EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 5, Number 4 (2014), 56 – 69

RECOVERING STURM-LIOUVILLE OPERATORS ON HEDGEHOG-TYPE GRAPHS WITH GENERAL MATCHING CONDITIONS

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Communicated by Ya.T. Sultanaev

Key words: Hedgehog-type graphs, differential operators, inverse spectral problems

AMS Mathematics Subject Classification: 34A55, 34B45, 34B07, 47E05.

Abstract. We study boundary value problems on hedgehog-type graphs for second-order ordinary differential equations with general matching conditions. We establish properties of spectral characteristics and investigate the inverse spectral problem of recovering the coefficients of a differential equation from the spectral data. For this inverse problem we prove a uniqueness theorem and provide a procedure for constructing its solution.

1 Introduction

We study an inverse spectral problem for Sturm-Liouville differential operators on the so-called hedgehog-type graphs with general matching conditions in the interior vertices. Inverse spectral problems consist in recovering operators from their spectral characteristics. The main results on inverse spectral problems for Sturm-Liouville operators on an *interval* are presented in the monographs [7], [11], [12] and other works. Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [6], [9], [10], [14], [15], [18] and the references therein). Most of the results in this direction are devoted to direct problems of studying properties of the spectrum and the root functions for operators on graphs. Inverse spectral problems, because of their nonlinearity, are more difficult to investigate, and nowadays there exists only a small number of papers in this area. In particular, inverse spectral problems of recovering *the coefficients* of differential operators *on trees* (i.e on graphs without cycles) were solved in [1], [3], [22]. Inverse problems for Sturm-Liouville operators on graphs with a cycle were studied in [23], [24], [25] and other papers but only in the case of the so-called *standard matching conditions.* In particular, in this case the uniqueness result was obtained in [24] for hedgehog-type graphs.

In the present paper we consider Sturm-Liouville operators on hedgehog-type graphs with generalized matching conditions (see Section 2 for definitions). This class of matching conditions appears in applications and produces new qualitative difficulties in investigating nonlinear inverse coefficient problems. For studying this class of inverse problems we develop the ideas of the method of spectral mappings [20], [21]. We prove a uniqueness theorem for this class of nonlinear inverse problems, and provide a constructive procedure for finding their solution. In order to construct the solution, we solve, in particular, an important auxiliary inverse problem for a quasi-periodic boundary value problem on the cycle with discontinuity conditions in interior points. The obtained results are natural generalizations of the well-known results on inverse problems for differential operators on an interval and on graphs with standard matching conditions.

We note that results and methods of the inverse spectral problem theory can be useful for investigating inverse problems for partial differential equations (see [7]). Inverse problems for partial differential equations are reflected in the monographs [5], [8], [16], [17] and others.

The paper is organized as follows. In Section 2 we introduce the main notions and give a formulation of the inverse problem. In Section 3 spectral properties are studied. Section 4 is devoted to the solution of the inverse problem.

2 Formulation of the inverse problem

Consider a compact graph G in $\mathbf{R}^{\mathbf{m}}$ with the set of edges $\mathcal{E} = \{e_0, \dots, e_r\}$, where e_0 is a cycle, $\mathcal{E}' = \{e_1, \dots, e_r\}$ are boundary edges. Let $\{v_1, \dots, v_{r+N}\}$ be the set of vertices, where $V = \{v_1, \dots, v_r\}$, $v_k \in e_k$, are boundary vertices, and $U = \{v_{r+1}, \dots, v_{r+N}\}$ are internal vertices, $U = \mathcal{E}' \cap e_0$. The cycle e_0 consists of N parts:

$$e_0 = \bigcup_{k=1}^{N} e_{r+k}\,, \quad e_{r+k} = [v_{r+k}, v_{r+k+1}],\ k = \overline{1, N},\ v_{r+N+1} := v_{r+1}.$$

Each boundary edge e_j, $j = \overline{1, r}$ has the initial point at v_j, and the end point in the set U. The set $\mathcal{E}'$ consists of N groups of edges: $\mathcal{E}' = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_N\,, \mathcal{E}_k \cap e_0 = v_{r+k}$. Let r_k be the number of edges in $\mathcal{E}_k$; hence $r = r_1 + \cdots + r_N$. Denote $m_0 = 1$, $m_k = r_1 + \cdots + r_k$, $k = \overline{1, N}$. Then

$$\mathcal{E}_k = \{e_j\},\ j = \overline{m_{k-1} + 1, m_k}, \quad v_{r+k} = \bigcap_{j=m_{k-1}+1}^{m_k} e_j\,, \quad k = \overline{1, N}.$$

Thus, the boundary edge $e_j \in \mathcal{E}_k$ can be viewed as the segment $e_j = [v_j, v_{r+k}]$. For example, the graph G with $N = 3$ and $r = 4$ is on fig.1.

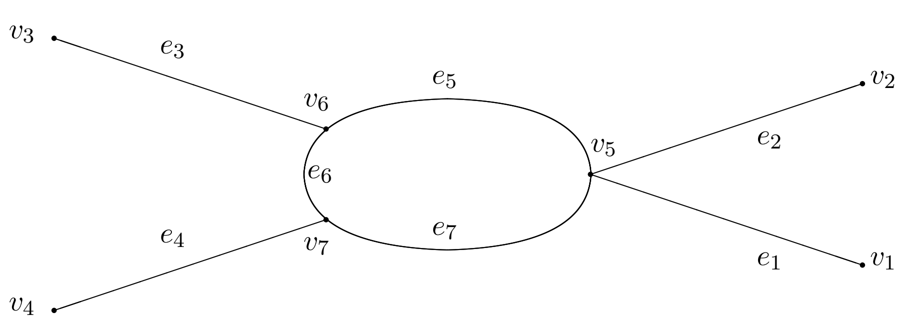

Fig. 1

Let T_j be the length of the edge e_j, $j = \overline{1, r+N}$, and let $T := T_{r+1} + \ldots + T_{r+N}$ be the length of the cycle e_0. Put $b_0 = 0$, $b_k = T_{r+1} + \ldots + T_{r+k}$, $k = \overline{1, N}$. Then $b_N = T$.

Each edge e_j, $j = \overline{1, r+N}$ is parameterized by the parameter $x_j \in [0, T_j]$, and $x_j = 0$ corresponds to the vertex v_j. The whole cycle e_0 is parameterized by the parameter $x \in [0, T]$, where $x = x_{r+j} + b_{j-1}$ for $x_{r+j} \in [0, T_{r+j}]$, $j = \overline{1, N}$.

An integrable function Y on G may be represented as $Y = \{y_j\}_{j=\overline{1,r+N}}$, where the function $y_j(x_j)$, $x_j \in [0, T_j]$, is defined on the edge e_j. The function $y(x)$, $x \in [0, T]$ on the cycle e_0 is defined by $y(x) = y_{r+j}(x_{r+j})$, $j = \overline{1, N}$.

Let $Q = \{q_j\}_{j=\overline{1,r+N}}$ be an integrable real-valued function on G; Q is called the potential. The function $q(x)$, $x \in [0, T]$ is defined by $q(x) = q_{r+j}(x_{r+j})$, $j = \overline{1, N}$. Denote $U_j(Y) := y_j'(0) - h_j y_j(0)$, $j = \overline{1, r+N}$, $U_{r+N+1} := U_{r+1}$, where h_j are real numbers. Consider the following differential equation on G:

$$-y_j''(x_j) + q_j(x_j) y_j(x_j) = \lambda y_j(x_j), \quad x_j \in [0, T_j], \quad j = \overline{1, r+N}, \tag{2.1}$$

where λ is the spectral parameter, the functions y_j, y_j', $j = \overline{1, r+N}$, are absolutely continuous on $[0, T_j]$ and satisfy the following matching conditions in each internal vertex $v_{\mu+1}$, $\mu = \overline{r+1, r+N}$:

$$\left.\begin{aligned} y_{\mu+1}(0) = \alpha_j y_j(T_j) \quad \text{for all} \quad e_j \in \mathcal{E}'_{\mu-r+1}, \\ U_{\mu+1}(Y) = \sum_{e_j \in \mathcal{E}'_{\mu-r+1}} \beta_j y_j'(T_j), \end{aligned}\right\} \tag{2.2}$$

$$y_{r+N+1} := y_{r+1},\ h_{r+N+1} := h_{r+1},\ \mathcal{E}_{N+1} := \mathcal{E}_1,\ \mathcal{E}'_{\mu-r+1} := \mathcal{E}_{\mu-r+1} \cup e_\mu,$$

where α_j and b_j are real numbers, and $\alpha_j \beta_j \neq 0$. For definiteness, let $\alpha_j \beta_j > 0$. The matching conditions (2.2) are a generalization of the standard matching conditions (see [24]), where $\alpha_j = \beta_j = 1$, $h_j = 0$.

Let us consider the boundary value problem B_0 on G for equation (2.1) with the matching conditions (2.2) and with the following boundary conditions at the boundary vertices $v_1, \ldots, v_r$:

$$U_j(Y) = 0, \quad j = \overline{1, r}.$$

Denote by $\Lambda_0 = \{\lambda_{n0}\}_{n \geq 0}$ the eigenvalues (counting with multiplicities) of B_0. Moreover, we also consider the boundary value problems $B_{\nu_1, \ldots, \nu_p}$, $p = \overline{1, r}$, $1 \leq \nu_1 < \ldots \nu_p \leq r$ for equation (2.1) with the matching conditions (2.2) and with the boundary conditions

$$y_k(0) = 0,\ k = \nu_1, \ldots, \nu_p, \quad U_j(Y) = 0,\ j = \overline{1, r},\ j \neq \nu_1, \ldots, \nu_p.$$

Denote by $\Lambda_{\nu_1, \ldots, \nu_p} := \{\lambda_{n, \nu_1, \ldots, \nu_p}\}_{n \geq 0}$ the eigenvalues (counting with multiplicities) of $B_{\nu_1, \ldots, \nu_p}$.

It will be shown in Section 4 that an important role for solving inverse problems on graphs is played by an auxiliary quasi-periodic boundary value problem on the cycle with discontinuity conditions in interior points. The parameters of this auxiliary problem depend on the parameters of B_0. More precisely, let us introduce real numbers

$\gamma_j, \eta_j,\ (j=\overline{1,N-1}),\ h, \alpha, \beta$ by the formulae

$$\left.\begin{aligned} \gamma_j = \sqrt{\frac{\alpha_{r+j}}{\beta_{r+j}}}\,,\ \eta_j = \gamma_j h_{r+j+1},\ j = \overline{1,N-1}, \quad h = h_{r+1}, \\ \alpha = \alpha_{r+N} \prod_{j=1}^{N-1} \gamma_j \prod_{j=1}^{N-1} \beta_{r+j}, \quad \beta = \prod_{j=1}^{N-1} \gamma_j \prod_{j=1}^{N} \beta_{r+j}. \end{aligned}\right\} \tag{2.3}$$

Clearly, $\alpha\beta > 0,\ \gamma_j > 0,\ j = \overline{1,N-1}$. Using these parameters we consider the following quasi-periodic discontinuity boundary value problem B on the cycle e_0:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0,T), \tag{2.4}$$

$$y(0) = \alpha y(T), \quad y'(0) - hy(0) = \beta y'(T), \tag{2.5}$$

$$y(b_j+0) = \gamma_j y(b_j-0),\ y'(b_j+0) = \gamma_j^{-1} y'(b_j-0) + \eta_j y(b_j-0),\ j = \overline{1,N-1}, \tag{2.6}$$

$$0 < b_1 < \ldots < b_{N-1} < b_N = T.$$

Let $S(x,\lambda)$ and $C(x,\lambda)$ be solutions of equation (2.4) satisfying discontinuity conditions (2.6) and the initial conditions $S(0,\lambda) = C'(0,\lambda) = 0,\ S'(0,\lambda) = C(0,\lambda) = 1$. Put $\varphi(x,\lambda) = C(x,\lambda) + hS(x,\lambda)$. Eigenvalues $\{\lambda_n\}_{n\ge 1}$ of B coincide with zeros of the characteristic function

$$a(\lambda) = \alpha\varphi(T,\lambda) + \beta S'(T,\lambda) - (1+\alpha\beta). \tag{2.7}$$

Put $d(\lambda) := S(T,\lambda),\ Q(\lambda) = \alpha\varphi(T,\lambda) - \beta S'(T,\lambda)$. All zeros $\{z_n\}_{n\ge 1}$ of the entire function $d(\lambda)$ are simple, i.e. $\dot d(z_n) \ne 0$, where $\dot d(\lambda) := \frac{d}{d\lambda} d(\lambda)$. Denote $M_n = -\dfrac{d_1(z_n)}{\dot d(z_n)}$, where $d_1(\lambda) := C(T,\lambda)$. The sequence $\{M_n\}_{n\ge 1}$ is called the Weyl sequence. Put $\omega_n = \operatorname{sign} Q(z_n),\ \Omega = \{\omega_n\}_{n\ge 1}$.

We choose and fix one edge $e_{\xi_i} \in \mathcal{E}_i$ from each block $\mathcal{E}_i,\ i = \overline{1,N}$, i.e. $m_{i-1}+1 \le \xi_i \le m_i$. Denote by $\xi := \{k :\ k = \xi_1, \ldots, \xi_N\}$ the set of indices $\xi_i,\ i = \overline{1,N}$. Let α_j and $\beta_j,\ j = \overline{1,r+N}$, be known a priori. The inverse problem is formulated as follows.

Inverse problem 1. Given $2^N + r - N$ spectra $\Lambda_j,\ j = \overline{0,r},\ \Lambda_{\nu_1,\ldots,\nu_p},\ p = \overline{2,N},\ 1 \le \nu_1 < \ldots < \nu_p \le r,\ \nu_j \in \xi$, and Ω, construct the potential Q on G and $H := [h_j]_{j=\overline{1,r+N}}$.

Obviously, in general it is not possible to recover also all coefficients α_j and β_j. Note that this inverse problem is a generalization of the classical inverse problems for Sturm-Liouville operators on an interval or on graphs.

Example 2.1. Let $N=3,\ r=4$ (see Fig.1).
Case 1. Take $\xi_1 = 2,\ \xi_2 = 3,\ \xi_3 = 4$. Then we specify Ω and the following spectra: $\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_{23}, \Lambda_{24}, \Lambda_{34}, \Lambda_{234}$.
Case 2. Take $\xi_1 = 1,\ \xi_2 = 3,\ \xi_3 = 4$. Then we specify Ω and the following spectra: $\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_{13}, \Lambda_{14}, \Lambda_{34}, \Lambda_{134}$.

Let us formulate the uniqueness theorem for the solution of Inverse problem 1. For this purpose together with q we consider a potential $\tilde q$. Everywhere below if a symbol α denotes an object related to q, then $\tilde\alpha$ will denote the analogous object related to $\tilde q$.

Theorem 2.1. *If $\Lambda_j = \tilde{\Lambda}_j,\ j = \overline{0, r},\ \Lambda_{\nu_1,\ldots,\nu_p} = \tilde{\Lambda}_{\nu_1,\ldots,\nu_p},\ p = \overline{2, N},\ 1 \le \nu_1 < \ldots < \nu_p \le r,\ \nu_j \in \xi,$ and $\Omega = \tilde{\Omega}$, then $Q = \tilde{Q}$ and $H = \tilde{H}$.*

This theorem will be proved in Section 4. We will also provide there a constructive procedure for the solution of Inverse problem 1. In Section 3 we study properties of spectral characteristics and prove some auxiliary assertions.

3 Properties of spectral characteristics

Let $S_j(x_j, \lambda),\ C_j(x_j, \lambda),\ j = \overline{1, r+N},\ x_j \in [0, T_j]$, be the solutions to equation (2.1) on the edge e_j with the initial conditions

$$S_j(0, \lambda) = C_j'(0, \lambda) = 0, \quad S_j'(0, \lambda) = C_j(0, \lambda) = 1. \tag{3.1}$$

Put $\varphi_j(x, \lambda) = C_j(x_j, \lambda) + h_j S_j(x_j, \lambda)$. For each fixed $x_j \in [0, T_j]$, the functions $S_j^{(\nu)}(x_j, \lambda),\ C_j^{(\nu)}(x_j, \lambda),\ \varphi_j^{(\nu)}(x_j, \lambda),\ j = \overline{1, r+N},\ \nu = 0, 1$, are entire in λ of order 1/2. Moreover,

$$\langle \varphi_j(x_j, \lambda),\ S_j(x_j, \lambda) \rangle \equiv 1,$$

where $\langle y, z \rangle := yz' - y'z$ is the Wronskian of y and z.

Lemma 3.1. *The following relations hold for $k = \overline{1, N-1},\ \nu = 0, 1$:*

$$S^{(\nu)}(b_{k+1} - 0, \lambda) = \gamma_k S(b_k - 0, \lambda) C_{r+k+1}^{(\nu)}(T_{r+k+1}, \lambda) + \gamma_k^{-1} S'(b_k - 0, \lambda) S_{r+k+1}^{(\nu)}(T_{r+k+1}, \lambda)$$
$$+ \eta_k S(b_k - 0, \lambda) S_{r+k+1}^{(\nu)}(T_{r+k+1}, \lambda), \tag{3.2}$$

$$C^{(\nu)}(b_{k+1} - 0, \lambda) = \gamma_k C(b_k - 0, \lambda) C_{r+k+1}^{(\nu)}(T_{r+k+1}, \lambda) + \gamma_k^{-1} C'(b_k - 0, \lambda) S_{r+k+1}^{(\nu)}(T_{r+k+1}, \lambda)$$
$$+ \eta_k C(b_k - 0, \lambda) S_{r+k+1}^{(\nu)}(T_{r+k+1}, \lambda), \tag{3.3}$$

Indeed, fix $k = \overline{1, N-1}$. Let $x \in [b_k, b_{k+1}]$, i.e. $x = x_{r+k+1} + b_k$, $x_{r+k+1} \in [0, T_{r+k+1}]$. Using the fundamental system of solutions $S_{r+k+1}(x_{r+k+1}, \lambda), C_{r+k+1}(x_{r+k+1}, \lambda)$, on e_{r+k+1}, one has

$$S^{(\nu)}(x, \lambda) = A(\lambda) C_{r+k+1}^{(\nu)}(x_{r+k+1}, \lambda) + B(\lambda) S_{r+k+1}^{(\nu)}(x_{r+k+1}, \lambda), \quad \nu = 0, 1.$$

Taking initial conditions (3.1) for $j = r + k + 1$ into account we find the coefficients $A(\lambda)$ and $B(\lambda)$, and arrive at (3.2). Relation (3.3) is proved similarly.

Let here and below $\lambda = \rho^2$, $\tau := Im\, \rho \ge 0$, $\Pi := \{\rho : \ \tau \ge 0\}$, $\Pi_\delta := \{\rho : \ \arg \rho \in [\delta, \pi - \delta]\}$, $\delta \in (0, \pi/2)$. The following theorem describes the asymptotic behavior of $S(x, \lambda)$ and $C(x, \lambda)$ on each interval $x \in (b_j, b_{j+1})$ (see [26]).

Theorem 3.1. *Fix $j = \overline{1, N-1}$. For $x \in (b_j, b_{j+1}),\ \nu = 0, 1,\ m = 1, 2,\ |\rho| \to \infty$,*

$$S^{(\nu)}(x, \lambda) = \Big(\prod_{k=1}^{j} \xi_k^+ \Big) \frac{d^\nu}{dx^\nu} \Big(\frac{\sin \rho x}{\rho} + \sum_{k=1}^{j} \sum_{1 \le \mu_1 < \ldots < \mu_k \le j} \Big(\prod_{i=1}^{k} \frac{\xi_{\mu_i}^-}{\xi_{\mu_i}^+} \Big) \frac{\sin(\rho \alpha_{\mu_1, \ldots, \mu_k}(x))}{\rho} \Big)$$

$$+O(\rho^{\nu+m-3}e^{\tau x}),$$

$$C^{(\nu)}(x,\lambda) = \Big(\prod_{k=1}^{j}\xi_k^+\Big)\frac{d^\nu}{dx^\nu}\Big(\cos\rho x + \sum_{k=1}^{j}\sum_{1\le\mu_1<\ldots<\mu_k\le j}\Big(\prod_{i=1}^{k}\frac{\xi_{\mu_i}^-}{\xi_{\mu_i}^+}\Big)\cos(\rho\alpha_{\mu_1,\ldots,\mu_k}(x))\Big)$$

$$+O(\rho^{\nu+m-3}e^{\tau x}),$$

where

$$\xi_j^{\pm} := \frac{\gamma_j+\gamma_j^{-1}}{2}, \quad \alpha_{\mu_1,\ldots,\mu_k}(x) := 2\sum_{i=1}^{k}(-1)^{i-1}b_{\mu_i} + (-1)^k x.$$

Using Theorem 3.1, we obtain for $|\rho|\to\infty,\ \rho\in\Pi_\delta$:

$$a(\lambda) = \frac{(\alpha+\beta)\xi}{2}\,e^{-i\rho T}[1], \quad d(\lambda) = -\frac{\xi}{2i\rho}\,e^{-i\rho T}[1], \quad \xi := \prod_{j=1}^{N-1}\xi_j^+\,. \tag{3.4}$$

Moreover,

$$a(\lambda) = O(e^{\tau T}), \quad d(\lambda) = O(\rho^{-1}e^{\tau T}), \quad |\rho|\to\infty, \quad \rho\in\Pi. \tag{3.5}$$

Fix $k=\overline{1,r}$. Let $\Phi_k = \{\Phi_{kj}\}_{j=\overline{1,r+N}}$, be the solution of equation (2.1) satisfying (2.2) and the boundary conditions

$$U_j(\Phi_k) = \delta_{jk}, \quad j=\overline{1,r}, \tag{3.6}$$

where δ_{jk} is the Kronecker symbol. Denote $M_k(\lambda) := \Phi_{kk}(0,\lambda)$, $k=\overline{1,r}$. The function $M_k(\lambda)$ is called the *Weyl function* with respect to the boundary vertex v_k. Clearly,

$$\Phi_{kk}(x_k,\lambda) = S_k(x_k,\lambda) + M_k(\lambda)\varphi_k(x_k,\lambda), \quad x_k\in[0,T_k], \quad k=\overline{1,r}, \tag{3.7}$$

and consequently,

$$\langle\varphi_k(x_k,\lambda),\,\Phi_{kk}(x_k,\lambda)\rangle \equiv 1. \tag{3.8}$$

Denote $M_{kj}^1(\lambda) := \Phi_{kj}(0,\lambda)$, $M_{kj}^0(\lambda) := \Phi_{kj}'(0,\lambda) - h_j\Phi_{kj}'(0,\lambda)$. Then

$$\Phi_{kj}(x_j,\lambda) = M_{kj}^1(\lambda)S_j(x_j,\lambda) + M_{kj}^0(\lambda)\varphi_j(x_j,\lambda), \quad x_j\in[0,T_j],\ j=\overline{1,r+N},\ k=\overline{1,r}. \tag{3.9}$$

In particular, $M_{kk}^1(\lambda)=1$, $M_{kk}^0(\lambda)=M_k(\lambda)$. Substituting (3.9) into (2.2) and (3.6) we obtain a linear algebraic system D_k with respect to $M_{kj}^\nu(\lambda)$, $\nu=0,1$, $j=\overline{1,r+N}$. The determinant $\Delta_0(\lambda)$ of D_k does not depend on k and has the form

$$\Delta_0(\lambda) = \sigma(\lambda)\Big(a_0(\lambda) + \sum_{k=1}^{N}\sum_{1\le\mu_1<\ldots<\mu_k\le N} a_{\mu_1\ldots\mu_k}(\lambda)\prod_{i=1}^{k}\Big(\sum_{e_j\in\mathcal{E}_{\mu_i}}\Omega_j(\lambda)\Big)\Big), \tag{3.10}$$

where

$$\sigma(\lambda) = \prod_{j=1}^{r}(\alpha_j\varphi_j(T_j,\lambda)), \quad \Omega_j(\lambda) = \frac{\beta_j\varphi_j'(T_j,\lambda)}{\alpha_j\varphi_j(T_j,\lambda)}\,, \tag{3.11}$$

$$a_0(\lambda) = a(\lambda), \quad a_1(\lambda) = \alpha d(\lambda). \tag{3.12}$$

We note that the coefficients $a_0(\lambda)$ and $a_{\mu_1\ldots\mu_k}(\lambda)$ in (3.10) depend only on $S_j^{(\nu)}(T_j,\lambda)$ and $C_j^{(\nu)}(T_j,\lambda)$, for $j=\overline{r+1,r+N}$, and (3.12) follows from Lemma 3.1. We do not need concrete formulae for the other coefficients $a_{\mu_1\ldots\mu_k}(\lambda)$. The function $\Delta_0(\lambda)$ is entire in λ of order 1/2, and its zeros coincide with the eigenvalues of the boundary value problem B_0. The function $\Delta_0(\lambda)$ is called the characteristic function for the boundary value problems B_0. Let $\Delta_{\nu_1,\ldots,\nu_p}(\lambda)$, $p=\overline{1,r}$, $1\le\nu_1<\ldots<\nu_p\le r$, be the function obtained from $\Delta_0(\lambda)$ by the replacement of $\varphi_j^{(\nu)}(T_j,\lambda)$ with $S_j^{(\nu)}(T_j,\lambda)$ for $j=\nu_1,\ldots,\nu_p$, $\nu=0,1$. More precisely,

$$\Delta_{\nu_1,\ldots,\nu_p}(\lambda)=\sigma_{\nu_1,\ldots,\nu_p}(\lambda)\Big(a_0(\lambda)+\sum_{k=1}^{N}\sum_{1\le\mu_1<\ldots<\mu_k\le N}a_{\mu_1\ldots\mu_k}(\lambda)$$

$$\times\prod_{i=1}^{k}\Big(\sum_{e_j\in\mathcal{E}_{\mu_i},\,j\ne\nu_1,\ldots,\nu_p}\Omega_j(\lambda)+\sum_{e_j\in\mathcal{E}_{\mu_i},\,j=\nu_1,\ldots,\nu_p}\Omega_j^0(\lambda)\Big)\Big), \tag{3.13}$$

where

$$\sigma_{\nu_1,\ldots,\nu_p}(\lambda)=\prod_{j=1,\,j\ne\nu_1,\ldots,\nu_p}^{r}(\alpha_j\varphi_j(T_j,\lambda))\prod_{j=\nu_1,\ldots,\nu_p}(\alpha_jS_j(T_j,\lambda)),\ \Omega_j^0(\lambda)=\frac{\beta_jS_j'(T_j,\lambda)}{\alpha_jS_j(T_j,\lambda)}. \tag{3.14}$$

The function $\Delta_{\nu_1,\ldots,\nu_p}(\lambda)$ is entire in λ of order 1/2, and its zeros coincide with the eigenvalues of the boundary value problem $B_{\nu_1,\ldots,\nu_p}$. The function $\Delta_{\nu_1,\ldots,\nu_p}(\lambda)$ is called the characteristic function for the boundary value problem $B_{\nu_1,\ldots,\nu_p}$.

Solving the algebraic system D_k we get by Cramer's rule: $M_{kj}^s(\lambda)=\Delta_{kj}^s(\lambda)/\Delta_0(\lambda)$, $s=0,1$, $j=\overline{1,r+N}$, where the determinant $\Delta_{kj}^s(\lambda)$ is obtained from $\Delta_0(\lambda)$ by the replacement of the column which corresponds to $M_{kj}^s(\lambda)$ with the column of free terms. In particular,

$$M_k(\lambda)=-\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)},\quad k=\overline{1,r}. \tag{3.15}$$

It is known (see [13]) that for each fixed $j=\overline{1,r+N}$, on the edge e_j, there exists a fundamental system of solutions of equation (1) $\{e_{j1}(x_j,\rho),e_{j2}(x_j,\rho)\}$, $x_j\in[0,T_j]$, $\rho\in\Pi$, $|\rho|\ge\rho^*$ with the properties:

1) the functions $e_{js}^{(\nu)}(x_j,\rho)$, $\nu=0,1$, are continuous for $x_j\in[0,T_j]$, $\rho\in\Pi$, $|\rho|\ge\rho^*$;

2) for each $x_j\in[0,T_j]$, the functions $e_{js}^{(\nu)}(x_j,\rho)$, $\nu=0,1$, are analytic for $\operatorname{Im}\rho>0$, $|\rho|>\rho^*$;

3) uniformly in $x_j\in[0,T_j]$, the following asymptotical formulae hold

$$e_{j1}^{(\nu)}(x_j,\rho)=(i\rho)^\nu\exp(i\rho x_j)[1],\ e_{j2}^{(\nu)}(x_j,\rho)=(-i\rho)^\nu\exp(-i\rho x_j)[1],\ \rho\in\Pi,\ |\rho|\to\infty, \tag{3.16}$$

where $[1]=1+O(\rho^{-1})$.

Fix $k=\overline{1,r}$. One has

$$\Phi_{kj}(x_j,\lambda)=A_{kj}^1(\rho)e_{j1}(x_j,\rho)+A_{kj}^0(\rho)e_{j2}(x_j,\rho),\quad x_j\in[0,T_j],\quad j=\overline{1,r+N}. \tag{3.17}$$

Substituting (3.17) into (2.2) and (3.6) we obtain a linear algebraic system D_k^0 with respect to $A_{kj}^{\nu}(\rho)$, $\nu = 0, 1,\ j = \overline{1, r+N}$. The determinant $\delta(\rho)$ of D_k^0 does not depend on k, and has the form

$$\delta(\rho) = \Big(\delta_0 + O\Big(\frac{1}{\rho}\Big)\Big)\rho^{r+N} \exp\Big(- i\rho \sum_{j=1}^{r+N} T_j\Big), \tag{3.18}$$

where δ_0 is the determinant obtained from $\delta(\rho)$ by the replacement of $e_{j1}^{(\nu)}(0,\rho)$, $e_{j1}^{(\nu)}(T_j,\rho)$, $e_{j2}^{(\nu)}(0,\rho)$, $e_{j2}^{(\nu)}(T_j,\rho)$ and h_j with $1, 0, (-1)^{\nu}, (-1)^{\nu}$ and 0, respectively. We assume that $\delta_0 \neq 0$. This condition is called the *regularity condition* for matching. Differential operators on G which do not satisfy the regularity condition, possess qualitatively different properties in connection with the formulation and investigation of inverse problems, and are not considered in this paper; they require a separate investigation. We note that for classical Kirchhoff's matching conditions we have $\alpha_j = \beta_j = 1,\ h_j = 0$, and the regularity condition is satisfied obviously. Solving the algebraic system D_k^0 and using (3.16)-(3.18) we get for each fixed $x_k \in [0, T_k)$:

$$\Phi_{kk}^{(\nu)}(x_k,\lambda) = (i\rho)^{\nu-1}\exp(i\rho x_k)[1], \quad \rho \in \Pi_\delta,\ |\rho| \to \infty. \tag{3.19}$$

In particular, $M_k(\lambda) = (i\rho)^{-1}[1],\ \rho \in \Pi_\delta,\ |\rho| \to \infty$. Moreover, uniformly in $x_k \in [0, T_k]$,

$$\varphi_k^{(\nu)}(x_k,\lambda) = \frac{1}{2}\Big((i\rho)^{\nu}\exp(i\rho x_k)[1] + (-i\rho)^{\nu}\exp(-i\rho x_k)[1]\Big),\ \rho \in \Pi,\ |\rho| \to \infty. \tag{3.20}$$

Using (3.10), (3.20), (3.4) and (3.5), by the well-known method (see, for example, [2]), one can obtain the following properties of the characteristic function $\Delta_0(\lambda)$ and the eigenvalues Λ_0 of the boundary value problem B_0.

1) For $\rho \in \Pi,\ |\rho| \to \infty$,

$$\Delta_0(\lambda) = O\Big(\exp\Big(\tau \sum_{j=1}^{r+N} T_j\Big)\Big).$$

2) There exist $h > 0,\ C_h > 0$ such that

$$|\Delta_0(\lambda)| \geq C_h \exp\Big(\tau \sum_{j=1}^{r+N} T_j\Big)$$

for $\tau \geq h$. Hence, the eigenvalues $\lambda_{n0} = \rho_{n0}^2$ lie in the domain $0 \leq \tau < h$.

3) The number N_ξ of zeros of $\Delta_0(\lambda)$ in the rectangle $\Lambda_\xi = \{\rho:\ \tau \in [0,h],\ \mathrm{Re}\,\rho \in [\xi, \xi+1]\}$ is bounded with respect to ξ.

4) For $n \to \infty$,

$$\rho_{n0} = \rho_{n0}^0 + O\Big(\frac{1}{\rho_{n0}^0}\Big),$$

where $\lambda_{n0}^0 = (\rho_{n0}^0)^2$ are the eigenvalues of the boundary value problem B_0 with $Q = 0$ and $H = 0$.

The characteristic functions $\Delta_{\nu_1,\dots,\nu_p}(\lambda)$ have similar properties. In particular, for $\rho \in \Pi,\ |\rho| \to \infty$,

$$\Delta_{\nu_1,\dots,\nu_p}(\lambda) = O\Big(|\rho|^{-p} \exp\Big(\tau \sum_{j=1}^{r+N} T_j\Big)\Big).$$

Using the properties of the characteristic functions and Hadamard's factorization theorem [4, p. 289], one gets that the specification of the spectrum Λ_0 uniquely determines the characteristic function $\Delta_0(\lambda)$, i.e. if $\Lambda_0 = \tilde\Lambda_0$, then $\Delta_0(\lambda) \equiv \tilde\Delta_0(\lambda)$. Analogously, if $\Lambda_{\nu_1,\dots,\nu_p} = \tilde\Lambda_{\nu_1,\dots,\nu_p}$, then $\Delta_{\nu_1,\dots,\nu_p}(\lambda) \equiv \tilde\Delta_{\nu_1,\dots,\nu_p}(\lambda)$. The characteristic functions can be constructed as the corresponding infinite products (see [7] for details).

4 Solution of Inverse problem 1

In this section we provide a constructive procedure for the solution of Inverse problem 1, and prove its uniqueness.

Fix $k = \overline{1,r}$, and consider the following auxiliary inverse problem on the edge e_k, which is called IP(k).

IP(k). Given two spectra Λ_0 and Λ_k, construct $q_k(x_k)$, $x_k \in [0, T_k]$, and h_k.

In IP(k) we construct the potential only on the edge e_k, but the spectra bring a global information from the whole graph. In other words, IP(k) is not a local inverse problem related to the edge e_k.

Let us prove the uniqueness theorem for the solution of IP(k).

Theorem 4.1. *Fix $k = \overline{1,r}$. If $\Lambda_0 = \tilde\Lambda_0$ and $\Lambda_k = \tilde\Lambda_k$, then $q_k(x_k) = \tilde q_k(x_k)$, a.e. on $[0, T_k]$, and $h_k = \tilde h_k$. Thus, the specification of two spectra Λ_0 and Λ_k uniquely determines the potential q_k on the edge e_k, and the coefficient h_k.*

Proof. Since $\Lambda_0 = \tilde\Lambda_0,\ \Lambda_k = \tilde\Lambda_k$, it follows that

$$\Delta_0(\lambda) \equiv \tilde\Delta_0(\lambda), \quad \Delta_k(\lambda) \equiv \tilde\Delta_k(\lambda),$$

and according to (3.15),

$$M_k(\lambda) = \tilde M_k(\lambda). \tag{4.1}$$

Consider the functions

$$P^k_{1s}(x_k,\lambda) = (-1)^{s-1}\Big(\varphi_k(x_k,\lambda)\tilde\Phi^{(2-s)}_{kk}(x_k,\lambda) - \tilde\varphi^{(2-s)}_k(x_k,\lambda)\Phi_{kk}(x_k,\lambda)\Big), \quad s = 1,2. \tag{4.2}$$

Using (3.8) we calculate

$$\varphi_k(x_k,\lambda) = P^k_{11}(x_k,\lambda)\tilde\varphi_k(x_k,\lambda) + P^k_{12}(x_k,\lambda)\tilde\varphi_k'(x_k,\lambda). \tag{4.3}$$

It follows from (3.19), (3.20) and (4.2) that

$$P^k_{1s}(x_k,\lambda) = \delta_{1s} + O(\rho^{-1}), \quad \rho \in \Pi_\delta,\ |\rho| \to \infty,\ x_k \in (0, T_k]. \tag{4.4}$$

According to (3.7) and (4.2),

$$P_{1s}^k(x_k,\lambda)=(-1)^{s-1}\Big(\Big(\varphi_k(x_k,\lambda)\tilde S_k^{(2-s)}(x_k,\lambda)-\tilde\varphi_k^{(2-s)}(x_k,\lambda)S_k(x_k,\lambda)\Big)$$

$$+(M_k(\lambda)-\tilde M_k(\lambda))\varphi_k(x_k,\lambda)\tilde\varphi_k^{(2-s)}(x_k,\lambda)\Big).$$

It follows from (4.1) that for each fixed x_k, the functions $P_{1s}^k(x_k,\lambda)$ are entire in λ of order 1/2. Together with (4.4) this yields $P_{11}^k(x_k,\lambda)\equiv 1$, $P_{12}^k(x_k,\lambda)\equiv 0$. Substituting these relations into (4.3) we get $\varphi_k(x_k,\lambda)\equiv\tilde\varphi_k(x_k,\lambda)$ for all x_k and λ, and consequently,

$$q_k(x_k)=\tilde q_k(x_k)\ \text{a.e. on } [0,T_k],\quad h_k=\tilde h_k\,.$$

□

Using the method of spectral mappings [21] for the Sturm-Liouville operator on the edge e_k one can get a constructive procedure for finding q_k and h_k. Here we only explain ideas briefly; for details and proofs see [21]. Take a boundary value problem $\tilde B_0$ with $\tilde Q=0$, $\tilde H=0$. Take a fixed $c_1>0$ such that $|\mathrm{Im}\,\rho_{n0}|,|\mathrm{Im}\,\tilde\rho_{n0}|<c_1$. In the ρ- plane we consider the contour γ (with counterclockwise circuit) of the form $\gamma=\gamma^+\cup\gamma^-$, where $\gamma^\pm=\{\rho:\ \pm\mathrm{Im}\,\rho=c_1\}$. Denote

$$\tilde r_k(x_k,\rho,\theta)=\frac{\langle\tilde\varphi_k(x_k,\lambda),\tilde\varphi_k(x_k,\theta)\rangle}{\lambda-\theta}\Big(M_k(\theta)-\tilde M_k(\theta)\Big).$$

For each fixed $x_k\in(0,T_k)$, the function $\varphi_k(x_k,\lambda)$ is the unique solution of the following linear integral equation

$$\tilde\varphi_k(x_k,\lambda)=\varphi_k(x_k,\lambda)+\frac{1}{2\pi i}\int_\gamma\tilde r_k(x_k,\lambda,\theta)\varphi_k(x_k,\theta)\,d\theta. \tag{4.5}$$

Using the solution $\varphi_k(x_k,\lambda)$ of equation (4.5) one can easily construct the coefficients $q_k(x_k)$ and h_k (for details see [7]).

Let us study the following auxiliary inverse problem on the cycle e_0, which is called IP(0). Consider the boundary value problem B of the form (2.4)-(2.6), where the parameters of B_0 are defined by (2.3), and α,β are known.

IP(0). Given $a(\lambda)$, $d(\lambda)$ and Ω, construct $q(x)$, $x\in[0,T]$, h,γ_j and η_j, $j=\overline{1,N-1}$.

This inverse problem is a generalization of the classical periodic inverse problem. Moreover, for the standard matching conditions ($\alpha_j=\beta_j=1,h_j=0$), IP(0) coincides with the classical periodic inverse problem.

This inverse problem IP(0) was solved in [26], where the following theorem is established.

Theorem 4.2. *The specification $a(\lambda),d(\lambda)$ and Ω uniquely determines $q(x),h,\gamma_j$ and η_j, $j=\overline{1,N-1}$. The solution of IP(0) can be found by the following algorithm.*

Algorithm 4.1.
1) Construct $D(\lambda) = a(\lambda) + (1 + \alpha\beta)$.
2) Find zeros $\{z_n\}_{n\geq 1}$ of the entire function $d(\lambda)$.
3) Calculate $Q(z_n)$ via

$$Q(z_n) = \omega_n \sqrt{D^2(z_n) - 4\alpha\beta}.$$

4) Construct $d_1(z_n)$ by

$$d_1(z_n) = \frac{1}{2\alpha}\,(D(z_n) + Q(z_n)).$$

5) Find $\dot{d}(z_n)$.
6) Calculate the Weyl sequence $\{M_n\}_{n\geq 1}$ via $M_n = -\dfrac{d_1(z_n)}{\dot{d}(z_n)}$.
7) From the given data $\{z_n, M_n\}_{n\geq 1}$ construct $q(x), \gamma_j, \eta_j$, $j = \overline{1, N-1}$, by solving the inverse Dirichlet problem with discontinuities inside the interval (see [19]).
8) Find $S(T,\lambda)$, $S'(T,\lambda)$ and $C(T,\lambda)$.
9) Calculate h, using (2.7).

Let us go on to the solution of Inverse problem 1. Firstly, we give the proof of Theorem 2.1.

Assume that $\Lambda_k = \tilde{\Lambda}_k$, $k = \overline{0,r}$, $\Lambda_{\nu_1,\dots,\nu_p} = \tilde{\Lambda}_{\nu_1,\dots,\nu_p}$, $p = \overline{2,N}$, $1 \leq \nu_1 < \ldots < \nu_p \leq r$, $\nu_j \in \xi$, and $\Omega = \tilde{\Omega}$. Then one has

$$\Delta_k(\lambda) \equiv \tilde{\Delta}_k(\lambda), \quad k = \overline{0,r},$$

$$\Delta_{\nu_1,\dots,\nu_p}(\lambda) \equiv \tilde{\Delta}_{\nu_1,\dots,\nu_p}(\lambda), \quad p = \overline{2,N},\ 1 \leq \nu_1 < \ldots < \nu_p \leq r,\ \nu_j \in \xi.$$

Moreover, according to (2.3), $\gamma_j = \tilde{\gamma}_j$, $j = \overline{1,N-1}$, and $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$. Using Theorem 4.1, we get $q_k(x_k) = \tilde{q}_k(x_k)$ a.e. on $[0,T_k]$ and $h_k = \tilde{h}_k$, $k = \overline{1,r}$, and consequently,

$$C_k(x_k,\lambda) \equiv \tilde{C}_k(x_k,\lambda),\ S_k(x_k,\lambda) \equiv \tilde{S}_k(x_k,\lambda),\ \varphi_k(x_k,\lambda) \equiv \tilde{\varphi}_k(x_k,\lambda), \quad k = \overline{1,r}. \tag{4.6}$$

By virtue of (3.11), (3.14) and (4.6) one has

$$\sigma(\lambda) \equiv \tilde{\sigma}(\lambda), \quad \sigma_{\nu_1,\dots,\nu_p}(\lambda) \equiv \tilde{\sigma}_{\nu_1,\dots,\nu_p}(\lambda), \quad \Omega_j(\lambda) \equiv \tilde{\Omega}_j(\lambda), \quad \Omega_j^0(\lambda) \equiv \tilde{\Omega}_j^0(\lambda), \quad j = \overline{1,r}.$$

Using (3.10) and (3.13), we obtain, in particular, $a_0(\lambda) = \tilde{a}(\lambda)$, $a_1(\lambda) = \tilde{a}_1(\lambda)$. In view of (3.12), this yields

$$a(\lambda) = \tilde{a}(\lambda), \quad d(\lambda) = \tilde{d}(\lambda).$$

It follows from Theorem 4.1 that $q_k(x_k) = \tilde{q}_k(x_k)$ a.e. on $[0,T_k]$, $k = \overline{r+1, r+N}$, and $h = \tilde{h}$, $\eta_j = \tilde{\eta}_j$, $j = \overline{1,N-1}$. Taking (2.3) into account, we get $H = \tilde{H}$. Theorem 2.1 is proved.

The solution of Inverse problem 1 can be constructed by the following algorithm.

Algorithm 4.2. Given Λ_k, $k = \overline{0,r}$, $\Lambda_{\nu_1,\dots,\nu_p}$, $p = \overline{2,N}$, $1 \leq \nu_1 < \ldots < \nu_p \leq r$, $\nu_j \in \xi$, and Ω.
1) Construct $\Delta_k(\lambda)$ and $\Delta_{\nu_1,\dots,\nu_p}(\lambda)$.
2) Calculate γ_j, $j = \overline{1,N-1}$, α and β, using (2.3).

3) For each fixed $k=\overline{1,r}$, solve the inverse problem IP(k) and find $q_k(x_k),\ x_k\in[0,T_k]$ on the edge e_k and h_k.
4) For each fixed $k=\overline{1,r}$, construct $C_k(x_k,\lambda), S_k(x_k,\lambda)$ and $\varphi_k(x_k,\lambda)$, $x_k\in[0,T_k]$.
5) Calculate $a(\lambda)$ and $d(\lambda)$, using (3.10), (3.12) and (3.13).
6) From the given $a(\lambda), d(\lambda)$ and Ω, construct $q_k(x_k),\ [0,T_k]$, $k=\overline{r+1,r+N}$, h and η_j, $j=\overline{1,N-1}$.
7) Find H, using (2.3).

Acknowledgements

This work was supported by Grant 1.1436.2014K of the Russian Ministry of Education and Science and by Grant 13-01-00134 of Russian Foundation for Basic Research.

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Received: 03.01.2014