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INVARIANT SETS OF CONTROL SYSTEMS WITH DISTRIBUTED PARAMETERS WITH TIME DELAY

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Abstract. In this paper we consider the problem of finding conditions ensuring that a given set is strong or weak invariant with respect to control system with time delay. System is described by heat conductivity equation in right-hand side of which there is the control in additive form. Necessary and sufficient conditions were obtained for the invariance of a given set under a geometric restriction and sufficient conditions under an integral restriction. The given conditions differ from results for the control with time delay obtained earlier.

1 Introduction

In this work the problem of strong and weak invariance of sets is considered for control systems with distributed parameters with time delay. The goal of study of invariant sets is holding solutions of equations for as long as possible in a given set (survivability region). Earlier in the works of J.-P. Aubin, A. Feuer, J. Heymann and other authors interesting results on invariance of given sets were obtained for systems with lumped parameters [1-3]. Note, that the theory of differential-delay equations (especially of equations with retarded argument and to the less extent of equations of neutral type) was deeply developed in different directions, the natural settings of problems were found and the adequate terminology was established. Hundreds of the works devoted to the theory and applications of differential-delay equations are published [4, 5]. However, there are theoretical and practical questions about control systems with distributed parameters which cannot be solved by known methods. Typical examples of such problems are the preservation of temperature in permissible limits in a given volume, deviations from undesirable states, etc.

Let Ω be a bounded domain with a piecewise smooth boundary. Let us denote by A the following differential operator

$$A\varphi = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial \varphi}{\partial x_j}\Big), \tag{1.1}$$

where functions $a_{ij} \in L^{\infty}(\Omega)$ satisfy the conditions: $a_{ij}(x) = a_{ji}(x),\ x \in \Omega$ and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^{n} \xi_i^2 \tag{1.2}$$

for any vector $(\xi_1, \xi_2, \ldots, \xi_n) \in R^n$. Inequality (1.2) is the condition of uniform ellipticity of the operator A. We take the space $C_0^2(\Omega)$ - the set of all twice continuously differentiable functions in Ω, vanishing in some boundary strip (for each function there is its own strip), as the domain of the operator A . We consider the following heat exchange equation with time delay

$$\dot{z}(t) + Az(t) = z(t-h) + u(t), \quad 0 < t \leq T, \tag{1.3}$$

where $z(t), u(t)$ are abstract functions, that is, for any $t \geq 0$ they are unique elements of the space H_r , and h is a positive constant. Initial states of equation (1.3) are functions from the set

$$X = \{z(t) : z(t) \in H_r,\ -h \leq t \leq 0\},$$

i.e.

$$z(t) = z_0(t),\ -h \leq t \leq 0, \tag{1.4}$$

where $z_0(t) \in X,\ -h \leq t \leq 0$.

We consider problem (1.3)-(1.4) from the point of view of control, i.e. consider the functions $u(\cdot)$ as control functions. They satisfy the conditions:

$$||u(t)||_{H_r} \leq \rho,\ 0 \leq t \leq T, \tag{1.5}$$

or $u(\cdot) \in L_2([0,T]; H_r)$ and

$$||u(\cdot)||_{L_2([0,T];\, H_r)} \leq \rho. \tag{1.6}$$

Controls $u(\cdot)$ satisfying either (1.5) or (1.6), are called *admissible controls*.

2 Main results

Definition 2.1. A set $W \subset \mathbb{R}$ is called strongly invariant on the time segment $[0, T]$ with respect to problem (1.3)-(1.5), if for any $z_0(t)$ with $\|z_0(t\|_{H_r} \in W, -h \leq t \leq 0)$ and $u(t)$ with $||u(t)||_{H_r} \leq \rho$, $t \in [0, T]$ the inclusion $||z(t)||_{H_r} \in W$ holds for all $0 \leq t \leq T$.

Definition 2.2. A set $W \subset \mathbb{R}$ is called weakly invariant on the time segment $[0, T]$ with respect to problem (1.3)-(1.5), if for any $z_0(t)$ with $||z_0(t)||_{H_r} \in W$, $-h \leq t \leq 0$, there is $u(t)$ with $||u(t)||_{H_r} \leq \rho$, $0 \leq t \leq T$, such that the inclusion $\|z(t)\|_{H_r} \in W$ holds for all $0 \leq t \leq T$.

We study the strong and weak invariance of the set $W = [0, b]$, where b is some positive number. Our goal is to find relations between the parameters T, b, ρ providing the strong or weak invariance of the set W on the time segment $[0, T]$ with respect to problem (1.3)-(1.5) (or (1.3), (1.4), (1.6)).

Proposal 1. For any function $f(t)$ with $||f(t)||_{H_r} \leq M,\ 0 \leq t \leq T$ the following inequality holds:

$$\sum_{k=1}^{\infty} \lambda_k^r \Big(\int_0^t e^{-\lambda_k (t-\tau)} f_k(\tau) d\tau \Big)^2 \leq \Big(\frac{1 - e^{-\lambda_1 t}}{\lambda_1} \Big)^2 M^2,\ 0 \leq t \leq T, \tag{2.1}$$

where $\lambda_1,\ \lambda_2, ...$ are the eigenvalues of the operator A and $f_k(\cdot)$ are the Fourier coefficients of the function $f(t)$.

Proof. We have

$$\sum_{k=1}^{\infty} \lambda_k^r \Big(\int_0^t e^{-\lambda_k (t-\tau)} f_k(\tau) d\tau \Big)^2 = \sum_{k=1}^{\infty} \lambda_k^r \Big(\int_0^t e^{-\frac{\lambda_k}{2} (t-\tau)} \cdot e^{-\frac{\lambda_k}{2} (t-\tau)} f_k(\tau) d\tau \Big)^2$$

$$\leq \sum_{k=1}^{\infty} \lambda_k^r \int_0^t e^{-\lambda_1 (t-\tau)} d\tau \int_0^t e^{-\lambda_1 (t-\tau)} f_k^2(\tau) d\tau \leq \int_0^t e^{-\lambda_1 (t-\tau)} d\tau \int_0^t e^{-\lambda_1 (t-\tau)}$$

$$\sum_{k=1}^{\infty} \lambda_k^r f_k^2(\tau) d\tau \leq \frac{1 - e^{-\lambda_1 t}}{\lambda_1} \int_0^t e^{-\lambda_1 (t-\tau)} M^2 d\tau = \Big(\frac{1 - e^{-\lambda_1 t}}{\lambda_1} \Big)^2 M^2,\ 0 \leq t \leq T.$$

Here the Cauchy-Bunyakovsky inequality was applied and the following relation was taken into account:

$$||f(\tau)||_{H_r}^2 = \sum_{k=1}^{\infty} \lambda_k^r f_k^2(\tau) \leq M^2,\ 0 \leq \tau \leq t \leq T.$$

□

Theorem 2.1. *Let $\rho, b, T > 0, \lambda_1 > 1$. The set $W = [0, b]$ is strongly invariant on the time segment $[0, T]$ with respect to problem (1.3)-(1.5) if and only if the following inequality holds:*

$$\rho \leq (\lambda_1 - 1) b. \tag{2.2}$$

Proof. Necessity. Let $z_0(t)$ satisfy the inequality $||z_0(t)||_{H_r} \leq b, -h \leq t \leq 0$ and $u(t)$ satisfy the inequality $||u(t)||_{H_r} \leq \rho, 0 \leq t \leq h$. Substituting these functions in equation (1.3) and expanding each of them in the Fourier series with respect to the eigenfunctions of the operator A, we get the following infinite system of differential equations

$$\dot{z}_k(t) + \lambda_k z_k(t) = z_{0k}(t-h) + u_k(t), 0 < t \leq T, \tag{2.3}$$

with the initial conditions

$$z_k(0) = z_{0k}(0) = z_{0k}^0,\ k = 1, 2, ... \tag{2.4}$$

Problem (2.3) - (2.4) has a solution in the form of the Cauchy formula:

$$z_k(t) = e^{-\lambda_k t} z_{0k}^0 + \int\limits_0^t e^{-\lambda_k(t-\tau)}(z_{0k}(\tau - h) + u_k(\tau))d\tau. \tag{2.5}$$

Introducing the notation $f_k(\tau) = z_{0k}(\tau - h) + u_k(\tau),\ k = 1, 2, ...,$ for the function $f(\cdot)$ with the Fourier coefficients $f_k(\cdot)$ we have

$$||f(\tau)||_{H_r} = ||z_0(\tau - h) + u(\tau)||_{H_r} \leq ||z_0(\tau - h)||_{H_r} + ||u(\tau)||_{H_r} \leq b + \rho. \tag{2.6}$$

Raising to the second power equalities (2.5), multiplying each of them by λ_k^r, and summarizing them in k, we get

$$||z(t)||_{H_r}^2 = \sum_{k=1}^{\infty} \lambda_k^r \Big(e^{-\lambda_k t} z_{0k}^0 + \int\limits_0^t e^{-\lambda_k(t-\tau)} f_k(\tau)d\tau\Big)^2$$

$$= \sum_{k=1}^{\infty} \lambda_k^r \Big(e^{-2\lambda_k t}|z_{0k}^0|^2 + 2e^{-\lambda_k t} z_{0k}^0 \int\limits_0^t e^{-\lambda_k(t-\tau)} f_k(\tau)d\tau + \Big(\int\limits_0^t e^{-\lambda_k(t-\tau)} f_k(\tau)d\tau\Big)^2\Big)$$

Hence, by applying the Cauchy-Bunyakovsky inequality, Proposal 1 and inequality (2.6), we have

$$\sum_{k=1}^{\infty} \lambda_k^r \Big(e^{-2\lambda_k t}|z_{0k}^0|^2 + 2e^{-\lambda_k t} z_{0k}^0 \int\limits_0^t e^{-\lambda_k(t-\tau)} f_k(\tau)d\tau + \Big(\int\limits_0^t e^{-\lambda_k(t-\tau)} f_k(\tau)d\tau\Big)^2\Big)$$

$$\leq e^{-2\lambda_1 t} \sum_{k=1}^{\infty} \lambda_k^r |z_{0k}^0|^2 + 2e^{-\lambda_1 t} \sum_{k=1}^{\infty} \lambda_k^{\frac{r}{2}} z_{0k}^0 \lambda_k^{\frac{r}{2}} \int\limits_0^t e^{-\lambda_k(t-\tau)} f_k(\tau)d\tau$$

$$+ \sum_{k=1}^{\infty} \lambda_k^r \Big(\int\limits_0^t e^{-\lambda_k(t-\tau)} f_k(\tau)d\tau\Big)^2 \leq e^{-2\lambda_1 t} b^2$$

$$+2e^{-\lambda_1 t} \sqrt{\sum_{k=1}^{\infty} \lambda_k^r |z_{0k}^0|^2} \sqrt{\sum_{k=1}^{\infty} \lambda_k^r \Big(\int\limits_0^t e^{-\lambda_k(t-\tau)} f_k(\tau)d\tau\Big)^2} + \Big(\frac{1 - e^{-\lambda_1 t}}{\lambda_1}\Big)(b + \rho)^2$$

$$\leq e^{-2\lambda_1 t} b^2 + 2e^{-\lambda_1 t} b \frac{1 - e^{-\lambda_1 t}}{\lambda_1}(b + \rho) + \Big(\frac{1 - e^{-\lambda_1 t}}{\lambda_1}\Big)^2 (b + \rho)^2$$

$$= \Big(e^{-\lambda_1 t} b + \frac{1 - e^{-\lambda_1 t}}{\lambda_1}(b + \rho)\Big)^2.$$

Let

$$\chi(t) = e^{-\lambda_1 t} b + \frac{1 - e^{-\lambda_1 t}}{\lambda_1}(b + \rho),\ 0 \leq t \leq h.$$

Then we have

$$\chi(0) = b,\ \chi'(t) = -\lambda_1 e^{-\lambda_1 t} b + e^{-\lambda_1 t}(b+\rho) = e^{-\lambda_1 t}(-\lambda_1 b + b + \rho).$$

This implies that $\chi'(t) \le 0$, i.e. $\chi(t)$ is a non-increasing function, only if condition (8) holds. Thereby, $||z(t)||_{H_r} \le b,\ 0 \le t \le h$. Arguing analogously for the time segment $[h, 2h]$ we have $||z(t)||_{H_r} \le b$. It is clear that continuing this process we can obtain that

$$||z(t||_{H_r} \le b,\ 0 \le t \le T$$

for any number$T > 0$.

Sufficiency. We will conduct the proof of sufficiency assuming the contrary, that is, let W be strongly invariant, but in spite of this inequality (2.2) be violated. We choose the functions $z_0(\cdot),\ u(\cdot)$ as follows:

$$z_{01}(t-h) = \frac{b}{\lambda_1^{\frac{r}{2}}},\ u_1(t) = \frac{\rho}{\lambda_1^{\frac{r}{2}}},\ z_{0k}(t-h) = 0,\ u_k(t) = 0,\ k \ge 2,\ 0 \le t \le h.$$

Then we have

$$||z(t)||^2_{H_r} = \lambda_1^r\Big(e^{-\lambda_1 t} z^0_{01} + \int\limits_0^t e^{-\lambda_1(t-\tau)}(z_{01}(t-\tau) + u_1(\tau))d\tau\Big)^2 = \Big(e^{-\lambda_1 t} b + \frac{1-e^{-\lambda_1 t}}{\lambda_1}(b+\rho)\Big)^2$$

Using calculus, we can show that the function

$$\chi(t) = e^{-\lambda_1 t} b + \frac{1-e^{-\lambda_1 t}}{\lambda_1}(b+\rho)$$

is increasing in the time interval $[0, h]$. Since $\chi(0) = b$, it is obvious that for sufficiently small values of t, $\chi(t) > b$, that is $||z(t)||_{H_r} > b$. This contradicts the strong invariance of the set W. □

Theorem 2.2. *Let $\rho, b > 0, \lambda_1 \ge 1$. Then the set $W = [0, b]$ is weakly invariant with respect to problem (1.3)-(1.5) on the segment $[0, T]$, for any $T > 0$.*

Proof. Let $\lambda_1 \ge 1$. We show that the set $W = [0, b]$ is weakly invariant with respect to problem (1.3)-(1.4) under condition (1.5). Let $z_0(\cdot)-$ be an arbitrary function in X, let us take $u(\cdot) = 0$. Then from solution (2.5) of problem (2.3)-(2.4) we have

$$||z(t)||^2_{H_r} = \sum_{k=1}^{\infty} \lambda_k^r\Big(e^{-\lambda_k t} z^0_{0k} + \int\limits_0^t e^{-\lambda_k(t-\tau)} z_{0k}(t-\tau)d\tau\Big)^2$$

$$\le e^{-2\lambda_1 t}\sum_{k=1}^{\infty} \lambda_k^r |z^0_{0k}|^2 + 2e^{-\lambda_1 t}\sum_{k=1}^{\infty} \lambda_k^r |z^0_{0k}| \int\limits_0^t e^{-\lambda_k(t-\tau)}|z_{0k}(t-\tau)|d\tau$$

$$+\sum_{k=1}^{\infty}\lambda_k^r\Big(\int_0^t e^{-\lambda_k(t-\tau)}z_{0k}^0(t-\tau)d\tau\Big)^2$$

$$\le e^{-2\lambda_1 t}b^2+2e^{-\lambda_1 t}b\frac{1-e^{-\lambda_1 t}}{\lambda_1}b+\Big(\frac{1-e^{-\lambda_1 t}}{\lambda_1}\Big)^2 b^2$$

$$=\Big(e^{-\lambda_1 t}+\frac{1-e^{-\lambda_1 t}}{\lambda_1}\Big)^2 b^2. \tag{2.7}$$

Here Proposal 1 and the following inequality

$$\sum_{k=1}^{\infty}\lambda_k^r|z_{0k}^0|\int_0^t e^{-\lambda_k(t-\tau)}|z_{0k}(t-\tau)|d\tau\le\sqrt{\sum_{k=1}^{\infty}\lambda_k^r|z_{0k}^0|^2}\sqrt{\sum_{k=1}^{\infty}\lambda_k^r\Big(\int_0^t e^{-\lambda_k(t-\tau)}|z_{0k}(t-\tau)|d\tau\Big)^2}$$

were used. If

$$\chi(t)=e^{-\lambda_1 t}+\frac{1-e^{-\lambda_1 t}}{\lambda_1},\ 0\le t\le h,$$

then

$$\chi(0)=1,\ \chi'(t)=(1-\lambda_1)e^{-\lambda_1 t}\le 0.$$

So $0<\chi(t)\le 1$. Hence by (13) we get $\|z(t)\|_{H_r}\le b,\ 0\le t\le h$. Applying the same reasoning to the time segment $[jh,\ (j+1)h],\ j=1,2,...$ (if necessary), we obtain that $\|z(t)\|_{H_r}\le b,\ 0\le t\le T$. □

Remark 5. In Theorem 2.2 the case $T=\infty$ is allowed.

Now let us consider problem (1.3), (1.4), (1.6).

Definition 2.3. A set $W\subset\mathbb{R}$ is called strongly invariant on the time segment $[0,T]$ with respect to problem (1.3), (1.4), (1.6), if for any $z_0(t)$ with $||z_0(t)||_{H_r}\in W,\ t\in[-h,0]$ and $u(t)$ with $||u(\cdot)||_{L_2([0,T];H_r)}\le\rho$ the inclusion $||z(t)||_{H_r}\in W$ holds for all $t\in[0,T]$.

Definition 2.4. A set $W\subset\mathbb{R}$ is called weakly invariant on the time segment $[0,T]$ with respect to problem (1.3), (1.4), (1.6), if for any $z_0(t)$ with $||z_0(t)||_{H_r}\in W,\ t\in[-h,0]$, there is $u(t)$ with $||u(\cdot)||_{L_2([0,T];H_r)}\le\rho$, such that the inclusion $||z(t)||_{H_r}\in W$ holds for all $t\in[0,T]$.

Proposal 2. For any function $g(t),\ 0\le t\le T$ with $\|g(\cdot)\|_{L_2([0,T];H_r)}\le N$ the following inequality holds:

$$\sum_{k=1}^{\infty}\lambda_k^r\Big(\int_0^t e^{-\lambda_k(t-\tau)}g_k(\tau)d\tau\Big)^2\le\frac{1-e^{-2\lambda_1 t}}{2\lambda_1}N^2,\ 0\le t\le T.$$

Proof. We have the following chain of relations

$$\sum_{k=1}^{\infty}\lambda_k^r\Big(\int_0^t e^{-\lambda_k(t-\tau)}g_k(\tau)d\tau\Big)^2\le\sum_{k=1}^{\infty}\lambda_k^r\int_0^t e^{-2\lambda_k(t-\tau)}d\tau\cdot\int_0^t g_k^2(\tau)d\tau$$

$$\leq \frac{1-e^{-2\lambda_1 t}}{2\lambda_1}\sum_{k=1}^{\infty}\lambda_k^r\int\limits_0^t g_k^2(\tau)d\tau = \frac{1-e^{-2\lambda_1 t}}{2\lambda_1}||g(\cdot)||_{L_2([0,t];H_r)} \leq \frac{1-e^{-2\lambda_1 t}}{2\lambda_1}N^2,$$

where the Cauchy-Bunyakovsky inequality and the following inequality $e^{-\lambda_k t} \leq e^{-\lambda_1 t},\ t \geq 0,\ k = 1, 2, \ldots$. were applied. □

Theorem 2.3. *If $\rho, b > 0$, then for any $T > 0$ the set $W = [0, b]$ is not strongly invariant with respect to problem (1.3), (1.4), (1.6) on the segment $[0, T]$.*

Proof. Let $\rho > 0$. In proof of this theorem we take into account the fact that the control value can be made arbitrarily large for sufficiently small time interval. For this purpose, we choose the initial function and control in a special way, namely

$$z_{01}(\tau-h) = \frac{b}{\lambda_1^{\frac{r}{2}}},\ u_{1t}(\tau) = \frac{1}{\lambda_1^{\frac{r}{2}}\sqrt{t}},\ z_{0k}(\tau-h) = 0,\ u_k(\tau) = 0,\ k \geq 2,\ 0 \leq \tau < t \leq h.$$

Then

$$z(t) = \Big(e^{-\lambda_1 t}z_{01}^0 + \int\limits_0^t e^{-\lambda_1(t-\tau)}(z_{01}(\tau-h)+u_{1t}(\tau))d\tau\Big)\varphi_1,\ 0 \leq t \leq h.$$

So

$$||z(t)||_{H_r}^2 = \lambda_1^r\Big(e^{-\lambda_1 t}z_{01}^0 + \int\limits_0^t e^{-\lambda_1(t-\tau)}(z_{01}(\tau-h)+u_{1t}(\tau))d\tau\Big)^2,\ 0 \leq t \leq h,$$

or

$$||z(t)||_{H_r} = e^{-\lambda_1 t}b + b\int\limits_0^t e^{-\lambda_1(t-\tau)}d\tau + \lambda_1^{\frac{r}{2}}\int\limits_0^t e^{-\lambda_1(t-\tau)}u_{1t}(\tau)d\tau.$$

Hence we obtain

$$||z(t)||_{H_r} \geq e^{-\lambda_1 t}b + \frac{1-e^{-\lambda_1 t}}{\lambda_1}b + e^{-\lambda_1 t}\sqrt{t}\rho. \tag{2.8}$$

If

$$\eta(t) = e^{-\lambda_1 t}b + \frac{1-e^{-\lambda_1 t}}{\lambda_1}b + e^{-\lambda_1 t}\sqrt{t}\rho,$$

then

$$\eta(0) = b,\ \eta'(t) = -\lambda_1 e^{-\lambda_1 t}b + e^{-\lambda_1 t}b - \lambda_1\rho e^{-\lambda_1 t}\sqrt{t} + \frac{1}{2\sqrt{t}}\rho e^{-\lambda_1 t} =$$

$$e^{-\lambda_1 t}(b - \lambda_1 b - \lambda_1\rho\sqrt{t} + \frac{\rho}{2\sqrt{t}} > 0$$

for sufficiently small $t > 0$. By this and (2.8) it follows that for sufficiently small $t > 0$ we have the inequality $\|z(t)\| > b$, which contradicts the strong invariance of the set W. This contradiction proves the validity of Theorem 2.3. □

Theorem 2.4. *Let $\rho, b > 0, \lambda_1 \geq 1$. Then the set $W = [0, b]$ is weakly invariant with respect to problem (1.3), (1.4), (1.6) on the segment $[0, T]$ for any $T > 0$.*

The proof of this theorem is similar to the proof of Theorem 2.2.

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