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ON SOLVABILITY OF SOME BOUNDARY VALUE PROBLEMS FOR ELLIPTIC TYPE OPERATOR-DIFFERENTIAL EQUATIONS

R.Z. Humbataliev

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Abstract. Conditions on the coefficients of a class of higher order operator-differential equations that provide solvability of some boundary value problems well-posed for these equations, are obtained in the paper. These boundary value problems are generalizations of some mixed problems for a semiharmonic equation. Relation of solvability of boundary value problems with the sharp value of the norms of intermediate derivative operators in some subspaces are found. The obtained results are applied to the proof of multiple completeness of some system of derivatives of chains of eigen and adjoint vectors corresponding to eigenvalues of the appropriate polynomial operator bundle on the half-plane, and to the completeness of decreasing elementary solutions of a homogeneous equation.

1 Introduction

Many problems of mechanics, mathematical physics, and the theory of partial differential equations are reduced to investigation of solvability of boundary value problems for operator-differential equations in different spaces, and also to investigation of completeness of a part of a system of eigen and adjoint vectors of polynomial operator bundles corresponding to the given operator-differential equation and completeness of elementary decreasing solutions of a homogeneous equation [1], [3], [4], [7-12], [14], [15], [19-21], [25-28].

Note that some problems of the theory of elasticity in the half-strip [19], [20], [25] the problems of the theory of vibrations of mechanical systems, vibrations of an elastic cylinder [10] are reduced to investigation of solvability of some boundary value problems for operator-differential equations and construction of the spectral theory of quadratic bundles and higher order bundles. For example, the stress-strain state of a plate is reduced to solving of the problems of the theory of elasticity in the half-strip. In the papers of P.F. Popkovich [19], [20], Yu.A. Ustinov and Yu.I. Yudovich [25], M.B. Orazov [18], the boundary value problem of the theory of elasticity in the strip $t > 0$, $|x| \leq 1$ is reduced to the solvability of different boundary value problems for a second order equation, and the solution is obtained in the form of limits of decreasing elementary solutions of a homogeneous equation which is closely connected with double completeness of the system of eigen and adjoint vectors.

Further, note that many problems of mechanics and physics are reduced to investigation of solvability of operator- differential equations and spectral problems of various type operator bundles [9-11], [12], [19], [20], [25], [26-28]. In the papers [7], [8] M.V. Keldysh gave the notion of multiple completeness of the system of eigen and adjoint vectors for some classes of operator bundles and showed its relation with solvability of the Cauchy problem for the appropriate operator-differential equations. Further, in this area, sufficiently important results were obtained [2-4], [10-12], [14], [15], [21]. In his paper [3], M.G. Gasymov suggested a method connecting the multiple completeness of a part of root vectors corresponding to eigenvalues on the left half-plane with solvability of some boundary value problems. Later on, in the papers [11], [14], [15] these ideas were developed and new theorems on solvability of boundary value problems and multiple completeness of a part of systems of eigen and adjoint vectors were obtained. Relation of solvability of boundary value problems with the sharp value of the norms of intermediate derivatives operators, that enables to find a wider class of operator-differential equations for which the stated problem is well posed, is shown in the paper [15].

Note that finding of the sharp values of the norms of intermediate derivatives operators is of an independent interest, and has numerous applications in different fields of analysis [13], [14], [16], [22-24], [25], for example, in approximation theory [23, 24].

For substantiation of the Fourier method for solving boundary value problems, it is necessary to use the completeness of the system of root vectors in the space of traces of solutions. This enables to prove the completeness of the system of elementary solutions of a homogeneous equation [4].

In papers [6], [17] conditions on the coefficients of a class of higher order operator-differential equations that provide solvability of some boundary value problems well-posed for these equations. These boundary value problems are generalizations of some mixed problems for the semiharmonic and the finite segment equation. The obtained results are applied to the proof of multiple completeness of some system of derivatives of chains of eigen and adjoint vectors corresponding to eigenvalues on the half-plane and finite segment of the appropriate polynomial operator bundle, and completeness of decreasing elementary solutions of a homogeneous equation.

2 Auxiliary facts and problem statement

Let H be a separable Hilbert space, A be a positive-definite operator with the domain of definition $D(A)$. By H_γ denote the scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma)$, $x, y \in H_\gamma$, $\gamma \geq 0$. For $\gamma = 0$ assume that $H_0 = H$.

By $L_2(R_+; H)$ we denote the Hilbert space of all vector-function $f(t)$ defined on $R_+ = (0, \infty)$ almost everywhere with the values in H such that

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^\infty \|f(t)\|^2 \, dt \right)^{\frac{1}{2}} < \infty.$$

Further, define the following Hilbert spaces ($n \geq 1$) [13]:

$$W_2^{2n}(R_+;H) = \left\{u : u^{(2n)} \in L_2(R_+;H), A^{2n}u \in L_2(R_+;H)\right\}$$

with the norm

$$\|u\|_{W_2^{2n}(R_+;H)} = \left(\left\|u^{(2n)}\right\|^2_{L_2(R_+;H)} + \left\|A^{2n}u\right\|^2_{L_2(R_+;H)}\right)^{\frac{1}{2}}.$$

Here and in the sequel, the derivatives are understood in the sense of distribution theory [13].

We define some subspaces of the space $W_2^{2n}(R_+;H)$, namely

$$W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right) = \left\{u : u \in W_2^{2n}(R_+;H), u^{(s_\nu)}(0) = 0, \nu = \overline{0, n-1}\right\},$$

where the integer $0 \leq s_0 < s_1 < ... < s_{n-1} \leq 2n-1$, and

$$W_2^{2n}\left(R_+;H;\{\nu\}_{\nu=0}^{2n-1}\right) = \left\{u : u \in W_2^{2n}(R_+;H), u^{(\nu)}(0) = 0, \nu = \overline{0, 2n-1}\right\}.$$

It follows by the trace theorem that $W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$ and $W_2^{2n}\left(R_+;H;\{\nu\}_{\nu=0}^{2n-1}\right)$ are complete Hilbert spaces [13].

The spaces $L_2(R;H)$ and $W_2^{2n}(R;H)$, where $R = (-\infty, \infty)$, are defined similarly.

We also define the spaces $D(R_+;H)$ and $D(R;H)$ as the set of infinitely-differentiable functions with compact supports on $[0, \infty)$, R respectively. Assume that

$$D\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right) = \left\{u : u \in D(R_+;H), u^{(s_\nu)}(0) = 0\right\}.$$

It follows by the density theorem [13] that the linear set $D(R_+;H)$ is dense in $W_2^{2n}(R_+;H)$, $D(R;H)$ is dense in $W_2^{2n}(R;H)$, and $D\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$ is dense in the space $W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$.

In the separable Hilbert space H, consider the initial boundary value problem

$$\left(-\frac{d^2}{dt^2} + A^2\right)^n u(t) + \sum_{j=0}^{2n-1} A_{2n-j}u^{(j)}(t) = f(t), t \in R_+ = (0, \infty), \tag{2.1}$$

$$u^{(s_\nu)}(0) = 0, \nu = \overline{0, n-1},\ 0 \leq s_0 \leq s_1 \leq ... \leq s_{n-1} \leq 2n-1, \tag{2.2}$$

where the operator coefficients satisfy the conditions :

1) A is a positive-definite self-adjoint operator;

2) $B_j = A_j A^{-j}$ $\left(j = \overline{1, 2n}\right)$ are bounded operators in H.

Definition 2.1. If for $f(t) \in L_2(R_+;H)$ there exists a vector-function $u(t) \in W_2^{2n}(R_+;H)$ that satisfies equation (2.1) almost everywhere in R_+, then it is said to be a regular solution of equation (2.1).

Definition 2.2. If for any $f(t) \in L_2(R_+;H)$ there exists a regular solution $u(t)$ of equation (2.1) that satisfies boundary condition (2.2) in the sense of convergence

$$\lim_{t\to+0}\left\|u^{(s_\nu)}(t)\right\|_{2n-s_\nu-\frac{1}{2}}=0,\ \nu=\overline{0,n-1},$$

and the estimate

$$\|u\|_{W_2^{2n}(R_+;H)}\le const\,\|f\|_{L_2(R_+;H)}$$

holds, then problem (2.1) ,(2.2) is said to be regularly solvable.

We state the following problem: under which conditions on the coefficients of operator-differential equation (2.1), problem (2.1), (2.2) is regularly solvable?

Operator coefficients should be chosen in such a way that problem (2.1), (2.2) were regularly solvable for sufficiently wide class of operator-differential equations. For obtaining such conditions, it is necessary to find the sharp values of the norms of intermediate derivative operators.

Note that for $A_j=0\ \left(j=\overline{1,2n}\right)$, the boundary value problem

$$\left(-\frac{d^2}{dt^2}+A^2\right)^n u(t)=f(t), t\in R_+=(0,\infty)\,, \tag{2.3}$$

$$u^{(s_\nu)}(0)=0,\nu=\overline{0,n-1}, \tag{2.4}$$

covers in particular, mixed problems for the semiharmonic equation in infinite cylinder. Really, if in the cylinder $\Omega=R_+\times x\,(0,\pi)$ we consider the semiharmonic equation

$$\left(-\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)^n u\,(t,x)=f\,(t,x)\,,\ \ (t,x)\in\Omega=R_+\times x\,(0,\pi)\,, \tag{2.5}$$

$$u_x^{(2k)}(t,0)=u_x^{(2k)}(t,\pi)=0, u_t^{(s_\nu)}(0,x)=0,\ t\in R_+, x\in(0,\pi)\,, k=\overline{0,n-1},\ \nu=\overline{0,n-1}. \tag{2.6}$$

We assume that $A^2y=-\frac{\partial^2 y}{\partial x^2}$ with the domain of definition

$$D\left(A^2\right)=\{y\in L_2\,(0,\pi):\ \ y' \text{ is absolutely continuous, } y''\in L_2\,(0,\pi)\,, y(0)=y\,(\pi)=0\}\,.$$

It follows by Lemma 2.1 that mixed problem (2.5), (2.6) is regularly solvable in the space $L_2\,(R_+\times[0,\pi])$. We state the following problem: under which conditions on the function on the coefficients $P_j\,(x)$ the problem

$$\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)^n u\,(x,t)+\sum_{j=0}^{2n-1}P_{2n,j}\,(x)\,\frac{\partial^{2n}u\,(x,t)}{\partial t^j\partial x^{2n-j}}=f\,(t,x)\,,(t,x)\in\Omega=R_+\times(0,\pi) \tag{2.7}$$

$$u_x^{(2k)}\,(t,0)=u_x^{(2k)}\,(t,\pi)=0, u_t^{(s_\nu)}\,(0,x)=0,\ \ k=\overline{0,n-1},\nu=\overline{0,n-1} \tag{2.8}$$

is also regularly solvable in the space $L_2\,(R_+\times[0,\pi])$?

Further, we are interested in how problem (2.1), (2.2) can be solved by the Fourier method. For that we shall prove the n- fold completeness of the system of eigen and adjoint vectors of the bundle $P(\lambda)=\left(-\lambda^2E+A^2\right)^n+\sum_{j=0}^{2n-1}\lambda^jA_{2n-j}$ corresponding to eigen-values on the left half-plane in the space of traces, and completeness of elementary solutions of the homogeneous equation $P(\frac{d}{dt})u(t) = 0$ in the space of regular solutions.

Obviously, the considered bundle

$$P(\lambda) = \left(-\lambda^2 E + A^2\right)^n + \sum_{j=0}^{2n-1} \lambda^j A_{2n-j}$$

corresponds to operator-differential equation (2.1).

Definition 2.3. If a non-zero vector $\varphi_{i,j,0} \neq 0$ is a solution of the equation $P(\lambda_i)\varphi = 0$, then λ_i is said to be an eigenvalue of the bundle $P(\lambda)$, and $\varphi_{i,j,0}\left(j = \overline{1,p}\right)$ is an eigenvector of the bundle $P(\lambda)$ corresponding to eigenvalue λ_i. If the system $\left\{\varphi_{i,j,1}, \varphi_{i,j,2}, ..., \varphi_{i,j,m_{ij}}\right\}$ satisfies the equation

$$\sum_{s=0}^{q} \frac{1}{s!}\left(\frac{d^s}{d\lambda^s} P(\lambda)\right)\Big|_{\lambda=\lambda_i} \cdot \varphi_{i,j,q-s} = 0,\ q = \overline{1, m_{ij}},$$

then the system $\left\{\varphi_{i,j,1}, \varphi_{i,j,2}, ..., \varphi_{i,j,m_{ij}}\right\}$ is said to be a system of eigen and adjoint vectors corresponding to the eigenvalue λ_i.

Definition 2.4. If the system $\left\{\varphi_{i,j,0}, ..., \varphi_{i,j,m_{ij}}\right\}$ is a chain of eigen and adjoint vectors corresponding to the eigenvalue λ_i, then the vector-functions

$$u_{i,j,h}(t) = e^{\lambda_i t}\left(\frac{t^h}{h!}\varphi_{i,j,0} + \frac{t^{h-1}}{(h-1)!}\varphi_{i,j,1} + ... + \varphi_{i,j,h}\right),\ h = \overline{0, m_{ij}}$$

satisfy the equation $P(d/dt)u(t) = 0$ and are said to be its elementary solutions. If $Re\,\lambda_i < 0$, we shall call them decreasing elementary solutions of this homogeneous equation.

Definition 2.5. Let $Re\lambda_i < 0$. Denote by

$$u^{(s_\nu)}_{i,j,h}(0) \equiv \frac{d^{s_\nu}}{dt^{s_\nu}} u_{i,j,h}(t)/_{t=0} = \varphi^{(s_\nu)}_{i,j,h},\ \nu = \overline{0, n-1}.$$

If the system $\left\{\left(\varphi^{(s_\nu)}_{i,j,h}\right)_{\nu=0}^{n-1}\right\} \subset H^n_{2n} = \underbrace{H_{2n} \times H_{2n} \times \cdots \times H_{2n}}_{n}$ is complete in the space $\tilde{H} = \overset{n-1}{\underset{\nu=0}{\oplus}} H_{2n-s_\nu-\frac{1}{2}}$, we shall say that the $K(\Pi_-)$ - system of eigen and adjoint vectors corresponding to eigenvalues from the left half-plane is n- fold complete in the space of traces.

It follows from the expansion of the resolvent of the operator bundle $P^{-1}(\lambda)$ in the vicinity of the point λ_i that, under conditions 1), 2) and $A^{-1} \in \sigma_\infty(H)$, for n-fold completeness of $K(\Pi_-)$ in the space of traces to hold, it is necessary and sufficient that it follows from the holomorphy property of the vector-function

$$R(\lambda) = \left(A^{2n-s_\nu-\frac{1}{2}} P^{-1}\left(\bar{\lambda}\right)\right)^* \sum_{\nu=0}^{n-1} \lambda^{s_\nu} A^{n-s_\nu-\frac{1}{2}} f_\nu, \tag{2.9}$$

in the left half-plane where $f_\nu = 0, \nu = \overline{0, n-1}$ [2-4], [8].

First we prove the following lemma.

Lemma 2.1. *Let condition 1) be satisfied. Then problem* (2.3), (2.4) *is regularly solvable.*

Proof. Write problem (2.3), (2.4) in the form of the equation $P_0u = f$, where $f \in L_2(R_+;H)$, $u \in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{2n-1}\right)$ and $P_0u = P_0(d/dt)u$ for $u \in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{2n-1}\right)$. Show that $KerP_0 = \{0\}$ and $ImP_0 = L_2(R_+;H)$.

It is obvious that the equation $P_0(d/dt)u = 0$ has a general solution from the space $W_2^{2n}(R_+;H)$ in the form

$$u_0(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k e^{-tA} c_k,$$

where e^{-tA} is a semi-group of bounded operators generated by the operator $(-A)$, the vectors $c_k \in H_{2n-\frac{1}{2}}$ (see [4], [5]). By using the conditions

$$u_0 \in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)\left(u_0^{(s_\nu)} = 0, \nu = \overline{0,n-1}\right),$$

we shall find the vectors $c_k \left(k = \overline{1,n}\right)$. We set

$$\Delta_0 = \begin{bmatrix} E & 0 & 0 & \cdots & 0 \\ -E & E & 0 & \cdots & 0 \\ E & -2E & E & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -E & (2n-1)E & -\ C_{2n-1}^{p}\ E & \cdots & E \end{bmatrix},$$

and let $\Delta_0\left((s_\nu)_{\nu=0}^{n-1}\right)$ be the matrix of dimension $n \times n$ obtained from Δ_0 by deleting the $(s_{\nu+1})$-th rows and columns $\left(\nu = \overline{0,n-1}\right)$. Since Δ_0 is a triangular operator-matrix, then $\Delta_0\left((s_\nu)_{\nu=0}^{n-1}\right)$ will also be a triangular operator-matrix. Then, by the conditions $u_0^{(s_\nu)}(0) = 0$ we get

$$\Delta_0\left((s_\nu)_{\nu=0}^{n-1}\right)\tilde{c} = 0,\ \tilde{c} = (c_0, c_1, ..., c_{n-1}).$$

Hence, we get $\tilde{c} = 0$, i.e. $u_0(t) = 0$.

Now, let us show that $ImP_0 = L_2(R_+;H)$. Denote by $f_1(t)$ a continuation of the vector-function $f(t)$ on $(-\infty, 0]$ as a zero vector-function. Then, by using the Fourier transformation, we see that the vector-function

$$u_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\xi^2E + A^2\right)^{-n}\hat{f}_1(\xi)e^{i\xi t}d\xi,\ t \in R = (-\infty, \infty)$$

satisfies equation (2.3) in R_+ almost everywhere. Show that $u_1(t) \in W_2^{2n}(R;H)$ $(R = (-\infty,\infty))$. By the Plancherel theorem, it suffices to prove that $A^{2n}\hat{u}_1(\xi) \in L_2(R;H)$ and $\xi^{2n}\hat{u}_1(\xi) \in L_2(R;H)$. Obviously,

$$\begin{aligned} \left\|A^{2n}\hat{u}_1(\xi)\right\|_{L_2(R;H)} &= \left\|A^{2n}\left(\xi^2E + A^2\right)^{-n}\hat{f}_1(\xi)\right\|_{L_2(R;H)} \\ &\le \sup_{\xi\in R}\left\|A^{2n}\left(\xi^2E + A^2\right)^{-n}\right\|\left\|\hat{f}_1(\xi)\right\|_{L_2(R;H)} \\ &= \sup_{\xi\in R}\left\|A^{2n}\left(\xi^2E + A^2\right)^{-n}\right\|\left\|f_1\right\|_{L_2(R;H)}. \end{aligned}$$

By the spectral expansion of the operator A it follows that for $\xi \in R$

$$\left\| A^{2n} \left(\xi^2 E + A^2\right)^{-n} \right\| = \sup_{\mu \in \sigma(A)} \left\| \mu^{2n} \left(\xi^2 + \mu^2\right)^{-n} \right\| \leq 1.$$

Therefore the previous inequality implies that $A^{2n} \hat{u}_1(\xi) \in L_2(R; H)$. The inclusion $\xi^{2n} \hat{u}_1(\xi) \in L_2(R; H)$ is proved similarly. Consequently, $u_1(t) \in W_2^{2n}(R; H)$. By $\omega_1(t)$ we denote the restriction of the vector-function $u_1(t) \in W_2^{2n}(R; H)$ on $[0, \infty)$, i.e. $\omega_1(t) = u_1(t)/_{[0,\infty)}$. Obviously, $\omega_1(t) \in W_2^{2n}(R_+; H)$, and it follows by theorem on traces that $\omega_1^{(s_\nu)}(0) \in H_{2n - s_\nu - \frac{1}{2}} \left(\nu = \overline{0, n-1}\right)$. Now, we shall look the solution of the equation $P_0 u = f$ in the form

$$u(t) = \omega_1(t) + \sum_{\nu=0}^{n-1} \frac{t^\nu}{\nu!} A^\nu e^{-tA} c_\nu \equiv \omega_1(t) + u_0(t),$$

where the unknown vectors c_ν are to be defined. It follows by condition (2.4) that

$$\Delta\left((s_\nu)_{\nu=0}^{n-1}\right) \tilde{c} = \tilde{\varphi}, \ \tilde{c} = (c_\nu)_{\nu=0}^{n-1}, \ \tilde{\varphi} = \left(A^{s_\nu}\left(u^{(s_\nu)}(0)\right) - \omega_1^{(s_\nu)}(0)\right)_{\nu=0}^{n-1}.$$

Each component of the vector $\tilde{\varphi}$ belongs to the space $H_{2n-\frac{1}{2}}$, $c_\nu \in H_{2n-\frac{1}{2}} \left(\nu = \overline{0, n-1}\right)$. Hence, $\frac{t^\nu}{\nu!} A^\nu e^{-tA} c_\nu \in W_2^{2n}(R_+; H)$ (see [4], [5]), therefore $u(t)$ is the desired solution. □

Since the operator $P_0 : \overset{0}{W}{}_2^{2n}(R_+; H) \to L_2(R_+; H)$ is bounded, it realizes an isomorphism between these spaces. Then, in the space $W_2^{2n}\left(R_+; H; \{s_\nu\}_{\nu=0}^{n-1}\right)$, the norms $\|u\|_{W_2^{2n}(R_+;H)}$ and $\|P_0 u\|_{L_2(R_+;H)}$ are equivalent. Therefore, by the theorem on intermediate derivatives [13], the following numbers, i.e. the norms of the operators of intermediate derivatives are finite

$$N_j\left(R_+; H; \{s_\nu\}_{\nu=0}^{n-1}\right)$$

$$= \sup_{0 \neq W_2^{2n}\left(R_+; H; \{s_\nu\}_{\nu=0}^{n-1}\right)} \left\| A^{2n-j} u^{(j)} \right\|_{L_{2(R_+;H)}} \|P_0 u\|_{L_{2(R_+;H)}}^{-1} < +\infty, \ j = \overline{0, 2n-1}. \tag{2.10}$$

For determining the norms of intermediate derivatives operators, we consider the following $4n$-th order operator bundle depending on a real parameter β :

$$P_j(\lambda; \beta; A) = \left(-\lambda^2 E + A^2\right)^{2n} - \beta (i\lambda)^{2j} A^{2n-2j}, \ j = \overline{0, 2n-1}, \tag{2.11}$$

where $\beta \in \left(0, d_{2n,j}^{-2n}\right)$ and

$$d_{2n,j} = \begin{cases} \left(\frac{j}{2n}\right)^{\frac{j}{2n}} \left(\frac{2n-j}{2n}\right)^{\frac{2n-j}{2n}}, & j = \overline{0, 2n-1} \\ 1, & j = 0 \end{cases}. \tag{2.12}$$

The following lemma is true.

Lemma 2.2. *Let $\beta \in \left(0, d_{2n,j}^{-2n}\right)$. Then the operator bundle $P_j(\lambda;\beta;A)$ has no spectrum on the imaginary axis and is represented in the form*

$$P_j(\lambda;\beta;A) = F_j(\lambda;\beta;A)\cdot F_j(-\lambda;\beta;A). \tag{2.13}$$

Moreover,

$$F_j(\lambda;\beta;A) = \prod_{k=1}^{2n}\left(\lambda E - \omega_{j,k}(\beta)A\right) \equiv \sum_{m=0}^{2n}\alpha_{m,j}(\beta)\lambda^m A^{n-m}, \tag{2.14}$$

where $Re\,\omega_{j,k}(\beta) < 0$ are the coefficients $\alpha_{m,j}(\beta) > 0$, and they satisfy

$$\sum_{\nu=-\infty}^{\infty}(-1)^{\nu}\alpha_{m+\nu,j}(\beta)\alpha_{m-\nu,j}(\beta) =$$

$$= \begin{cases} \sum\limits_{\nu=-\infty}^{\infty}(-1)^{\nu}c_{m+\nu}c_{m-\nu} - \beta, \ if \ \ m = j \\ \sum\limits_{\nu=-\infty}^{\infty}(-1)^{\nu}c_{m+\nu}c_{m-\nu}, \ \ if \ \ m \neq j, m = \overline{0, 2n-1}. \end{cases} \tag{2.15}$$

Here, the numbers $\alpha_{n,j}(\beta)$ are determined by (2.11) *and*

$$c_m = \begin{cases} (-1)^{\frac{m}{2}}C_n^m, if \ m = 2k, & k = \overline{1, n-1} \\ 0, \ \ if \ m = 2k-1, & k = \overline{1, n-1} \end{cases},$$

and we assume that $\alpha_{m,j}(\beta) = 0, c_m = 0$ for $m > 2n$ and $m < 0$.

Proof. Let $\mu \in \sigma(A)$, where $\sigma(A)$ is the spectrum of the operator$A$. Then, for $\beta \in \left(0, d_{2n,j}^{-2n}\right)$ and for $\lambda = i\xi, \xi \in R$ $j = \overline{1, 2n-1}$, the polynomial

$$P_j(\lambda;\beta;\mu) = \left(-\lambda^2 + \mu^2\right)^{2n} - \beta(i\lambda)^{2j}\mu^{2n-2j}, j = \overline{0, 2n-1}, \tag{2.16}$$

is positive. Indeed,

$$\begin{aligned} P_j(\lambda;\beta;\mu) &= \left(\xi^2+\mu^2\right)^{2n} - \beta\xi^{2j}\mu^{2n-2j} \\ &= \left(\xi^2+\mu^2\right)^{2n}\left(1 - \beta\frac{\xi^{2j}\mu^{2n-2j}}{\left(\xi^2+\mu^2\right)^{2n}}\right) \\ &= \left(\xi^2+\mu^2\right)^{2n}\left(1 - \beta\frac{\left(\frac{\xi}{\mu}\right)^{2j}}{\left(\left(\frac{\xi}{\mu}\right)^2+1\right)^{2n}}\right) \\ &\geq \left(\xi^2+\mu^2\right)^{2n}\left(1 - \beta\sup_{\tau\geq 0}\frac{\tau^{2j}}{\left(\tau^2+1\right)^{2n}}\right) \\ &= \left(\xi^2+\mu^2\right)^{2n}\left(1 - \beta\cdot d_{2n,j}^{2n}\right) > 0. \end{aligned}$$

Thus, the polynomial $P_j(\lambda;\beta;\mu)$ has no roots on the imaginary axis and therefore it has exactly n roots in the left half-plane and exactly n roots in the right half-plane.

Since the roots of the polynomial $P_j(\lambda;\beta;\mu)$ are symmetric with respect to the origin and real axis, and it is homogeneous with respect to λ and μ, we have

$$P_j(\lambda;\beta;\mu)=\prod_{k=1}^{2n}(\lambda-\omega_{j,k}(\beta)\mu)\cdot\prod_{k=1}^{2n}(\lambda+\omega_{j,k}(\beta)\mu), \tag{2.17}$$

where for $Re\omega_{j,k}(\beta)<0$, $\beta\in\left(0,d_{2n,j}^{-2n}\right)$. Denoting

$$F_j(\lambda;\beta;\mu)=\prod_{k=1}^{2n}(\lambda-\omega_{j,k}(\beta)\mu)\equiv\sum_{l=0}^{2n}\alpha_{l,j}(\beta)\lambda^l\mu^{n-l}, \tag{2.18}$$

we obtain

$$P_j(\lambda;\beta;A)=F_j(\lambda;\beta;A)\cdot F_j(-\lambda;\beta;A). \tag{2.19}$$

It follows by Viet's theorem that $\alpha_{n,j}(\beta)>0$, since $\omega_{j,k}(\beta)$ are either real or mutually adjoint complex numbers in the left half-plane.

It is obvious that $\alpha_{n,j}(\beta)=1$. On the other hand, $\alpha_{0,j}(\beta)=(-1)^n\prod_{k=1}^{n}\omega_{j,k}(\beta)$. Since $Re\ \omega_{j,k}(\beta)<0$, then $\alpha_{0,j}(\beta)>0$, because if $\omega_{j,k}(\beta)$ is a complex root, then $\overline{\omega_{j,k}(\beta)}$ is also a root of the polynomial $P_j(\lambda;\beta;\mu)$ (the coefficients are real). On the other hand, by (2.16) and (2.17) we get that $\alpha_{0,j}^2(\beta)=1$. Then $\alpha_{0,j}(\beta)=1$ or $\alpha_{0,j}(\beta)=-1$, but $\alpha_{0,j}(\beta)>0$, and therefore $\alpha_{0,j}(\beta)=1$.

Further, comparing the coefficients of the polynomial $P_j(\lambda;\beta;\mu)$, by (2.16) and (2.19) we get the validity of equality (2.15). Using spectral decomposition of the operator A, by (2.17) and (2.19) we get the statement of the lemma. □

In the case $j=0$, Lemma 2 is proved similarly .

Using expansion (2.17) and representation (2.18), first let us calculate the norms

$$N_j(R_+;H)=\sup_{0\neq u\in W_2^{2n}(R_+;H)}\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R;H)}\|P_0u\|_{L_2(R;H)}^{-1},\ j=\overline{0,2n-1},$$

$$N_j\left(R_+;H;\{\nu\}_{\nu=0}^{2n-1}\right)=\sup_{0\neq u\in W_2^{2n}(R_+;H)}\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R_+;H)}\|P_0u\|_{L_2(R_+;H)}^{-1},\ j=\overline{0,2n-1}.$$

and

$$N_j\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)=$$

$$=\sup_{0\neq u\in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)}\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R_+;H)}\cdot\|P_0u\|_{L_2(R_+;H)}^{-1},\ j=\overline{0,2n-1}.$$

Theorem 2.1. *For the norms $N_j(R;H)$ and $N_j\left(R;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$ the following equalities hold:*

$$N_j(R_+;H)=d_{2n,j}^{-n},\ j=\overline{0,2n-1},$$

and

$$N_j\left(R;H;\{\nu\}_{\nu=0}^{2n-1}\right)=d_{2n,j}^{-n},\ j=\overline{0,2n-1}.$$

Proof. For $u \in D(R;H)$, after simple calculations we get

$$\|P_0(d/dt)u\|^2_{L_2(R;H)} = \left\|\left(-\frac{d^2}{dt^2}+A^2\right)^n\right\|^2_{L_2(R;H)}$$
$$\left\|\sum_{m=0}^{n}(-1)^m C_n^m A^{2n-2m}u^{(2m)}\right\|^2_{L_2(R;H)}$$
$$\left\|\sum_{k=0}^{2n}c_k A^{2n-k}u^{(k)}\right\|^2_{L_2(R;H)}$$

$$= \sum_{k=0}^{n}\left(\sum_{s=-\infty}^{\infty}(-1)^s c_{k+s}c_{k-s}\right)\left\|A^{2n-k}u^{(k)}\right\|^2_{L_2(R;H)}, \tag{2.20}$$

where $c_k = (-1)^k C_n^{\frac{k}{2}}$ for $k=2m$, $m=\overline{0,n}$ and $c_k=0$, for $k=2m-1$, $m=\overline{1,n}$. For $k>2n$ and $k<0$, we assume that $c_k=0$. Similarly we have

$$\|F_j(d/dt;\beta;A)u\|^2_{L_2(R;H)} = \left\|\sum_{k=0}^{2n}\alpha_{k,j}(\beta)A^{2n-k}u^{(k)}\right\|^2_{L_2(R;H)}$$

$$= \sum_{k=0}^{2n}\left(\sum_{s=-\infty}^{\infty}(-1)^s\alpha_{k+s,j}(\beta)\alpha_{k-s,j}(\beta)\right)\left\|A^{2n-k}u^{(k)}\right\|^2_{L_2(R;H)}. \tag{2.21}$$

Then, taking into account equalities (2.15) and (2.20), we get

$$\|F_j(d/dt;\beta;A)u\|^2_{L_2(R;H)} = \|P_0(d/dt)u\|^2_{L_2(R;H)} - \beta\left\|A^{2n-j}u^{(j)}\right\|^2_{L_2(R;H)}. \tag{2.22}$$

Hence, we have $\beta \in \left(0, d_{2n,j}^{-2n}\right)$ and $u \in D(R;H)$, and they satisfy the inequality

$$\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R;H)} \le \frac{1}{\beta^{\frac{1}{2}}}\|P_0(d/dt)u\|^2_{L_2(R;H)}.$$

Passing to limit as $\beta \to d_{2n,j}^{-2n}$, we get that for all $u \in W_2^{2n}(R;H)$ following the inequality is satisfied:

$$\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R;H)} \le d_{2n,j}^{n}\|P_0 u\|_{L_2(R;H)},\ j=\overline{0,2n-1}. \tag{2.23}$$

Let us show that inequality (2.23) is exact. Show this in the case $j=\overline{1,2n-1}$. For $j=0$, it is proved in a similar way, but with some little alterations.

Let $\varepsilon>0$. Denote

$$\mathrm{E}(u_\varepsilon) \equiv \|P_0 u\|^2_{L_2(R;H)} - \left(d_{2n,j}^{-2n}+\varepsilon\right)\left\|A^{2n-j}u^{(j)}\right\|^2_{L_2(R;H)}.$$

Show that there exists a vector-function $u_\varepsilon(t) = g_\varepsilon(t)\varphi_\varepsilon$, where $\varphi_\varepsilon \in H_{4n}$, $\|\varphi_\varepsilon\|=1$, and $g_\varepsilon(t)$ is a scalar function such that $g_\varepsilon(t) \in W_2^{2n}(R)$ and $\mathrm{E}(g_\varepsilon(t)\varphi_\varepsilon)<0$. Using the

Fourier transformation and the Plancherel theorem, we can write it in the form

$$\begin{aligned}
\mathrm{E}(u_\varepsilon) &= \int_{-\infty}^{\infty}\left(\left(\xi^2E+A^2\right)^n g_\varepsilon(\xi)\varphi_\varepsilon, \left(\xi^2E+A^2\right)^n g_\varepsilon(\xi)\varphi_\varepsilon\right) - \left(d_{2n,j}^{-2n}+\varepsilon\right)\times \\
&\quad \times\left(A^{2n-j}\xi^{2j}g_\varepsilon(\xi)\varphi_\varepsilon, A^{2n-j}\xi^{2j}g_\varepsilon(\xi)\varphi_\varepsilon\right)d\xi \\
&= \int_{-\infty}^{\infty}\left(\left(\left(\xi^2E+A^2\right)^{2n}-\left(d_{2n,j}^{-2n}+\varepsilon\right)\right)\xi^{2j}A^{2n-j}\varphi_\varepsilon,\varphi_\varepsilon\right)\left|g_\varepsilon(\xi)\right|^2 d\xi \\
&\equiv \int_{-\infty}^{\infty}\left(P_j\left(i\xi, d_{2n,j}^{-2n}+\varepsilon, A\right)\varphi_\varepsilon,\varphi_\varepsilon\right)\left|g_\varepsilon(\xi)\right|^2 d\xi < 0,
\end{aligned} \tag{2.24}$$

where $\hat{g}_\varepsilon(\xi)$ is the Fourier transformation of the function $g_\varepsilon(t)$. Obviously, if the operator A has even one eigenvalue $\mu > 0$ and eigenvector φ, $\|\varphi\|_0 = 1$, then

$$\begin{aligned}
\left(P_j\left(i\xi, d_{2n,j}^{-2n}+\varepsilon, A\right)\varphi,\varphi\right) &= \left(\xi^2+\mu^2\right)^{2n} - \left(d_{2n,j}^{-2n}+\varepsilon\right)\xi^{2j}\mu^{2n-2j} \\
&= \left(\xi^2+\mu^2\right)\left(1-\left(d_{2n,j}^{-2n}+\varepsilon\right)\frac{\xi^{2j}\mu^{2n-2j}}{\left(\xi^2+\mu^2\right)^{2n}}\right).
\end{aligned}$$

Thus, in some neighbourhood of $\tau = \mu\frac{j}{2n-j}$, the integrand function in (2.24) is negative. Then, by continuity of $\left(P_j\left(i\xi, d_{2n,j}^{-2n}+\varepsilon, A\right)\varphi,\varphi\right)$ with respect to ξ, we get that there exists a neighbourhood (η_1,η_2) of the point $\xi = \mu\frac{j}{2n-j}$, where $\left(P_j\left(i\xi,\, d_{2\,n,j}^{-2\,n}+\varepsilon, A\right)\varphi\,,\,\varphi\right) < 0$. Now, assume

$$g_\varepsilon(t) = \frac{1}{\sqrt{2\pi}}\int_{\eta_1}^{\eta_2}\hat{g}(\xi)d\xi,$$

where $\hat{g}(\xi)$ is an infinitely differentiable function with the support in the interval (η_1,η_2) $(\hat{g}(\xi)=0,\ \xi\in R\backslash(\eta_1,\eta_2))$. It is obvious that $\hat{g}_\varepsilon(\xi)\in W_2^{2n}(R;H)$ and $\mathrm{E}(u_\varepsilon) = \mathrm{E}(g_\varepsilon(t)\varphi) < 0$. If the operator has no eigenvalues, then for $\mu\in\sigma(A)$ we can find a vector φ_δ, $\|\varphi_\delta\| = 1$, such that $A^{(m)}\varphi_\delta = \mu^{(m)}\varphi_\delta + 0(1)$, for $\delta\to 0, m = 1,2,\ldots.$

In this case,

$$\left(P_j\left(i\xi, d_{2n,j}^{-2n}+\varepsilon, A\right)\varphi_\delta,\varphi_\delta\right) = P_j\left(i\xi, d_{2n,j}^{-2n}+\varepsilon,\mu\right)+0(1),$$

as $\delta\to 0$. Then, choosing sufficiently small δ, and using the previous reasonings, we get that $\mathrm{E}(u_\varepsilon) = \mathrm{E}(g_\varepsilon(t)\varphi_\delta) < 0$. Thus, $N_j(R;H) = d_{2n,j}^{-n}\left(j=\overline{0,2n-1}\right)$.

Now, show that $N_j\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right) = d_{2n,j}^{-n}\left(j=\overline{0,2n-1}\right)$. It is obvious that for $u\in W_2^{2n}\left(R_+;H;\{\nu\}_{\nu=0}^{2n-1}\right)$ equality (2.22) holds (since $u^{(\nu)}=0, \nu=\overline{0,2n-1}$). Then for all $u\in W_2^{2n}\left(R_+;H;\{\nu\}_{\nu=0}^{2n-1}\right)$ the inequality

$$\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R_+;H)} \le d_{2n,j}^{-n}\left\|P_0\,u\right\|_{L_2(R_+;H)}, j=\overline{0,2n-1}, \tag{2.25}$$

holds, i.e. $N_j\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)\le d_{2n,j}^{-n}$. We showed that there exists a vector-function $u_\varepsilon(t)\in W_2^{2n}(R;H)$, such that

$$\mathrm{E}(u_\varepsilon) = \left\|P_0\,u_\varepsilon\right\|_{L_2(R;H)}^2 - \left(d_{2n,j}^{-2n}+\varepsilon\right)\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R;H)}^2 < 0.$$

Since $\mathrm{E}(\cdot)$ is a continuous functional in $W_2^{2n}(R;H)$, then there exists $v_\varepsilon(t) \in D(R;H)$, $t \in R_+$ such that $v_\varepsilon(t) = 0$, for $|t| \geq N\,(N > 0)$ and $\mathrm{E}(v_\varepsilon) < 0$. Then assuming $\tilde{u}_\varepsilon(t) = v_\varepsilon(t - N)$, we get that $\tilde{u}_\varepsilon(t) \in W_2^{2n}\left(R_+;H;\{\nu\}_{\nu=0}^{2n-1}\right)$ and

$$\|P_0\,\tilde{u}_\varepsilon\|_{L_2(R_+;H)} - \left(d_{2n,j}^{-2n} + \varepsilon\right)\left\|A^{2n-j}\tilde{u}_\varepsilon^{(j)}\right\|_{L_2(R_+;H)} = \mathrm{E}(\tilde{u}_\varepsilon) < 0.$$

Consequently, inequality (2.16) is exact. □

Now, find $N_j\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$. With the help of coefficients of the polynomial $F_j(\lambda;\beta;A)$ and $P_0(d/dt)A$, we determine the following matrices $R_j(\beta) = \left(r_{p,q,j}(\beta)\right)_{p,q=1}^{2n} : \mathbb{C}^n \to \mathbb{C}^n$ $T = (c_{p,q})_{p,q=1}^{2n} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, where $p \geq q$

$$r_{p,q,j}(\beta) = \sum_{s=0}^{\infty}(-1)^s\,\alpha_{p+s,j}(\beta)\,\alpha_{q-s-1,j}(\beta)\,,\left(\alpha_s(\beta) = 0,\ s < 0, s > 2n\right),$$

and for $p < q$ we assume $r_{p\,q,j}(\beta) = r_{q\,p,j}(\beta)$. For $p \geq q$

$$c_{pq} = \sum_{s=0}^{\infty}(-1)^s\,c_{p+s}c_{q-s-1.}\left(c_s = 0, s < 0, s > 2n\right),$$

and for $p < q$ $c_{p\,q} = c_{q\,p}$, $p,q = \overline{1,n}$, where

$$c_s = \begin{cases} (-1)^{\frac{s}{2}}\,C_n^{\frac{s}{2}}, & \text{if } s = 2k,\ k = \overline{1,2n-1} \\ 0, & \text{if } s = 2k-1,\ k = \overline{1,2n-1}. \end{cases}$$

Let

$$S_j(\beta) = R_j(\beta) - T,\ \tilde{R}_j(\beta) = R_j(\beta) \otimes E^{2n},\ \tilde{T} = T \otimes E^{2n},\ \tilde{S}(\beta) = S(\beta) \otimes E^{2n},$$

where E^{2n} is the unit operator-matrix in $\mathbb{C}^{2n}$, and $\otimes$ means that each element of $\tilde{S}(\beta)$ is an element of the matrix $S_j(\beta)$ multiplied by the unit operator E. Similarly to equality (2.22) for the vector-function $u \in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$, we prove the following statement.

Lemma 2.3. *For any $u \in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$ and $\beta \in \left(0, d_{2n,j}^{-2n}\right)$, the equality*

$$\|F_j(d/dt;\beta;A)\,u\|^2_{L_2(R_+;H)} + \tilde{S}_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right) =$$

$$= \|P_0 u\|^2_{L_2(R_+;H)} - \beta\left\|A^{2n-j}u^{(j)}\right\|^2_{L_2(R_+;H)}, \tag{2.26}$$

holds, where the operator matrix $\tilde{S}_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)$ is obtained from the matrix $\tilde{S}_j(\beta)$ of dimension $n \times n$ by deleting the (s_0+1)-th, (s_1+1)-th,..., $(s_{n-1}+1)$-th rows and columns.

It is obvious that $\tilde{S}_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right) = S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right) \otimes E^n$ is a symmetric operator matrix.

Thus, we have the following.

Lemma 2.4. $\sigma\left(\tilde{S}_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)\right)=\sigma\left(S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)\right)$ *as sets.*

Now, it is obvious that $N_j\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)\geq N_j\left(R_+;H;\{\nu\}_{\nu=0}^{n-1}\right)=d_{2n,j}^{-n}$, since $W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)\supset W_2^{2n}\left(R_+;H;\{\nu\}_{\nu=0}^{n-1}\right)$. The following theorems holds for the case when $N_j\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)=d_{2n,j}^{-n}$.

Theorem 2.2. $N_j\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)=d_{2n,j}^{-n}\left(j=\overline{0,2n-1}\right)$ *if and only if for all* $\beta\in\left(0,d_{2n,j}^{-2n}\right)$

$$S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)>0.$$

Proof. Let $N_j\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)=d_{2n,j}^{-n}$. Then, for $u\in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$, it follows by equality (26) that

$$\|F_j\left(d/dt;\beta;A\right)u\|_{L_2(R_+;H)}^2+\left(\tilde{S}_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)\tilde{\varphi},\tilde{\varphi}\right)$$

$$=\|P_0\,u\|_{L_2(R_+;H)}^2\left(1-\beta\frac{\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R_+;H)}^2}{\|P_0\,u\|_{L_2(R_+;H)}^2}\right)$$

$$\geq\|P_0\,u\|_{L_2(R_+;H)}^2\left(1-\beta\sup_{0\neq u\in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)}\frac{\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R_+;H)}^2}{\|P_0\,u\|_{L_2(R_+;H)}^2}\right)$$

$$=\|P_0\,u\|_{L_2(R_+;H)}^2\left(1-\beta d_{2n,j}^{-2n}\right). \tag{2.27}$$

Since all the roots of the polynomial $F_j\left(\lambda;\beta;A\right)$ are in the left half-plane, this means that the Cauchy problem

$$F_j\left(d/dt;\beta;A\right)u=0,u^{(s_\nu)}(0)=0,u^{(k)}(0)=A^{2n-k-\frac{1}{2}}\varphi_k,k\neq s_\nu, \tag{2.28}$$

has a unique solution in the space $W_2^{2n}\left(R_+;H\right)$ for any collection of $\varphi_k\in H_{2n-k-\frac{1}{2}},k\neq s_\nu\left(\nu=\overline{0,n-1}\right)$. Then, if in (2.27), instead of $u(t)$ we write this solution, we get that for any collection of n vectors $\varphi_k\in H_{2n-k-\frac{1}{2}}$ we have $\left(\tilde{S}_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)\tilde{\varphi},\tilde{\varphi}\right)>0$. Hence, it follows that $S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)>0$.

Conversely, if $S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)>0$, for all $\beta\in\left(0,d_{2n,j}^{-2n}\right)$, then it follows from (2.26) that

$$\|P_0\,u\|_{L_2(R_+;H)}^2-\beta\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R_+;H)}^2>0$$

for $\beta\in\left(0,d_{2n,j}^{-2n}\right)$ and $u\in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)$. Passing to limit as $\beta\to d_{2n,j}^{-n}$, we get $N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\leq d_{2n,j}^{-n}$. But, we showed that $N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\geq d_{2n,j}^{-n}$, i.e. $N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)=d_{2n,j}^{-n}$. □

Thus, if for all $\beta\in\left(0,d_{2,n,j}^{-2n}\right)$, $S_j\left(\beta,\{s_\nu\}_{\nu=0}^{\,n}\right)$ is not positive, then $N_j\left(R_+;\{s_\nu\}_{\nu=0}^{\,n-1}\right)>d_{2,n,j}^{-n}$. Then, $N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\in\left(0,d_{2n,j}^{-2n}\right)$.

In this case the following statement holds.

Theorem 2.3. *If* $S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)$ *is not positive for all* $\beta\in\left(0,d_{2n,j}^{-2n}\right)$, *then* $N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)=\left(\mu\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)\right)^{\frac{1}{2}}$, *where* $\mu\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)$ *is the smallest root of the equation* $\det S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)=0$ *in the interval* $\left(0,d_{2n,j}^{-2n}\right)$.

Proof. As we noted, in this case $N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\in\left(0,d_{2\,n,j}^{-2\,n}\right)$. Then, for $\beta\in\left(0,N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\right)$ and for any $u\in W_2^{2n}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)$, it follows from (2.27) that

$$\|F_j\left(d/dt;\beta;A\right)u\|_{L_2(R_+;H)}^2+\left(\tilde{S}_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)\tilde{\varphi},\tilde{\varphi}\right)$$
$$\geq\|P_0\,u\|^2\left(1-\beta\sup_{0\neq u\in W_2^{2n}\left(R_+;H;\{s_\nu\}_{\nu=0}^{n-1}\right)}\frac{\left\|A^{2n-j}u^{(j)}\right\|_{L_2}^2}{\|P_0\,u\|_{L_2}^2}\right)$$
$$\geq\|P_0\,u\|^2\left(1-\beta N_j^{-2}\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)\right)>0.$$

By considering the Cauchy problem (2.28), for $\beta\in\left(0,N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\right)$ we get $S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)>0$. Then, the first eigenvalue of the matrix $S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)$ is positive for all $\beta\in\left(0,N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\right)$. On the other hand, it follows from definition of $N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)$ that for all $\beta\in\left(0,N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\right)$ there exists a vector-function $u_\beta\in W_2^{2n}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)$ such that

$$\|P_0\,u_\beta\|_{L_2(R_+;H)}^2-\beta\left\|A^{2n-j}u_\beta^{(j)}\right\|_{L_2(R_+;H)}^2<0.$$

Then, it follows by equality (2.26) that for $\beta\in\left(N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right),d_{2n,j}^{-2n}\right)$

$$\left(\tilde{S}_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)\tilde{\varphi}_\beta,\tilde{\varphi}_\beta\right)<0,$$

where $\tilde{\varphi}_\beta=\left(\varphi_{\beta,k}=A^{2n-k-\frac{1}{2}}u_\beta^{(k)}(0)\right)$, $k\neq s_\nu,\nu=\overline{0,n-1}$. Hence, it follows that the first eigenvalue of the matrix $S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)$ is negative for $\beta\in\left(N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right),d_{2n,j}^{-2n}\right)$. As for $\lambda_1(\beta)$, this is a continuous function such that $\lambda_1\left(N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\right)=0$, i.e. $N_j^{-2}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)$ is the smallest root of the equation $\det S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)=0$. □

Theorems 2.2 and 2.3 yield the following statement.

Theorem 2.4. *The norms of the operators of intermediate derivatives satisfy the equality*

$$N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)=\begin{cases}d_{2n,j}^{n}, & \text{if } \det S_j(\beta)\neq0,\ \beta\in\left(0,d_{2n,j}^{-2n}\right),\\ \mu_j^{-\frac{1}{2}}\left((s_\nu)_{\nu=0}^{n-1}\right), & \text{if } \det S_j(\beta_0)=0,\ \beta_0\in\left(0,d_{2n,n}^{-2n}\right).\end{cases}$$

Thus, for finding the exact values of the norm $N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)$, we have to solve the equations $\det S_j\left(\beta,\{s_\nu\}_{\nu=0}^{n}\right)=0$. If this equation has a solution in the interval $\left(0,d_{2n,j}^{-2n}\right)$, then the smallest of them μ_0 will satisfy the condition $\mu_j^{-\frac{1}{2}}=N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)$. Here, we must take into account that $\alpha_j(\mu_0)>0$. If under the condition $\alpha_j(\mu_0)>0$ this equation has no solution in the interval $\left(0,d_{2n,j}^{-2n}\right)$, then $N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)=d_{2n,j}^{-n}$.

3 On solvability of boundary value problem (2.1), (2.2)

Now, we shall find conditions that provide regular solvability of problems (2.1), (2.2). The following theorem is true.

Theorem 3.1. *Let conditions 1), 2) be satisfied, and the operators* $A_j\left(j=\overline{1,2n}\right)$ *satisfy the conditions*

$$\alpha=\sum_{j=0}^{2n-1} N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\|B_{2n-j}\|<1,$$

where $B_j=A_jA^{-j}\left(j=\overline{1,2n}\right)$. *Then problem* (2.1), (2.2) *is regularly solvable.*

Proof. Write problem (2.1), (2.2) in the form of the equation

$$Pu=P_0\,u+P_1\,u=f,$$

where $f\in L_2\left(R_+;H\right),u\in W_2^{2n}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)$. By Lemma 1, the operator

$$P_0:W_2^{2n}\left(R_+;\{L\}\right)\to L_2\left(R_+;H\right)$$

is continuous and boundedly invertible. It follows by condition 2) that the operator $P_1:W_2^{2n}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\to L_2\left(R_+;H\right)$ is also bounded. Indeed,

$$\begin{aligned}\|P_1u\| &= \|P_1(d/dt)u\|=\sum_{j=0}^{2n-1}\left\|A_{2n-j}u^{(j)}\right\|_{L_2(R_+;H)}\\ &\le \sum_{j=0}^{2n-1}\|B_{2n-j}\|\left\|A^{2n-j}u^{(j)}\right\|_{L_2(R_+;H)}\\ &\le const\,\|u\|_{W_2^{2n}(R_+;H)}\,.\end{aligned}$$

Thus, the operator $P:W_2^{2n}\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\to L_2\left(R_+;H\right)$ is bounded. Show that P is an invertible operator. If we denote $v=P_0\,u$, then we get the equation $\left(E+P_1P_0^{-1}\right)v=f$ with respect to v in the space $L_2\left(R_+;H\right)$. Then, for any v (the operator P_0 is an isomorphism),

$$\begin{aligned}\left\|P_1P_0^{-1}v\right\| &= \|P_1\,u\|\le\sum_{j=0}^{2n\,-1}\|B_{2n-j}\|\left\|A^{2n-j}u^{(j)}\right\|\\ &\le \sum_{j=0}^{2n-1}\|B_{2n-j}\|\,N_j\left(R_+;\{s_\nu\}_{\nu=0}^{n-1}\right)\|P_0\,u\|_{L_{2n}(R_+;H)}\\ &= \alpha\,\|P_0\,u\|_{L_{2n}(R_+;H)}=\alpha\,\|v\|_{L_{2n}(R_+;H)}\,.\end{aligned}$$

Since $0<\alpha<1$, the operator $\left(E+P_1P_0^{-1}\right)$ is invertible in $L_2\left(R_+;H\right)$, hence

$$v(t)=(E+P_1P_0^{-1})^{-1}f(t),$$

and

$$u(t) = P_0^{-1}(E + P_1 P_0^{-1})^{-1} f(t).$$

It is obvious that

$$\|u\|_{W_2^{2n}(R_+;H)} \leq const \, \|f\|_{L_2(R_+;H)} \, .$$

The theorem is proved. □

Corollary 3.1. *Let the conditions of Theorem 3.1 be satisfied. Then, the problem*

$$P(d/dt)u(t) = 0, t \in R_+ = (0, \infty) \, , \tag{3.1}$$

$$u^{(s_\nu)}(0) = \varphi_\nu, \nu = \overline{0, n-1}, \tag{3.2}$$

has a unique regular solution, and moreover, for any $\varphi_\nu \in H_{2n-\nu-\frac{1}{2}} \left(\nu = \overline{0, n-1}\right)$

$$\|u\|_{W_2^{2n}(R_+;H)} \leq const \sum_{\nu=0}^{n-1} \|\varphi_\nu\|_{2n-s_\nu-\frac{1}{2}} \, . \tag{3.3}$$

Proof. We write $u(t) = \omega(t) - u_0(t)$, where $u_0(t) = \sum\limits_{k=0}^{n-1} \frac{t^k}{k!} A^k e^{-tA} c_k$. Here c_k are the unknown vectors that are to be determined by the system $u^{(s_\nu)}(0) = -\varphi_\nu \left(\nu = \overline{0, n-1}\right)$. Then, with respect to ω, we get the following problem

$$P(d/dt)\omega(t) = g(t), t \in R_+ = (0, \infty) \, , \tag{3.4}$$

$$\omega^{(s_\nu)}(0) = 0, \nu = \overline{0, n-1}, \tag{3.5}$$

where $g(t) = P_1(d/dt)u_0(t)$. Hence,

$$\|g(t)\|_{L_2(R_+;H)} = \|P_1(d/dt)u_0(t)\|_{L_2(R_+;H)} \leq const \sum_{k=0}^{n-1} \|\varphi_\nu\|_{2n-s_\nu-\frac{1}{2}} \, ,$$

i.e. $g(t) \in L_2\left(R_+; H\right)$. Applying Theorem 3.1, we get that problem $(32), (33)$ is regularly solvable. Then,

$$\|\omega\|_{W_2^{n}(R_+;H)} \leq const \sum_{\nu=0}^{n-1} |\varphi_\nu|_{2n-s_\nu-\frac{1}{2}} \, .$$

On the other hand,

$$\|u_0(t)\| \leq const \sum_{k=0}^{n-1} \|c_\nu\|_{2n-\frac{1}{2}} \leq const \sum_{k=0}^{n-1} \|\varphi_\nu\|_{2n-s_\nu-\frac{1}{2}} \, ,$$

and then the desired solution will be $u = \omega - \, u_0$. □

Note that it follows by Theorem 3.1 that problem (2.7), (2.8) has a unique regular solution for any $f\left(x, t\right) \in L_2\left(R_+ \times [0, \pi]\right)$, if the condition

$$\sum_{j=0}^{2n-1} N_j \left(R_+; (s_\nu)_{\nu=0}^{n-1}\right) \sup_{x \in [0,\pi]} \left|P_{2n-j}\left(x\right)\right| < 1$$

is satisfied.

4 n-fold completeness

Now, we investigate the n-fold completeness of a system of derivatives of the chains of eigen and adjoint vectors corresponding to eigenvalues in the left half-plane.

First, we estimate the resolvent $P^{-1}(\lambda)$ in some sectors.

Theorem 4.1. *Let conditions 1), 2) be satisfied, and*

$$\alpha = \sum_{j=0}^{2n-1} d_{2n,j}^{n} \left\| B_{2n-j} \right\| < 1,$$

where $d_{2n,j}$ are defined by (2.12)*. Then on the sectors*

$$S_{\pm\theta} = \left\{ \lambda : \left| \arg \lambda \pm \frac{\pi}{2} \right| < \theta \right\}$$

for small $\theta > 0$, the operator bundle $P(\lambda)$ is invertible and in these sectors the estimate

$$\sum_{j=0}^{2n-1} \left\| \lambda^j A^{2n-j} P^{-1}(\lambda) \right\| \leq const \tag{4.1}$$

holds.

Proof. Let $\lambda \in S_\theta$. It suffices to prove the desired inequality for the sectors $S_{-\theta} = \left\{ \lambda : \lambda = re^{i\varphi}, r > 0, \frac{\pi}{2} + \theta < \varphi < \frac{\pi}{2} \right\}$. For other parts of the sector $S_{\pm\theta}$, this inequality is proved similarly. Let $P(\lambda) = P_0(\lambda) + P_1(\lambda)$, where

$$P_0(\lambda) = \left(-\lambda^2 + A^2 \right)^n ; P_1(\lambda) = \sum_{j=0}^{2n-1} \lambda^j A_{2n-j}.$$

For $\lambda = re^{i\frac{\pi}{2}}$, we have

$$P(\lambda) = P_0(\lambda) + P_1(\lambda) = \left(E + P_1(\lambda) P_0^{-1}(\lambda) \right) P_0^{-1}(\lambda).$$

It is obvious that for $\lambda = re^{i\frac{\pi}{2}}$

$$\begin{aligned} \left\| P_1(\lambda) P_0^{-1}(\lambda) \right\| &= \left\| \sum_{j=0}^{2n-1} r^j e^{i(2n-j)\frac{\pi}{2}} A_{2n-j} \left(r^2 E + A^2 \right)^{-n} \right\| \\ &\leq \sum_{j=0}^{2n-1} \left\| B_{2n-j} \right\| \left\| r^j A^{2n-j} \left(r^2 E + A^2 \right)^{-n} \right\|. \end{aligned}$$

Using the spectral expansion of the operator A, we have

$$\begin{aligned} \left\| r^j A^{2n-j} \left(r^2 E + A^2 \right)^{-n} \right\| &= \sup_{\mu \in \sigma(A)} \left| r^j \mu^{2n-j} \left(r^2 + \mu^2 \right)^{-n} \right| \\ &\leq \sup_{\tau \geq 0} \left| \tau^j \left(\tau^2 + 1 \right)^{-n} \right| = d_{2n,j}^{n}. \end{aligned}$$

Then, we get that for $\lambda = re^{i\frac{\pi}{2}}$, $r > 0$

$$\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\| < \alpha < 1.$$

Consequently, for $\lambda = re^{i\frac{\pi}{2}}$

$$P^{-1}(\lambda) = P_0^{-1}(\lambda)\left(E + P_1(\lambda)P_0^{-1}(\lambda)\right)^{-1}, \ \left\|P^{-1}(\lambda)\right\| \leq \left\|P_0^{-1}(\lambda)\right\| \cdot \frac{1}{1-\alpha}. \qquad (4.2)$$

Then, we have

$$\begin{aligned} \left\|\lambda^j A^{2n-j} P^{-1}(\lambda)\right\| &= \left\|\lambda^j A^{2n-j} P^{-1}(\lambda)\left(E + P_1(\lambda)P_0^{-1}(\lambda)\right)^{-1}\right\| \\ &\leq \frac{1}{1-\alpha}\left\|\lambda^j A^{2n-j} P_0^{-1}(\lambda)\right\| \leq \frac{d_{2n,j}^n}{1-\alpha}. \qquad (4.3) \end{aligned}$$

Thus, for $\lambda = re^{i\frac{\pi}{2}}$, inequality (4.1) is true. Now, let $\lambda = re^{i\varphi}$, $\frac{\pi}{2} \leq \varphi < \frac{\pi}{2} + \theta$. Then it is obvious that for $\lambda = re^{i\left(\frac{\pi}{2}+\varphi\right)} = ire^{i\varphi}$

$$\begin{aligned} P(\lambda) &= P\,(ir) + (ir)^{(2n-1)}\,Q_1\left(e^{i(2n-1)\varphi} - 1\right) \\ &\quad + (ir)^{(2n-2)}\,Q_2\left(e^{i(2n-2)\varphi} - 1\right) + ... + (ir)\,Q_k\left(e^{i\varphi} - 1\right), \end{aligned}$$

where $Q_{2k-1} = A_{2k-1}$, $k = \overline{1,n}$, $Q_{2k} = (-1)^k C_n^k A^{2k} + A_{2k}$, $k = \overline{1,n-1}$. Then

$$\begin{aligned} P(\lambda) &= (E + (ir)^{2n-1}\,Q_1 P^{-1}(ir)\left(e^{i(2n-1)\varphi} - 1\right) + ... \\ &\quad + irQ_k P^{-1}(ir)\left(e^{i\varphi} - 1\right))P\,(ir) \\ &= (E + T\,(ir,\varphi))\,P\,(ir)\,. \end{aligned}$$

Since $\theta > 0$ is sufficiently small and $0 \leq \varphi < \theta$, then by using estimate (4.3) we get $\|T\,(ir,\varphi)\| < \frac{1}{2}$, for small θ and $r > 0$. Then we get that for $\lambda = re^{i\varphi}$, $\frac{\pi}{2} \leq \varphi < \frac{\pi}{2} + \theta$, where θ is a sufficiently small number inequality (4.1) holds. □

Lemma 4.1. *Let conditions 1), 2) be satisfied and*
3) A^{-1} *be a completely continuous operator, i.e.* $A^{-1} \in \sigma_\infty(H)$*;*
4) the operator $(E + B_{2n})$ *be invertible in* H*.*

Then the operator bundle $P(\lambda)$ *has only a discrete spectrum with a unique limit point at infinity. In addition, if* $A^{-1} \in \sigma_p\ (0 < p < \infty)$*, the operator bundle* $A^{2n}P^{-1}(\lambda)$ *is represented in the form of ratio of two entire functions of order* ρ *and of minimal type* ρ*.*

Proof. It is obvious that $P(\lambda)$ may be represented in the form

$$P(\lambda) = (-1)^n\,\lambda^{2n}E + A^{2n} + \sum_{j=1}^{2n-1}\lambda^j Q_{2n-j} + A^{2n},$$

where $Q_{2k} = (-1)^k C_n^k A^{2k} + A^{2k}$, $Q_{2k-1} = A_{2k-1}$. Then

$$\begin{aligned} P(\lambda) &= \left((-1)^n \lambda^{2n} A^{-2n} + \sum_{j=1}^{2n-1} \lambda^j Q_{2n-j} A^{-2n} + E + A^{2n} A^{-2n} \right) A^{2n} \\ &= (E + B_{2n}) \left((-1)^n \lambda^{2n} (E + B_{2n})^{-1} A^{-2n} + \right. \\ &\quad \left. + \sum_{j=1}^{2n-1} \lambda^j (E + B_{2n})^{-1} Q_{2n-j} A^{-2n+j} A^{-j} + E \right) A^{2n} \\ &= (E + B_{2n}) \times (E + M(\lambda)) A^{2n}, \end{aligned}$$

where

$$M(\lambda) = \left((-1)^n \lambda^{2n} (E + B_{2n})^{-1} A^{-2n} + \sum_{j=1}^{2n-1} \lambda^j (E + B_{2n})^{-1} Q_{2n-j} A^{-2n+j} A^{-j} + E \right).$$

It is obvious that

$$T_{2n} = (-1)^n (E + B_{2n}) A^{-2n} \in \sigma_\infty (H),$$

$$T_j = (E + B_{2n})^{-1} \left(A_{2n-j} A^{-2n+j} \right) A^{-j} \in \sigma_\infty (H)$$

for all $\lambda \in \mathbb{C}$, i.e. $M(\lambda)$ is a compact-valued operator-function and $M(0) = 0$. It follows by the Keldysh lemma [7] that the operator bundle $E + M(\lambda)$ is invertible for all $\lambda \in \mathbb{C}$ except for a countable number of points λ_k, that will be eigenvalues of $E + M(\lambda)$ and have a limit point only at infinity. Obviously, this refers to the bundle $P(\lambda) = (E + B_{2n}) (E + M(\lambda)) A^{2n}$ as well. On the other hand, by the boundedness of the operators $A^{-1} \in \sigma_p$, $(E + B_{2n})$ and $(E + B_{2n})^{-1} Q_{2n-j} A^{-2n+j}$ we get that $T_{2n} \in \sigma_{\frac{p}{2n}}$, à $T_j \in \sigma_{\frac{p}{j}}$ $\left(j = \overline{1, 2n-1} \right)$. Therefore, it follows by the Keldysh lemma [2], [8] that $(E + M(\lambda))^{-1}$ is represented in the form of ratio of two entire functions of orderρ and of minimal type for order ρ.This property holds for the bundle $P(\lambda)$ as well, since $A^{2n} P^{-1}(\lambda) = (E + M(\lambda))^{-1} (E + B_{2n})^{-1}$. □

Now, we prove a theorem on completeness of the system $K(\Pi_-)$.

Theorem 4.2. *Let the conditions of Theorem 3.1 be satisfied, $A^{-1} \in \sigma_\infty (H)$ and let the following conditions be satisfied:*

a) $A^{-1} \in \sigma_p (0 < p \leq 1), B_j \in L(H), j = \overline{1, 2n}$;

b) $A^{-1} \in \sigma_p (0 < p < \infty), B_j \in \sigma_\infty (H)$.

Then the system $K(\Pi_-)$ is n-fold complete in the space of traces, and the system of elementary decreasing solutions of the equation $P(d/dt)u = 0$ is complete in the space of regular solutions of problem (2.1), (2.2)

Proof. By Lemma 5, under the conditions of the theorem, $P(\lambda)$ has a discrete spectrum with a unique limit point at infinity, and $A^2 P^{-1}(\lambda)$ is a meromorphic function of order ρ and of minimal type ρ. Let there exist a vector $\tilde{f} = \{f_\nu\}_{\nu=0}^{k-1} \in \tilde{H}$ for which

$$\left(\tilde{f}, \tilde{\varphi}_{i,j,h} \right)_{\tilde{H}} = \sum_{\nu=0}^{n-1} \left(f_\nu, \varphi_{i,j,h}^{(s_\nu)} \right)_{H_{2n-s_\nu-\frac{1}{2}}} = 0, i = 1, 2, ...$$

Then, by the ortogonality condition and the expansion in a neighbourhood of the point λ_i, we easily get that (see (2.9) and [2], [4], [8])

$$R(\lambda)=\sum_{\nu=0}^{n-1}\left(A^{2n-s_\nu-\frac{1}{2}}P^{-1}\left(\bar{\lambda}\right)\right)^{*}\lambda^{s_\nu}A^{2n-s_\nu-\frac{1}{2}}f_\nu$$

is an analytic vector-function in the left half-plane Π_-. When the conditions of the theorem are satisfied, problem (2.1),(2.2) has a regular solution $u(t)\in W_{2n}^{2n}\left(R_+;H\right)$. We can represent it in the form

$$u(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}P^{-1}(\lambda)\sum_{s=0}^{2n-1}Q_s(\lambda)u^{(s)}(0)e^{\lambda t}d\lambda,$$

where $Q_s(\lambda)=\lambda^{2n-s-1}E+...+Q_{n-s-1}\left(s=\overline{0,2n-1}\right)$. It follows by the condition $\|P^{-1}(\lambda)\|\leq const|\lambda|^{-2n}$ that for $\lambda\in S_{\pm\theta}$ that at $t>0$ the integration contour may be changed by the contour Γ that for large $|\lambda|$ $(\lambda\in S_{\pm\theta})$ coincides with the rays $\Gamma_\pm=\left\{\lambda:\lambda=re^{\pm i\left(\frac{\pi}{2}+\theta\right)}\right\}$ for small $\theta>0$, i.e.

$$u(t)=\frac{1}{2\pi i}\int_{\Gamma}P^{-1}(\lambda)\sum_{s=0}^{2n-1}Q_s(\lambda)u^{(s)}(0)e^{\lambda t}d\lambda.$$

For $t>0$, we can differentiate this integral arbitrarily many times, therefore for $t>0$

$$\begin{aligned}\sum_{\nu=0}^{k-1}\left(u^{(s_\nu)}(t),f_\nu\right)_{2n-s_\nu-\frac{1}{2}} &= \frac{1}{2\pi i}\sum_{\nu=0}^{k-1}\left(\int_\Gamma P^{-1}(\lambda)\lambda^{s_\nu}\sum_{s=0}^{2n-1}Q_s(\lambda)u^{(s)}(0),f_\nu\right)_{2n-s_\nu-\frac{1}{2}}e^{\lambda t}d\lambda\\ &= \frac{1}{2\pi i}\int_\Gamma\left(A^{2n-s_\nu-\frac{1}{2}}P^{-1}(\lambda)\lambda^{s_\nu}\sum_{s=0}^{2n-1}Q_s(\lambda)u^{(s)}(0),f_\nu\right)e^{\lambda t}d\lambda\\ &= \frac{1}{2\pi i}\int_\Gamma\left(\sum_{s=0}^{2n-1}Q_s(\lambda)u^{(s)}(0),R\left(\bar{\lambda}\right)\right)e^{\lambda t}d\lambda=\frac{1}{2\pi i}\int_\Gamma v(\lambda)\,d\lambda,\end{aligned}$$

where $v(\lambda)=\left(\sum\limits_{s=0}^{2n-1}Q_s(\lambda)u^{(s)}(0),R\left(\bar{\lambda}\right)\right)$, and $R(\lambda)$ is defined from equality (2.9).

It follows by the conditions a) and b) of the theorem and by the properties of $R(\lambda)$ that $v(\lambda)$ is an entire function of order ρ and of minimal type ρ, and in the right half-plane it grows not quicker than $|\lambda|^{2n-1}$, on the imaginary axis does not grow quicker than $|\lambda|^{2n}$. Then by the Fragmen-Lindelof theorem $v(\lambda)$ is a polynomial. Therefore, $v(\lambda)=v_0+\lambda v_1+...+\lambda^m v_m$. On the other hand, for $t>0$

$$\int_\Gamma\lambda^j e^{\lambda t}d\lambda=0,j=\overline{0,m}.$$

Consequently, for $t>0$

$$\sum_{\nu=0}^{n-1}\left(u^{(s_\nu)}(t),f_\nu\right)_{2n-s_\nu-\frac{1}{2}}=0.$$

Passing to limit as $t \to 0$, we get $\sum\limits_{\nu=0}^{n-1} (\chi_\nu, f_\nu)_{2n-s_\nu-\frac{1}{2}} = 0$. Since $\chi_\nu \in H_{2n-s_\nu-\frac{1}{2}}$ are arbitrary elements, we have $f_\nu = 0$, $\nu = \overline{0, n-1}$. Thus, the system $K(\Pi_-)$ is n- fold complete in the space of traces. Then, for any $\varepsilon > 0$, we can find a number N and numbers $C_{i,j,h}^N(\varepsilon)$ such that

$$\left\| \chi_\nu - \sum_{i=1}^{N} \sum_{(j,h)} C_{i,j,h}^N(\varepsilon) \varphi_{i,j,h}^{(s_\nu)} \right\|_{2n-s_\nu-\frac{1}{2}} < \frac{\varepsilon \cdot const}{N}, \tag{4.4}$$

where *const* is as in inequality (3.3). Since

$$\varphi_{i,j,h}^{(s_\nu)}(0) = u_{i,j,h}^{(s_\nu)}(t)/_{t=0}, \chi_\nu = u^{(s_\nu)}(t)/_{t=0},$$

by using inequalities (4.4) and (3.3), we get

$$\left\| u(t) - \sum_{i=1}^{N} \sum_{(j,h)} C_{i,j,h}^N(\varepsilon) \right\| < \varepsilon_1, \left(\varepsilon_1 = \frac{\varepsilon N}{c_1} \right).$$

□

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Rovshan Zulfugar Humbataliev
Azerbaijan Cooperation University
Institute of Mathematics and Mechanics
National Academy of Sciences of Azerbaijan
9, B. Vahabzade St.,
AZ 1141, Baku, Azerbaijan
E-mail: rovshangumbataliev@rambler.ru

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