + \int_{a_3}^{a_2} \chi_{(E_1\cup E_2)^*}(t_1,t_2)dt + ... \\
&\quad + \int_{a_{n+1}}^{a_n} \chi_{(E_1\cup E_2\cup ...\cup E_n)^*}(t_1,t_2)dt \\
&= \sum_{i=1}^{n-1} \left(\frac{n}{i^{1/p}} - \frac{n}{(i+1)^{1/p}}\right) \chi_{\left[0,\frac{1}{n}\right)\times\left[0,\frac{i}{n}\right)} + \frac{n}{n^{1/p}} \chi_{\left[0,\frac{1}{n}\right)\times[0,1)} \qquad (2.6)
\end{aligned}$$

If we follow the same steps to calculate $(f_2)^*$, $(f_3)^*$,...,$(f_n)^*$, we get the same expression, i.e., (2.6). Consequently, for $k = 1, 2, ..., n$, by using Minkowski inequality, we get

$$\begin{aligned}
\|f_k\|_{L_2^{p,q}} &= \left(\int_0^\infty \int_0^\infty t_2^{q/p-1} \left((f_k)^*(t_1,t_2)\right)^q dt_1 dt_2\right)^{1/q} \\
&\le \sum_{i=1}^{n-1} \left(\int_0^{\frac{1}{n}} \int_0^{\frac{i}{n}} t_2^{q/p-1} \left(\frac{n}{i^{1/p}} - \frac{n}{(i+1)^{1/p}}\right)^q dt_2 dt_1\right)^{1/q} \\
&\quad + \left(\int_0^{\frac{1}{n}} \int_0^1 t_2^{q/p-1} \left(\frac{n}{n^{1/p}}\right)^q dt_2 dt_1\right)^{1/q} \\
&= n^{1-1/p-1/q} \left(\frac{p}{q}\right)^{1/q} \left(\sum_{i=1}^{n-1} \left(\frac{(i+1)^{1/p} - i^{1/p}}{(i+1)^{1/p}}\right) + 1\right) \\
&\le n^{1-1/p-1/q} \left(\frac{p}{q}\right)^{1/q} \left(\frac{1}{p} \sum_{i=1}^{n-1} \left(\frac{1}{i+1}\right) + 1\right) \\
&< n^{1-1/p-1/q} \left(\frac{p}{q}\right)^{1/q} \left(\frac{1}{p} (\log n) + 1\right)
\end{aligned}$$

which on using (2.5) gives that

$$A_k = \frac{\left\|\sum_{k=1}^n f_k\right\|_{L_2^{p,q}}}{\sum_{k=1}^n \|f_k\|_{L_2^{p,q}}} > \left(\sum_{i=1}^n \frac{1}{i^{1/p}}\right) \left(\frac{n^{1/p-1}}{\frac{1}{p}(\log n) + 1}\right) \to \infty \quad \text{as} \quad n \to \infty.$$

The assertion is now proved completely. □

## 3 An equivalent norm

As mentioned in Section 1, the restriction $q < p$ for (1.2) to be a norm for the space $L_2^{p,q}$ is necessary. In the case of dimension one, this situation is handled by replacing $f^*$ with $f^{**}$ which is the integral average of $f^*$ taken over the interval of integration. A natural extension in dimension two is the operator $f^{**}$ defined by

$$f^{**}(t_1,t_2) = \frac{1}{t_1t_2}\int_0^{t_2}\int_0^{t_1} f^*(s_1,s_2)ds_1ds_2.$$

**Remark 6.** It can be noted that $f^*(t_1,t_2) \le f^{**}(t_1,t_2)$. Indeed, as $f^*(t_1,t_2)$ is a decreasing function, we have

$$\begin{aligned} f^{**}(t_1,t_2) &= \frac{1}{t_1t_2}\int_0^{t_2}\int_0^{t_1} f^*(s_1,s_2)ds_1ds_2 \\ &\ge \frac{1}{t_1t_2}\int_0^{t_2}\int_0^{t_1} f^*(t_1,t_2)ds_1ds_2 \\ &= \frac{1}{t_1t_2}f^*(t_1,t_2)\int_0^{t_2}\int_0^{t_1} ds_1ds_2 \\ &= f^*(t_1,t_2). \end{aligned}$$

Unfortunately, $f^{**}$ is not the right operator to work with because of the fact that this is not sublinear, see [1]. The correct operator which is also sublinear, also considered in [1], is $f_{2,1}^{**}$ defined by

$$f_{2,1}^{**}(t_1,t_2) = \frac{1}{t_1}\int_0^{t_1}\left(\frac{1}{t_2}\int_0^{t_2} f_2^*(.,\tau)d\tau\right)_1^*(\sigma)d\sigma.$$

We replace $f^{**}$ in (1.2) by $f_{2,1}^{**}$ and write

$$\|f\|'_{L_2^{p,q}} := \left(\int_{\mathbf{R}_+^2} t_2^{q/p-1} f_{2,1}^{**}(t_1,t_2)^q dt_1dt_2\right)^{1/q}. \tag{3.1}$$

We prove the following

**Proposition 3.1.** *For $1 < p,q < \infty$, the space $L_2^{p,q}$ is a normed space with the norm given by (3.1).*

*Proof.* This is straightforward in view of the fact that $f_{2,1}^{**}$ is sublinear and using Minkowski's inequality. □

It is known ([1], Proposition 2.1) that

$$f_{2,1}^{*}(t_1,t_2) \le f_{2,1}^{**}(t_1,t_2) \tag{3.2}$$

and that

$$f_{2,1}^{**}(t_1,t_2) \le f^{**}(t_1,t_2). \tag{3.3}$$

The equivalence of the quasi-norm (1.2) and the norm (3.1) is an easy consequence of ([1], Theorem 2.4) (see also [4]). The precise result is the following

**Proposition 3.2.** *Let $1 < p,q < \infty$. The quasi-norm $\|.\|_{L_2^{p,q}}$ and the norm $\|.\|'_{L_2^{p,q}}$ given respectively, in (1.2) and (3.1) are equivalent.*

## 4 Duality

The following result will be used in the main result of this section.
**Lemma A. ([2], Theorem 2.13)** *If $f$ is a measurable function on $\mathbf{R^2}$, then*

$$f^*(t_1,t_2) = f^*_{2,1}(t_1,t_2),$$

*where*

$$f^*_{2,1}(t_1,t_2) = \left[f^*_2(.,t_2)\right]^*_1(t_1).$$

The following result gives the description of the dual space of the space $L^{p,q}_2$.

**Theorem 4.1.** *For $1 < p,q < \infty$, the space $(L^{p,q}_2)^*$ is isomorphic to $L^{p',q'}_2$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$.*

*Proof.* In view of the definition of the space $L^{p,q}_2$ and ([2], Theorem 3.1), we have for $p=q$

$$(L^{p,p}_2)^* = (L^p_2)^* = L^{p'}_2 = L^{p',p'}_2,$$

where $L^p_2$ denote the standard two-dimensional Lebesgue space. Thus the assertion holds for $p=q$.

Assume that $p \neq q$. Let $g \in L^{p',q'}_2$ be arbitrary but fixed. Define for $f \in L^{p,q}_2$, the functional $\phi_g$ by

$$\phi_g(f) = \int_{\mathbf{R}^2} f(x_1,x_2)g(x_1,x_2)dx_1dx_2.$$

Clearly, $\phi_g$ is linear. It is known ([2], Theorem 2.11) that for measurable functions $f,g$ defined on $\mathbf{R}^2$,

$$\begin{aligned}\int_{\mathbf{R}^2} |f(x_1,x_2)g(x_1,x_2)|dx_1dx_2 &\le \int_{\mathbf{R}^2_+} f^*(t_1,t_2)g^*(t_1,t_2)dt_1dt_2 \\ &\le \int_0^\infty f^*(t_1)g^*(t_1)dt_1. \qquad (4.1)\end{aligned}$$

Now, by using (4.1), Lemma A, (3.2) and Hölder's inequality, we have

$$\begin{aligned}|\phi_g(f)| &= \left|\int_{\mathbf{R}^2} f(x_1,x_2)g(x_1,x_2)dx_1dx_2\right| \\ &\le \int_{\mathbf{R}^2} |f(x_1,x_2)g(x_1,x_2)|dx_1dx_2 \\ &\le \int_0^\infty\int_0^\infty f^*(t_1,t_2)g^*(t_1,t_2)dt_1dt_2 \\ &\le \int_0^\infty\int_0^\infty f^{**}_{2,1}(t_1,t_2)g^{**}_{2,1}(t_1,t_2)dt_1dt_2 \\ &\le \left(\int_0^\infty\int_0^\infty t_2^{q/p-1}\left(f^{**}_{2,1}(t_1,t_2)\right)^q dt_1dt_2\right)^{1/q} \\ &\quad\times\left(\int_0^\infty\int_0^\infty t_2^{q'/p'-1}\left(g^{**}_{2,1}(t_1,t_2)\right)^{q'} dt_1dt_2\right)^{1/q'} \\ &= \|f\|'_{L^{p,q}_2}\|g\|'_{L^{p',q'}_2}\end{aligned}$$

which gives that $\phi_g \in (L_2^{p,q})^*$. Moreover, the last inequality gives that

$$\|\phi_g\|'_{(L_2^{p,q})^*} := \sup_{\|f\|'_{L_2^{p,q}}=1} \frac{|\phi_g(f)|}{\|f\|'_{L_2^{p,q}}} \le \|g\|'_{L_2^{p',q'}}. \tag{4.2}$$

Conversely, let $\phi \in (L_2^{p,q})^*$ be arbitrary. Let $\Sigma$ denote the family of Lebesgue measurable subsets of $\mathbf{R}^2$. Define $\nu(A) = \phi(\chi_A)$, where $A \in \Sigma$. Note that

$$\nu(\emptyset) = \phi(\chi_\emptyset) = \phi(0) = 0.$$

If $\{A_n\}$ is a sequence of pairwise disjoint measurable subsets of $\mathbf{R}^2$ such that $A = \bigcup_{n=1}^{\infty} A_n$, then

$$\begin{aligned}
\sum_{k=1}^{\infty} \nu(A_k) &= \lim_{n\to\infty} \sum_{k=1}^{n} \nu(A_k) \\
&= \lim_{n\to\infty} \phi\left(\sum_{k=1}^{n} \chi_{A_k}\right) = \phi\left(\lim_{n\to\infty} \sum_{k=1}^{n} \chi_{A_k}\right) \\
&= \phi\left(\sum_{k=1}^{\infty} \chi_{A_k}\right) = \phi\left(\chi_A\right) \\
&= \nu(A).
\end{aligned}$$

Therefore, $\nu$ is a countably additive function on $\Sigma$. Also, if $\mu$ denote the Lebesgue measure on $\Sigma$, then for any set $A \in \Sigma$ such that $\mu(A) = 0$, we have that $\chi_A = 0$, $\mu$ almost everywhere on $\mathbf{R}^2$. So

$$\nu(A) = \phi(\chi_A) = 0.$$

Thus, $\mu(A) = 0$ implies $\nu(A) = 0$. So, $\nu$ is absolutely continues with respect to $\mu$. By the Radon-Nikodym Theorem, there exists $g \in L^1 = L_2^{1,1}$ such that

$$\nu(A) = \int_A g(x)dx.$$

Let $f$ be a simple function in $L_2^{p,q}$ such that $f = \sum_{i=1}^{n} c_i \chi_{A_i}$, where $A_1, A_2, ..., A_n$ are disjoint measurable subsets of $\mathbf{R}^2$ and $c_1, c_2, ..., c_n$ are real numbers. Then,

$$\begin{aligned}
\phi(f) &= \sum_{i=1}^{n} c_i \phi\left(\chi_{A_i}\right) \\
&= \sum_{i=1}^{n} c_i \nu(A_i) \\
&= \sum_{i=1}^{n} c_i \int_{A_i} g(x_1, x_2) dx_1 dx_2 \\
&= \int_{A_i} g(x_1, x_2) f(x_1, x_2) dx_1 dx_2
\end{aligned}$$

This holds for all simple functions in $L_2^{p,q}$. Simple functions are dense in $L_2^{p,q}$, therefore,

$$\phi(f)=\int_{\mathbf{R}^2} f(x_1,x_2)g(x_1,x_2)dx_1dx_2$$

for all $f\in L_2^{p,q}$. Let us take $f$ such that

$$f^*(t_1,t_2)=\int_{t_1/2}^{\infty}\int_{t_2/2}^{\infty} h(s_1,s_2)\frac{ds_1ds_2}{s_1s_2}$$

where $h(s_1,s_2)=s_2^{q'/p'-1}\left(g^*(s_1,s_2)\right)^{q'-1}$. Then, by using Proposition 3.2 and the two-dimensional Hardy inequality for the conjugate Hardy operator ([5], Theorem 2), we have

$$\begin{aligned}
\left(\|f\|'_{L_2^{p,q}}\right)^q &\le C\left(\|f\|_{L_2^{p,q}}\right)^q\\
&= C\int_0^\infty\int_0^\infty t_2^{q/p-1}\left(f^*(t_1,t_2)\right)^q dt_1dt_2\\
&= C\int_0^\infty\int_0^\infty t_2^{q/p-1}\left(\int_{t_1/2}^{\infty}\int_{t_2/2}^{\infty} h(s_1,s_2)\frac{ds_1ds_2}{s_1s_2}\right)^q dt_1dt_2\\
&= C\int_0^\infty\int_0^\infty \left(\int_{u_1}^{\infty}\int_{u_2}^{\infty} h(s_1,s_2)\frac{ds_1ds_2}{s_1s_2}\right)^q u_2^{q/p-1}du_1du_2\\
&= C\int_0^\infty\int_0^\infty u_2^{q'/p'-1}\left(g^*(u_1,u_2)\right)^{q'} du_1du_2 = C\|g\|^{q'}_{L_2^{p',q'}}.
\end{aligned}\tag{4.3}$$

Now, on using ([6], Theorem 3.10), (4.1), Proposition 3.2 and (4.3), we obtain

$$\begin{aligned}
\|\phi\|'_{(L_2^{p,q})^*} &= \sup_{f\in L_2^{p,q}}\frac{|\phi(f)|}{\|f\|'_{L_2^{p,q}}}\ge\frac{|\phi(f)|}{\|f\|'_{L_2^{p,q}}}\\
&= \frac{\int_{\mathbf{R}^2} f(x_1,x_2)g(x_1,x_2)dx_1dx_2}{\|f\|'_{L_2^{p,q}}}\\
&= \frac{\int_0^\infty f^*(t_1)g^*(t_1)dt_1}{\|f\|'_{L_2^{p,q}}}\\
&\ge \frac{\int_0^\infty\int_0^\infty f^*(t_1,t_2)g^*(t_1,t_2)dt_1dt_2}{\|f\|'_{L_2^{p,q}}}\\
&= \frac{1}{\|f\|'_{L_2^{p,q}}}\int_0^\infty\int_0^\infty\left(\int_{t_1/2}^{\infty}\int_{t_2/2}^{\infty} h(s_1,s_2)\frac{ds_1ds_2}{s_1s_2}\right)g^*(t_1,t_2)dt_1dt_2\\
&\ge \frac{1}{\|f\|'_{L_2^{p,q}}}\int_0^\infty\int_0^\infty\left(\int_{t_1/2}^{t_1}\int_{t_2/2}^{t_2} h(s_1,s_2)\frac{ds_1ds_2}{s_1s_2}\right)g^*(t_1,t_2)dt_1dt_2\\
&\ge \frac{1}{\|f\|'_{L_2^{p,q}}}\int_0^\infty\int_0^\infty\left(\int_{t_1/2}^{t_1}\int_{t_2/2}^{t_2} s_2^{q'/p'-2}s_1^{-1}ds_1ds_2\right)\left(g^*(t_1,t_2)\right)^{q'} dt_1dt_2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-2^{1-q'/p'}}{\|f\|'_{L_2^{p,q}}}\left(\frac{p'}{q'-p'}\right)(\log(t_1)-\log(t_1/2)) \qquad (4.4)\\
&\quad\times \int_0^\infty\int_0^\infty t_2^{q'/p'-1}\left(g^*(t_1,t_2)\right)^{q'} dt_1 dt_2\\
&= \frac{1-2^{1-q'/p'}}{\|f\|'_{L_2^{p,q}}}\left(\frac{p'}{q'-p'}\right)(\log 2)\,\|g\|^{q'}_{L_2^{p',q'}}\\
&\geq C\left(\|g\|'_{L_2^{p',q'}}\right)^{q'-q'/q}\\
&= C\|g\|'_{L_2^{p',q'}}
\end{aligned}
$$

i.e., $g \in L_2^{p',q'}$. The assertion now follows in view of (4.2) and (4.4). □

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