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INTEGRAL EQUATIONS WITH SUBSTOCHASTIC KERNELS

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Abstract. The non-homogeneous or homogeneous integral equation of the second kind with a substochastic kernel $W(x,t) = K(x-t) + T(x,t)$ is considered on the semi axis, where K is the density of distribution of some variate, and $T \geq 0$ satisfies the condition $\lambda(t) = \int_{-t}^{\infty} K(y)\,dy + \int_0^{\infty} T(x,t)\,dx < 1$, $\sup \lambda(t) = 1$.

The existence of a minimal positive solution of the non-homogeneous equation is proved. The existence of a positive solution of the homogeneous equation is also proved under some simple additional conditions. The results may be applied to the study of Random Walk on the semi axis with the reflection at the boundary.

1 Introduction

Consider the following integral equation with a substochastic kernel on the semi axis:

$$f(x) = g(x) + \int_0^{\infty} K(x-t) f(t)\,dt + \int_0^{\infty} T(x,t) f(t)\,dt\,, \qquad x > 0\,, \tag{1.1}$$

or

$$f(x) = g(x) + \int_0^{\infty} W(x,t) f(t)\,dt\,, \quad x > 0, \tag{1.2}$$

where

$$K,\, T \geq 0,\;\; W(x,t) = K(x-t) + T(x,t)\,. \tag{1.3}$$

Assume that the kernel-function K is conservative, i.e. it is a probability density for a variate:

$$0 \leq K \in L_1(-\infty,\infty)\,, \qquad \int_{-\infty}^{\infty} K(t)\,dt = 1. \tag{1.4}$$

The total kernel $W(x,t)$ is assumed to be a substochastic one:

$$W(x,t) \geq 0,\;\; \lambda(t) = \int_0^{\infty} W(x,t)\,dx \;\leq 1. \tag{1.5}$$

We shall call the function $\mu(x) = 1 - \lambda(x)$ indicatrix of dissipation (or functional of dissipation) for the kernel W.

Let $\hat{K}$ and $\hat{T}$ be the following integral operators entering (1.1):

$$\hat{K}f(x) = \int_0^\infty K(x-t)\, f(t)\, dt, \quad \hat{T}f(x) = \int_0^\infty T(x,t)\, f(t)\, dt\ , \qquad x > 0.$$

It is well known, that the Wiener-Hopf type operator $\hat{K}$ is a non-compact operator in the function space $L^+ \equiv L_1(0,\infty)$, as well as in some other spaces. In applications the "reflection" operator $\hat{T}$, as a rule, is compact in L^+.

The non-homogeneous and homogeneous integral equations of type (1.1), (1.4) arise in Random Walks' theory, as well as in some other problems of Markov and Semi-Markov stochastic processes. Several well-known problems of Radiative Transfer (RT), Kinetic theory of gases (KTG), etc. are reduced to such an equation.

Transport of particles is a special, complex form of a Random Walk. A large range of problems of Radiative Transfer in the semi-space with reflected boundary is reduced to an integral equation of form (1.1). In case of regular reflection, the kernel T depends on the sum of arguments (see for example [4]). In case of other reflection laws function T may have more complex structure. Detailed discussion of application of equation (1.1) to RT problems is beyond the scope of this paper.

If $T = 0$, then the equation (1.1) becomes the well-known Wiener-Hopf conservative equation, having various applications (see [8], [9], [2]):

$$h(x) = g(x) + \int_0^\infty K(x-t)\, h(t)\, dt \qquad x > 0.$$

In RT the summand $\int\limits_0^\infty T(x,t)\, f(t)\, dt$ in (1.1) takes into account the reflection of walking particles at the boundary $x = 0$. The law T of the reflection depends on concrete (specific) conditions.

Milne problem with a reflection is of essential interest in KTG (see [5], [1]). In some particular cases, the problem is reduced to the construction of positive solutions to the homogeneous equation of the form:

$$Q(x) = \int_0^\infty K(x-t)\, Q(t)\, dt + \int_0^\infty T(x,t)\, Q(t)\, dt\ , \quad x > 0. \tag{1.6}$$

The problems of Random Walk with a reflection at a barrier are of interest also in the molecular biology (see [7] and the literature cited there).

The present paper is devoted to the problems of solvability of non-homogeneous equation (1.1) and homogeneous equation (1.6).

2 The minimal solution of non-homogeneous equation (1.1)

Consider equation (1.1) under the following additional assumptions:
a) the substochastic kernel $W(x,t) = K(x-t) + T(x,t)$ satisfies the conditions

$$\lambda(t) = \int_0^\infty W(x,t)\, dx < 1, \quad \sup_{(0,\infty)} \lambda(x) = 1; \tag{2.1}$$

b) the free term g satisfies the conditions:

$$0 \leq g \in L^{+}. \tag{2.2}$$

Let us first consider the question of the fulfillment of the conditions (2.1). From (2.1) we have

$$\lambda(t) = \int_{-t}^{\infty} K(y)\, dy + \int_{0}^{\infty} T(x,t)\, dx. \tag{2.3}$$

It is clear from (1.4) and (2.3), for the fulfillment of the first condition in (2.1) it is necessary that

$$\int_{-\infty}^{-t} K(x)\, dx > 0, \quad \forall t > 0, \tag{2.4}$$

i.e., the restriction of the function K to the negative semi-axis should not be a compact one.

Suppose that condition (2.4) holds. Then for the fulfillment of conditions (2.1), it is sufficient to satisfy the inequality:

$$\int_{0}^{\infty} T(x,t)\, dx \leq q \int_{-\infty}^{-t} K(y)\, dy\,, \text{ for some } q < 1. \tag{2.5}$$

Conditions (2.1) have a simple physical sense. The coefficient q is related to the value of the albedo of the reflecting boundary $x = 0$.

We proceed to the consideration of equation (1.2).

Consider the simple iterations f_n for equation (1.2) defined by:

$$f_{n+1}(x) = g(x) + \int_{0}^{\infty} W(x,t)\, f_n(t)\, dt\ , \qquad f_0 = 0, \quad n = 0, 1, \cdots \tag{2.6}$$

We have

$$0 \leq f_n \in L^{+}, \quad f_n \uparrow \text{ in } n.$$

We will consider the iterative sequence f_n in the space $L_{loc} = L_{loc}[0,\infty)$. This space consists of functions that are integrable on every finite interval. The space L_{loc} equipped with the topology of the convergence in L_1 on every finite interval.

If a monotone sequence f_n converges in L_{loc}, then the limit f is a solution of equation (1.2) (see [2]) and it is called the basic solution (BS) of (1.2).

Let $\tilde{f} \in L_{loc}$ be an arbitrary positive solution of equation (1.2). Then the inequality $f_n \leq \tilde{f}$ is verified by the induction on n. Therefore, the sequence f_n converges in L_{loc} by virtue of the Lebesgue theorem:

$$f_n \to f \leq \tilde{f}.$$

It follows from the foregoing that the BS is a minimal positive solution of equation (1.2).

Let us consider the Banach space $L(\mu) \subset L_{loc}$ of the functions integrable with the weight μ, where $\mu(x) = 1 - \lambda(x) > 0$ is the indicatrix of dissipation of the kernel W. In this space, the norm is defined by the equality $\|f\| = \int_0^\infty |f(x)|\, \mu(x)\, dx$.

Theorem 2.1. *Equation* (1.2) *under conditions* (2.1), (2.2) *possesses the basic solution* $f \in L(\mu)$, $f \geq 0$. *The following inequality takes place*

$$\int_0^\infty f(x)\,\mu(x)\,dx \leq \int_0^\infty g(x)\,dx. \tag{2.7}$$

Proof. We will use the approach developed in the paper [6]. Consider the iterations f_n defined by (2.6). Integrating (2.6) from 0 to ∞, we have:

$$\int_0^\infty f_{n+1}(x)\,dx = \int_0^\infty \left[\int_0^\infty [K(x-t)+T(x,t)]\,dx\right] f_n(t)\,dt + \int_0^\infty g(x)\,dx =$$

$$= \int_0^\infty \left[\int_{-t}^\infty K(y)\,dy + \int_0^\infty T(x,t)\,dx\right] f_n(t)\,dt + \int_0^\infty g(x)\,dx =$$

$$= \int_0^\infty [1-\mu(t)]\; f_n(t)\,dt + \int_0^\infty g(x)\,dx \leq \int_0^\infty [1-\mu(t)]\; f_{n+1}(t)\,dt + \int_0^\infty g(x)\,dx\;.$$

From here, we have:

$$\int_0^\infty \mu(t)\,f_{n+1}(t)\,dt \leq \int_0^\infty g(x)\,dx.$$

This inequality and the monotonicity of the iterative sequence f_n imply the convergence $f_n \to f$ in $L(\mu)$, and inequality (2.7). □

Remarks

1. Inequality (2.7) has a simple physical and probabilistic meaning. The question concerning the terms, which make (2.7) an equality is of considerable interest. By simple examples, one can see that equality does not always take place.

2. If the indicatrix of dissipation μ decreases rapidly at infinity, then inequality (2.7) describes the asymptotic behaviour of the function f at infinity unsatisfactorily. In the case of slow decrease of μ, this equality may give fairly complete information about the asymptotics of f.

3. Inequality (2.7) implies the property of the continuous dependence of BS on g. Let f_1 and f_2 be solutions of equation (1.1) when $g = g_1$ and $g = g_2$ respectively, and $g_1 \geq g_2$. The following estimate is derived from (2.7) and $f_1 \geq f_2$:

$$\|f_1 - f_2\|_{L(\mu)} = \int_0^\infty [f_1(x) - f_2(x)]\,\mu(x)\,dx \leq \|g_1 - g_2\|_L\,.$$

3 On the solvability of homogeneous equation (1.6)

3.1 On Wiener-Hopf homogeneous equation

The non-trivial solvability of equation (1.6) depends on the solvability of the following Wiener-Hopf homogeneous equation with the conservative kernel K, entering (1.6):

$$S(x) = \int_0^\infty K(x-t)\,S(t)\,dt\,, \qquad x > 0. \tag{3.1}$$

This equation plays an important role in Mathematical Physics, in the Theory of Stochastic processes, etc.

By analogy to the terminology previously used in the theory of Random Walks (see [8], [9]), we will call a positive bounded continuous monotonic solution satisfying the normalization condition $S(\infty) = 1$ a P-solution of equation (1.6), and a monotonic unbounded positive solution will be called a P^*-solution.

The first mathematical results on the conservative equation with symmetric kernel were obtained in the classical work of N. Wiener and E. Hopf [10] in connection with solving the well-known Milne problem in Radiative Transfer theory. In that work they studied the kernel

$$K(x) = \frac{1}{2}E_1(x) = \frac{1}{2}\int_1^\infty e^{-|x|s}\frac{ds}{s},$$

which decreases exponentially at infinity.

The first general results on the existence of a P-solution to non-symmetric conservative equation (3.1) were obtained by D. V. Lindley in work [8].

A proof of the existence of a P^*-solution to equation (3.1) with general symmetric conservative kernel is given in F. Spitzer work [9]. In author's work [3], some inaccuracies and errors contained in the [9] are indicated (and partially corrected).

However, the following question remained open: does this approach allow the proof of the existence of a P^*-solution to equation (3.1) with a general symmetric conservative kernel, or not?

Complete solution to this problem was found by N. B. Yengibaryan, who applied the method of nonlinear factorization of equations.

In the following theorem, we state some basic facts for equation (3.1) (see [2]).

Theorem M. *Let the kernel function K in equation* (3.1) *satisfies the conditions of the conservativeness* (1.4) *and the following inequalities hold:*

$$-\infty \le \nu \le 0, \tag{3.2}$$

where $\nu = \int_{-\infty}^{\infty} xK(x)\,dx$ or K is an even function. Then, there exists an absolutely continuous solution $S > 0$ to equation (3.1), *and*
a) if $\nu < 0$, then S is a P-solution,
b) if the kernel function K is an even function or $\nu = 0$, then the solution S is a P^*-solution. The following asymptotics takes place:

$$S(x) = O(x)\,,\ x \to \infty. \tag{3.3}$$

For suitable a, $b > 0$ the following inequality is satisfied:

$$S(t) \le a + bt. \tag{3.4}$$

The function $S(x)$ can be normalized by the condition $S(0) = 1$.

3.2 An existence theorem for homogeneous equation (1.6).

Consider equation (1.6) under conditions (2.1). Suppose that, in addition, we have the following conditions.

a) One of the conditions of the theorem M on the kernel function K is satisfied, which ensures the existence of a positive solution $S(x)$ to homogeneous equation (3.1).

b) The kernel T is such that:

$$g(x) \equiv \int_0^\infty T(x,t)\, S(t)\, dt \in L_1(0,\infty) \tag{3.5}$$

i.e.

$$\int_0^\infty \left[\int_0^\infty T(x,t)\, dx\right] S(t)\, dt < +\infty. \tag{3.6}$$

Under the conditions of Statement a) of Theorem M it is sufficient for condition (3.5) that

$$\int_0^\infty T(x,t)\, dt \in L_1(0,\infty). \tag{3.7}$$

Under the conditions of Statement b) of Theorem M, according to inequality (3.4) not only condition (3.7) but also the condition below is sufficient for the fulfillment of condition (3.6)

$$\int_0^\infty T(x,t)\, t dt \in L_1(0,\infty). \tag{3.8}$$

Let us introduce the new function h in (1.6), which is connected with Q by the relation:

$$Q = S + h. \tag{3.9}$$

Substituting (3.9) into (1.6), we obtain the following non-homogeneous integral equation with respect to h:

$$h(x) = g(x) + \int_0^\infty K(x-t)\, h(t)\, dt + \int_0^\infty T(x,t)\, h(t)\, dt, \qquad x > 0, \tag{3.10}$$

where g is defined by (3.5). As an h, we will take the minimal positive solution of equation (3.10). Its existence follows by Theorem 1, in virtue of (3.5) and the presence of the positive functional of dissipation $\mu(t) = 1 - \lambda(t) > 0$ of the $\hat{W}$ operator. The following inequality takes place:

$$\int_0^\infty h(t)\, dt \int_{-\infty}^{-t} K(y)\, dy < +\infty. \tag{3.11}$$

We have proved

Theorem 3.1. *Let the following conditions hold.*

a) The kernel function K satisfies the conditions of the conservativeness (1.4).

b) One of the conditions of Theorem M on the existence of a positive solution $S(x)$ of the Wiener-Hopf homogeneous equation (3.1) *is satisfied.*

c) The indicatrix of dissipation μ *of the kernel* W *satisfies conditions* (2.1)*. In particular, if*

$$K(-x) > 0 \quad (x > 0), \qquad \int_0^\infty T(x,t)\, dx \le q \int_{-\infty}^{-t} K(y)\, dy\,, \quad for\ some\ q < 1.$$

d) The kernel T *satisfies condition* (3.6)*. In particular, this happens if the conditions* (3.7), (3.8) *take place.*

Then equation (1.6) *possesses a positive solution of the form* $Q = S + h$*, where the function* $S > 0$ *is the basic solution of equation* (3.1) *and* $h \ge 0$ *is defined by* (3.10).

Theorems 2.1 and 3.1 imply the following statements.

Corollary 3.1. *Let conditions* (1.4), (2.1), (2.2), (3.2) *and the condition*

$$\int_0^\infty T(x,t)t\, dt < +\infty$$

be satisfied.

Then both homogeneous equation (1.6) *and non-homogeneous equation* (1.1) *possess positive solutions.*

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References

[1] S.M. Andriyan, A.Kh. Khachatryan, *On a problem of physical kinetics,* Computational Mathematics and Mathematical Physics. 45 (2005), no. 11, 1982–1989.

[2] L.G. Arabadzhyan, N.B. Engibaryan, *Convolution equations and nonlinear functional equations*, J. Math. Sci. 36 (1987), no. 6, 745—791.

[3] A.G. Barseghyan, *On a distribution of sums of identically distributed random variables*, Teor. Veroyatnost. i Primenen. 57 (2012), no. 1, 155-–158.

[4] J. Casti, R. Kalaba, *Imbedding methods in applied mathematics,* Series Applied Mathematics and Computation, no. 2. Addison-Wesley Publishing Company, Reading, Mass.-London-Amsterdam, 1973.

[5] C. Cercingnani, *Theory and application of the Boltzmann equation*, Elsevier, New York, 1975.

[6] N.B. Engibaryan, *On the fixed points of monotonic operators in the critical case*, Izvestiya: Mathematics. 70 (2006), no. 5, 931–947.

[7] N.R. Hansen, *The maximum of a random walk reflected at a general barrier,* The Annals of Applied Probability. 16 (2006), no. 1, 15–29.

[8] D.V. Lindley, *The theory of queue with a single server.* Proc. Cambridge Phil. Soc. 48 (1952), 277–289.

[9] F. Spitzer, *The Wiener–Hopf equation whose kernel is a probability density.* I, II, Duke Math. J. 24 (1957), 327—343.

[10] N. Wiener, E. Hopf, *Uber eine Klasse singularer Integralgleichungen*, Sitz. Akad. Wiss. Berlin (1931), 696-706.

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