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SCHWARZ PROBLEM FOR FIRST ORDER ELLIPTIC SYSTEMS IN UNBOUNDED SECTORS

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Abstract. In this article we deal with a Schwarz-type boundary value problem for both the inhomogeneous Cauchy-Riemann equation and the generalized Beltrami equation on an unbounded sector with angle $\vartheta = \pi/n, n \in \mathbb{N}$. By the method of plane parqueting-reflection and the Cauchy-Pompeiu formula for the sector, the Schwarz-Poisson integral formula is obtained. We also investigate the boundary behaviour and the C^α-property of a Schwarz-type as well as of a Pompeiu-type operator. The solution to the Schwarz problem of the Cauchy-Riemann equation is explicitly expressed. Sufficient conditions on the coefficients of the generalized Beltrami equation are obtained under which the corresponding system of integral equations is contractive. This proves the existence of a unique solution to the Schwarz problem of the generalized Beltrami equation.

1 Introduction

A variety of boundary value problems (BVPs) for partial differential equations (PDEs) has been studied, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Explicit solutions on some special domains are obtained. Those special domains include the unit disc [4, 16, 17], the half-plane [7, 20], a circular ring [26], the quarter-plane [1, 2, 6], a quarter-ring and a half-hexagon [23], a half-disc and a half-ring [10], lens and lune [15], triangles [11, 29], and sectors [3, 28].

Generally speaking, the theory of boundary value problems for analytic and generalized analytic functions is as well closely connected with the theory of singular integral equations, index theory and many other theories, as it has a variety of applications in elasticity theory, fluid dynamic and shell theory [18, 19, 21, 22, 27]. The Schwarz boundary value problem is in the center of interest as a basic problem in complex analysis. It has some influence on the solvability of Dirichlet-type and Neumann-type BVPs. Solving the Schwarz problem for analytic functions serves to determine harmonic Green functions. Furthermore, the solutions of BVPs of higher order complex model equations are generally obtained by iterating the corresponding solutions of Cauchy-Riemann equation [5]. Besides, the Cauchy-Riemann equation $\partial_{\bar{z}} w = 0$ is the special form of an elliptic system of two real first order partial differential equations. The Beltrami equations present more general systems of the same type. These systems take a key part in geometrical function theory and many other problems of mathematical physics see, for example [27].

In this article, the Schwarz-Poisson formula for an unbounded sector is obtained by the plane parqueting-reflection method and the Cauchy-Pompeiu integral formula for the sector, in Section 2. In Section 3, a Schwarz-type operator and a Pompeiu-type operator for the sector are introduced, and their boundary behaviors are investigated. Especially their boundary values at the corner point of the sector are proved to exist. Finally the solution to the Schwarz problem for the inhomogeneous Cauchy-Riemann equation is explicitly expressed. Section 4 is devoted to the Schwarz problem for the generalized Beltrami equation in the sector. Firstly, the C^α-properties of both the Schwarz-type operator and the Pompeiu-type operator are investigated. Using the contraction mapping principle, we prove the existence of a unique solution of a Schwarz problem for the generalized Beltrami equation. This approach has been used, firstly by Tutschke [25, 24], to study a Schwarz problem in a certain Hölder space. Yüksel [30] applied the method to a Schwarz problem for the generalized Beltrami equation with Hölder continuous coefficients on a regular domain.

Next we introduce some notations used in the sequel. Let Ω be an unbounded sector domain in the complex plane $\mathbb{C}$ defined by

$$\Omega = \left\{ z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{n} \right\}, \tag{1.1}$$

with the boundary $\partial\Omega = [0, \infty) \cup L$ where $L = \left\{ te^{i\pi/n}, 0 < t < \infty \right\}$ is oriented towards the origin. We regard n as a fixed positive integer, $\vartheta = \pi/n$ and $\omega = e^{i\vartheta}$. By rotation, we define the domains

$$\Omega_k \equiv \omega^{2k}\Omega = \left\{ \omega^{2k} z \;:\; z \in \Omega \right\}, \;\; k = 0, \cdots, n-1. \tag{1.2}$$

Here $\Omega_0 = \Omega$ is the sector defined by (1.1). By reflection on the real axis, we define the domains

$$\underline{\Omega}_k = \left\{ \bar{z} \;:\; z \in \Omega_k \right\}, \;\; k = 0, \cdots, n-1. \tag{1.3}$$

Obviously, the domains Ω_k, $\underline{\Omega}_k$ are disjoint and

$$\mathbb{C} = \bigcup_{k=0}^{n-1} \left(\overline{\Omega_k} \cup \overline{\underline{\Omega}_k} \right). \tag{1.4}$$

One of the fundamental tools for solving boundary value problems of complex partial differential equations is the Cauchy-Pompeiu formula, which is valid for the bounded sector Ω_R defined by

$$\Omega_R = \left\{ z \in \mathbb{C} \;: |z| < R, \, 0 < \arg z < \frac{\pi}{n} \right\}, \tag{1.5}$$

for some $R > 0$.

Lemma 1.1. *Let* $w \in C^1(\Omega_R;\mathbb{C}) \cap C\left(\overline{\Omega_R};\mathbb{C}\right)$. *Then*

$$\frac{1}{2\pi i}\int_{\partial\Omega_R} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{\Omega_R} w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \begin{cases} w(z), & z \in \Omega_R \\ 0, & z \notin \overline{\Omega_R}, \end{cases} \tag{1.6}$$

and

$$-\frac{1}{2\pi i}\int_{\partial\Omega_R} w(\zeta)\frac{d\bar\zeta}{\overline{\zeta - z}} - \frac{1}{\pi}\int_{\Omega_R} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} = \begin{cases} w(z), & z \in \Omega_R \\ 0, & z \notin \overline{\Omega_R}, \end{cases} \tag{1.7}$$

where Ω_R *is the sector defined by* (1.5) *and* $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbb{R}$.

Theorem 1.1. *If* $w : \Omega \to \mathbb{C}$ *satisfies* $|w(z)| \leq C|z|^{-\epsilon}$ *for* $|z| > k$ *and* $w_{\bar z} \in L_{p,2}(\Omega;\mathbb{C})$, *then*

$$\frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{\Omega} w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \begin{cases} w(z), & z \in \Omega \\ 0, & z \notin \overline{\Omega}, \end{cases} \tag{1.8}$$

where Ω *is the sector defined by* (1.1), C, $\epsilon > 0$ *and* K *is a sufficiently large positive real number .*

Proof. The representation formula (1.6) gives

$$w(z) = \frac{1}{2\pi i}\int_{\partial\Omega_R} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{\Omega_R} w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z},\ z \in \Omega_R. \tag{1.9}$$

Since for $2|z| < R$

$$\left|\frac{1}{2\pi i}\int_0^{\pi/n} w\left(Re^{i\theta}\right)\frac{iRe^{i\theta}}{Re^{i\theta} - z}d\theta\right| \leq \frac{1}{2n}\frac{M(R,w)R}{R - |z|} \leq \frac{M(R,w)}{n}$$

with

$$M(R,w) = \sup_{|z|=R,\, 0<\arg z<\pi/n} |w(z)|$$

By the assumption, we get

$$\lim_{R\to\infty} M(R,w) = \lim_{R\to\infty}\frac{C}{R^\epsilon} = 0$$

Hence, the integral on the arc boundary of the sector Ω_R tends to zero as $R \to \infty$. Then letting R tend to infinity, on both sides of (1.9), implies

$$w(z) = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{\Omega} w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{1.10}$$

for $z \in \Omega$.

The existence of the second integral in (1.10) follows from $w_{\bar z} \in L_1(\Omega;\mathbb{C})$ [20]. While the first integral is improper and exists because of the estimate for $2|z| < r$,

$$\left|\int_r^{2k} w\left(\omega^\nu t\right)\frac{dt}{t - z\omega^{-\nu}}\right| \leq C\int_r^{2k}\frac{1}{t^{\epsilon+1}}\frac{t}{t - |z|}dt \leq 2C\int_r^{2k}\frac{1}{t^{\epsilon+1}}dt < \frac{2C}{\epsilon r^\epsilon},\ \ \nu \in \{0,1\}.$$

Hence the first equality in (1.8) holds. □

Remark 1. The boundedness condition on the function w at infinity can be weakened to $w \in L_2(\partial\Omega;\mathbb{C})$.

2 Schwarz-Poisson formula in the sector

The Schwarz-Poisson formula is derived from the Cauchy-Pompeiu formula for the sector.

Theorem 2.1. *Any $w:\Omega\to\mathbb{C}$, such that $w\in L_2(\partial\Omega;\mathbb{C})\cap C(\partial\Omega;\mathbb{C})$ and $w_{\bar z}\in L_{p,2}(\Omega;\mathbb{C}), p>2$, can be represented, for $z\in\Omega$, as*

$$w(z)=\frac{1}{2\pi i}\int_{\partial\Omega}w(\zeta)\sum_{k=0}^{n-1}\frac{1}{\zeta-z\omega^{2k}}d\zeta-\frac{1}{\pi}\int_{\Omega}w_{\bar\zeta}(\zeta)\sum_{k=0}^{n-1}\frac{1}{\zeta-z\omega^{2k}}d\xi d\eta, \tag{2.1}$$

and

$$\begin{aligned}w(z)&=\frac{1}{\pi i}\int_{\partial\Omega}\mathrm{Re}\ w(\zeta)\sum_{k=0}^{n-1}\frac{1}{\zeta-z\omega^{2k}}d\zeta\\&-\frac{1}{\pi}\int_{\Omega}\left[w_{\bar\zeta}(\zeta)\sum_{k=0}^{n-1}\frac{1}{\zeta-z\omega^{2k}}-\overline{w_{\bar\zeta}(\zeta)}\sum_{k=0}^{n-1}\frac{1}{\bar\zeta-z\omega^{2k}}\right]d\xi d\eta,\end{aligned} \tag{2.2}$$

Proof. By Theorem 1.1 for $z\in\Omega$, we have

$$w(z)=\frac{1}{2\pi i}\int_{\partial\Omega}w(\zeta)\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int_{\Omega}w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-z}, \tag{2.3}$$

$$0=\sum_{k=1}^{n-1}\left\{\frac{1}{2\pi i}\int_{\partial\Omega}w(\zeta)\frac{d\zeta}{\zeta-z\omega^{2k}}-\frac{1}{\pi}\int_{\Omega}w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-z\omega^{2k}}\right\}, \tag{2.4}$$

and

$$0=\sum_{k=0}^{n-1}\left\{\frac{1}{2\pi\, i}\int_{\partial\Omega}w(\zeta)\frac{d\zeta}{\zeta-\bar z\omega^{-2k}}-\frac{1}{\pi}\int_{\Omega}w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-\bar z\omega^{-2k}}\right\}. \tag{2.5}$$

Thus (2.1) follows directly by summing (2.3) and (2.4). We need only to prove (2.2). Taking complex conjugation on both sides of (2.5), gives

$$0=\sum_{k=0}^{n-1}\left\{\frac{1}{2\pi\, i}\int_{\partial\Omega}\overline{w(\zeta)}\frac{d\zeta}{\zeta-z\omega^{2k}}+\frac{1}{\pi}\int_{\Omega}\overline{w_{\bar\zeta}(\zeta)}\frac{d\xi d\eta}{\bar\zeta-z\omega^{2k}}\right\}. \tag{2.6}$$

Then adding (2.6) to (2.1) gives the claimed formula

$$\begin{aligned}w(z)&=\frac{1}{\pi i}\int_{\partial\Omega}\mathrm{Re}\ w(\zeta)\sum_{k=0}^{n-1}\frac{1}{\zeta-z\omega^{2k}}d\zeta\\&-\frac{1}{\pi}\int_{\Omega}\left[w_{\bar\zeta}(\zeta)\sum_{k=0}^{n-1}\frac{1}{\zeta-z\omega^{2k}}-\overline{w_{\bar\zeta}(\zeta)}\sum_{k=0}^{n-1}\frac{1}{\bar\zeta-z\omega^{2k}}\right]d\xi d\eta,\end{aligned}$$

this completes the proof.

□

3 Schwarz problem for the inhomogeneous Cauchy-Riemann equation

In this Section we study the Schwarz problem

$$\begin{aligned} &w_{\bar z} = f(z), \quad \text{in } \Omega, \\ &\text{Re } w(\zeta) = \varphi(\zeta), \quad \text{on } \partial\Omega, \\ &\text{Im } w\left(i^{1/n}\right) = c, \quad c \in \mathbb{R}. \end{aligned} \tag{3.1}$$

where $f \in L_{p,2}(\Omega;\mathbb{C}), p > 2$ and $\varphi \in L_2(\partial\Omega;\mathbb{R}) \cap C(\partial\Omega;\mathbb{R})$.
Firstly, the boundary behavior of some integrals is investigated. Let

$$K(z,\zeta) = \sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta - \bar z\omega^{-2k}}\right). \tag{3.2}$$

The following lemmas are valid

Lemma 3.1. *If $\varphi \in L_2([0,\infty);\mathbb{C}) \cap C([0,\infty);\mathbb{C})$, then, for $t, t_0 \in (0,\infty)$,*

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_0^\infty [\varphi(\zeta) - \varphi(t_0)]\, K(z,\zeta) d\zeta = \varphi(t) - \varphi(t_0), \tag{3.3}$$

and for $t \in L\backslash\{0\}$,

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_0^\infty \varphi(\zeta) K(z,\zeta) d\zeta = 0, \tag{3.4}$$

where $K(z,\zeta)$ is defined by (3.2)*.*

Proof. By simple computation, one can get

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\sum_{k=1}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta - \bar z\omega^{-2k}}\right) = 0, \;\; t \in (0,\infty)$$

because,

$$\frac{1}{\zeta - t\omega^{2j}} = \frac{1}{\zeta - t\omega^{-2(n-j)}}, \;\; 1 \le j \le n-1.$$

Hence, one gets

$$\begin{aligned} &\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_0^\infty [\varphi(\zeta) - \varphi(t_0)]\, K(z,\zeta) d\zeta \\ &= \lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_0^\infty [\varphi(\zeta) - \varphi(t_0)]\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \bar z}\right) d\zeta. \end{aligned}$$

Thus, from the boundary property of the Poisson kernel on the real axis, (3.3) follows.
If $\zeta \in [0,\infty)$ and $z \in L\backslash\{0\}$, then $K(z,\zeta) \equiv 0$, and hence (3.4) is true. □

Similarly, the following lemma can be proved.

Lemma 3.2. *If $\varphi \in L_2(L;\mathbb{C}) \cap C(L;\mathbb{C})$, then, for $t, t_0 \in L\backslash\{0\}$,*

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_L [\varphi(\zeta)-\varphi(t_0)]\, K(z,\zeta)d\zeta = \varphi(t)-\varphi(t_0), \tag{3.5}$$

and for $t \in (0,\infty)$,

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_L \varphi(\zeta)K(z,\zeta)d\zeta = 0, \tag{3.6}$$

where $K(z,\zeta)$ is defined by (3.2).

For the common corner point at $z=0$ we obtain the following result.

Lemma 3.3. *If $\varphi \in L_2(\partial\Omega;\mathbb{C}) \cap C(\partial\Omega;\mathbb{C})$, then*

$$\lim_{z\in\Omega,\, z\to 0} \frac{1}{2\pi i}\int_{\partial\Omega} [\varphi(\zeta)-\varphi(0)]\, K(z,\zeta)d\zeta = 0, \tag{3.7}$$

where $K(z,\zeta)$ is defined by (3.2).

Proof. Since,

$$\begin{aligned}
&\lim_{z\in\Omega,\, z\to 0} \frac{1}{2\pi i}\int_{\partial\Omega} [\varphi(\zeta)-\varphi(0)]\, K(z,\zeta)d\zeta \\
&\quad = \lim_{z\in\Omega,\, z\to 0} \frac{1}{2\pi i}\int_{\partial\Omega} [\varphi(\zeta)-\varphi(0)] \sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta-\bar{z}\omega^{-2k}}\right)d\zeta,
\end{aligned} \tag{3.8}$$

we investigate this limit in the following two cases.
Case I. If n is an even number, the right-hand side of (3.8) will be as

$$\begin{aligned}
&\lim_{z\in\Omega,\, z\to 0} \frac{1}{2\pi i}\int_0^\infty [\varphi(\zeta)-\varphi(0)] \sum_{k=0}^{\frac{n}{2}-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta-\bar{z}\omega^{-2k}}\right. \\
&\qquad\qquad \left. + \frac{1}{\zeta + z\omega^{2k}} - \frac{1}{\zeta+\bar{z}\omega^{-2k}}\right)d\zeta \\
&\quad - \lim_{z\in\Omega,\, z\to 0} \frac{1}{2\pi i}\int_0^\infty [\varphi(\zeta w)-\varphi(0)] \sum_{k=0}^{\frac{n}{2}-1}\left(\frac{1}{\zeta - z\omega^{2k-1}} - \frac{1}{\zeta-\bar{z}\omega^{-(2k-1)}}\right. \\
&\qquad\qquad \left. + \frac{1}{\zeta + z\omega^{2k-1}} - \frac{1}{\zeta+\bar{z}\omega^{-(2k-1)}}\right)d\zeta \\
&= \lim_{z\in\Omega,\, z\to 0} \frac{1}{2\pi i}\int_{-\infty}^\infty \Phi_1(\zeta) \sum_{k=0}^{\frac{n}{2}-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta-\bar{z}\omega^{-2k}}\right)d\zeta \\
&\quad - \lim_{z\in\Omega,\, z\to 0} \frac{1}{2\pi i}\int_{-\infty}^\infty \Phi_2(\zeta) \sum_{k=0}^{\frac{n}{2}-1}\left(\frac{1}{\zeta - z\omega^{2k-1}} - \frac{1}{\zeta-\bar{z}\omega^{-(2k-1)}}\right)d\zeta,
\end{aligned}$$

where Φ_1 and Φ_2 are continuous functions on $\mathbb{R}$, defined by

$$\Phi_1(\zeta) = \begin{cases} \varphi(\zeta) - \varphi(0), \ \zeta \in (0, \infty), \\ \varphi(0) - \varphi(-\zeta), \ \zeta \in (-\infty, 0), \end{cases}$$

$$\Phi_2(\zeta) = \begin{cases} \varphi(\zeta\omega) - \varphi(0), \ \zeta \in (0, \infty), \\ \varphi(0) - \varphi(-\zeta\omega), \ \zeta \in (-\infty, 0). \end{cases}$$

For $z \in \Omega$, all of the points $z\omega^{2k}\,(0 \le k \le \frac{n}{2} - 1)$, $z\omega^{2k-1}\,(1 \le k \le \frac{n}{2} - 1)$ are in the upper half plane, but $z\omega^{-1}$ lies in the lower half plane. Therefore, computing the above limits of these Poisson integrals on the real line shows

$$\lim_{z\in\Omega,\, z\to 0} \int_{\partial\Omega} [\varphi(\zeta) - \varphi(0)]\, K(z,\zeta) d\zeta = \frac{n}{2}\Phi_1(0) - \left(\frac{n}{2} - 1\right)\Phi_2(0) + \Phi_2(0) = 0.$$

Case II. If n is an odd number, the right-hand side of (3.8) equals

$$\lim_{z\in\Omega,\, z\to 0} \left\{ \frac{1}{2\pi i} \int_0^\infty (\varphi(\zeta) - \varphi(0)) \sum_{k=0}^{\frac{n-1}{2}} \left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta - \bar{z}\omega^{-2k}} \right) d\zeta \right.$$

$$+ \frac{1}{2\pi i} \int_0^\infty (\varphi(\zeta) - \varphi(0)) \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{1}{\zeta + z\omega^{2k-1}} - \frac{1}{\zeta + \bar{z}\omega^{-(2k-1)}} \right) d\zeta$$

$$- \frac{1}{2\pi i} \int_0^\infty (\varphi(\zeta\omega) - \varphi(0)) \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{1}{\zeta - z\omega^{2k-1}} - \frac{1}{\zeta - \bar{z}\omega^{-(2k-1)}} \right) d\zeta$$

$$\left. - \frac{1}{2\pi i} \int_0^\infty (\varphi(\zeta\omega) - \varphi(0)) \sum_{k=0}^{\frac{n-1}{2}} \left(\frac{1}{\zeta + z\omega^{2k}} - \frac{1}{\zeta + \bar{z}\omega^{-2k}} \right) d\zeta \right\},$$

$$= \lim_{z\in\Omega,\, z\to 0} \left\{ \frac{1}{2\pi i} \int_{-\infty}^\infty \Phi_3(\zeta) \sum_{k=0}^{\frac{n-1}{2}} \left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta - \bar{z}\omega^{-2k}} \right) d\zeta \right.$$

$$\left. - \frac{1}{2\pi i} \int_{-\infty}^\infty \Phi_4(\zeta) \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{1}{\zeta - z\omega^{2k-1}} - \frac{1}{\zeta - \bar{z}\omega^{-(2k-1)}} \right) d\zeta \right\},$$

where,

$$\Phi_3(\zeta) = \begin{cases} \varphi(\zeta) - \varphi(0), \ \zeta \in (0, \infty), \\ \varphi(-\zeta\omega) - \varphi(0), \ \zeta \in (-\infty, 0), \end{cases}$$

$$\Phi_4(\zeta) = \begin{cases} \varphi(\zeta\omega) - \varphi(0), \ \zeta \in (0, \infty), \\ \varphi(-\zeta) - \varphi(0), \ \zeta \in (-\infty, 0). \end{cases}$$

The continuity of the functions Φ_3 and Φ_4 at the origin $\zeta = 0$ implies that the above limit, and hence the left-hand side of (3.8), equal to

$$\frac{n+1}{2}\Phi_3(0) - \frac{n-1}{2}\Phi_4(0) = 0$$

$\square$

Lemma 3.4. *If* $\varphi \in L_2(\partial\Omega;\mathbb{C}) \cap C(\partial\Omega;\mathbb{C})$, *then for* $t \in \partial\Omega$,

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_{\partial\Omega} [\varphi(\zeta)-\varphi(0)]\, K(z,\zeta)d\zeta = \varphi(t)-\varphi(0), \tag{3.9}$$

where $K(z,\zeta)$ *is defined by* (3.2).

Proof. By Lemma 3.3 we need only to prove the result for all $t \in \partial\Omega\backslash\{0\}$. According to Lemmas 3.1 and 3.2 Eq.(3.9) is equivalent to

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_{L} [\varphi(\zeta)-\varphi(0)]\, K(z,\zeta)d\zeta = \varphi(t)-\varphi(0), \tag{3.10}$$

for $t \in L\backslash\{0\}$, and for $t \in (0,\infty)$

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_{0}^{\infty} [\varphi(\zeta)-\varphi(0)]\, K(z,\zeta)d\zeta = \varphi(t)-\varphi(0). \tag{3.11}$$

Since for $z \in L\backslash\{0\}$,

$$\frac{1}{\zeta-\bar{z}\omega^{-2k}} = \frac{1}{\zeta - z\omega^{2(n-k-1)}}, \qquad k=0,\cdots,n-2,$$

then the left-hand side of (3.10) equals

$$\begin{aligned}
&\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_{L} [\varphi(\zeta)-\varphi(0)]\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\bar{z}\omega^{2}}\right)d\zeta\\
&= \lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_{0}^{\infty} [\varphi(\zeta\omega)-\varphi(0)]\left(\frac{1}{\zeta-z\omega^{-1}}-\frac{1}{\zeta-\bar{z}\omega}\right)d\zeta\\
&= \varphi(t)-\varphi(0), \qquad t\in L\backslash\{0\}.
\end{aligned}$$

Thus (3.10) is true. Similarly (3.11) can be proved. □

The following result summarizes the fruits of the above lemmas.

Proposition 3.1. *If* $\varphi \in L_2(\partial\Omega;\mathbb{C}) \cap C(\partial\Omega;\mathbb{C})$, *then*

$$\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_{\partial\Omega} \varphi(\zeta)K(z,\zeta)d\zeta = \varphi(t),\ t\in\partial\Omega, \tag{3.12}$$

where $K(z,\zeta)$ *is defined by* (3.2).

Proof. By Lemma 3.4 we get

$$\begin{aligned}
&\lim_{z\in\Omega,\, z\to t} \frac{1}{2\pi i}\int_{\partial\Omega} \varphi(\zeta)K(z,\zeta)d\zeta\\
&\qquad = \varphi(t)-\varphi(0)+\frac{\varphi(0)}{2\pi i}\lim_{z\in\Omega,\, z\to t}\int_{\partial\Omega} K(z,\zeta)d\zeta.
\end{aligned} \tag{3.13}$$

Applying integral representation (2.2) for the function $w(z)=1$, gives

$$1=\frac{1}{\pi i}\int_{\partial\Omega}\sum_{k=0}^{n-1}\frac{1}{\zeta-z\omega^{2k}}d\zeta,\quad z\in\Omega. \tag{3.14}$$

Taking the real part on both sides of (3.14), shows

$$\begin{aligned}1 &= \frac{1}{2\pi i}\int_{\partial\Omega}\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{1}{\zeta-\bar{z}\omega^{-2k}}\right)d\zeta\\ &= \frac{1}{2\pi i}\int_{\partial\Omega}K(z,\zeta)d\zeta.\end{aligned}$$

Therefore, the right-hand side of (3.13) equals $\varphi(t)$. Hence, (3.12) is satisfied for any $t\in\partial\Omega$. □

Now, we introduce the Schwarz-type operator for the sector as

$$S_\Omega[\varphi](z)=\frac{1}{\pi i}\int_{\partial\Omega}\varphi(\zeta)\sum_{k=0}^{n-1}\frac{1}{\zeta-z\omega^{2k}}d\zeta,\quad z\in\Omega, \tag{3.15}$$

where $\varphi\in L_2(\partial\Omega;\mathbb{R})\cap C(\partial\Omega;\mathbb{R})$. Obviously, $S_\Omega[\varphi](z)$ is analytic in the sector Ω. Furthermore,

$$\text{Re}S_\Omega[\varphi](z)=\frac{1}{2\pi i}\int_{\partial\Omega}\varphi(\zeta)K(z,\zeta)d\zeta,\quad z\in\Omega, \tag{3.16}$$

for $\varphi\in L_2(\partial\Omega;\mathbb{R})\cap C(\partial\Omega;\mathbb{R})$, where $K(z,\zeta)$ is defined by (3.2).
According to Proposition 3.1 and (3.16), we obtain the following result.

Theorem 3.1. *If $\varphi\in L_2(\partial\Omega;\mathbb{R})\cap C(\partial\Omega;\mathbb{R})$, then $\{\text{Re}S_\Omega[\varphi]\}^+(t)=\varphi(t)$, $t\in\partial\Omega$, where S_Ω is the Schwarz-type operator defined by* (3.15).

A Pompeiu-type operator for the sector is introduced by

$$T_\Omega[f](z)=-\frac{1}{\pi}\int_\Omega\sum_{k=0}^{n-1}\left(\frac{f(\zeta)}{\zeta-z\omega^{2k}}-\frac{\overline{f(\zeta)}}{\bar{\zeta}-z\omega^{2k}}\right)d\xi d\eta,\ z\in\Omega, \tag{3.17}$$

where $f\in L_{p,2}(\Omega;\mathbb{C})$, $p>2$. Simple computation gives

$$\text{Re }T_\Omega[f](z)=-\frac{1}{2\pi}\int_\Omega\left[f(\zeta)K(z,\zeta)-\overline{f(\zeta)}K(z,\bar{\zeta})\right]d\xi d\eta,\ z\in\Omega, \tag{3.18}$$

where $K(z,\zeta)$ is defined by (3.2).

Theorem 3.2. *If $f\in L_{p,2}(\Omega;\mathbb{C})$, $p>2$, then $\partial_{\bar z}T_\Omega[f](z)=f(z)$, $z\in\Omega$, in the weak sense, and $\{\text{Re }T_\Omega[f]\}^+(t)=0$, $t\in\partial\Omega$, where T_Ω is the Pompeiu-type operator defined by* (3.17).

Proof. By (3.17), we see $\partial_{\bar z} T_\Omega[f](z) = \partial_{\bar z} T[f](z) = f(z)$, $z \in \Omega$ in the weak sense, where T is the classical Pompeiu integral operator, described in detail by Vekua[27]. On the other hand, $K(t,\zeta) = K(t,\bar\zeta) = 0$ for $(t,\zeta) \in \partial\Omega \times \Omega$, implies that $\{\text{Re } T_\Omega[f]\}^+(t) = 0$, $t \in \partial\Omega$. □

Theorem 3.3. *The Schwarz problem*

$$\begin{aligned} &\partial_{\bar z} w = f \;\; in\ \Omega, \;\; \text{Re } w = \varphi \text{ on } \partial\Omega, \\ &\text{Im } w\left(i^{1/n}\right) = c,\ c \in \mathbb{R}, \end{aligned} \tag{3.19}$$

for $f \in L_{p,2}(\Omega;\mathbb{C})$, $p > 2$, $\varphi \in L_2(\partial\Omega;\mathbb{R}) \cap C(\partial\Omega;\mathbb{R})$, *is uniquely solvable by*

$$\begin{aligned} w(z) = ic &+ \frac{1}{\pi i}\int_{\partial\Omega} \varphi(\zeta) \sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{\zeta - \text{Re}\left[i^{1/n}\omega^{2k}\right]}{\zeta^2 - 2\zeta\text{Re}\left[i^{1/n}\omega^{2k}\right] + 1}\right) d\zeta \\ &- \frac{1}{\pi}\int_{\partial\Omega}\left\{ f(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{\zeta - \text{Re}\left[i^{1/n}\omega^{2k}\right]}{\zeta^2 - 2\zeta\text{Re}\left[i^{1/n}\omega^{2k}\right] + 1}\right)\right. \\ &\left. - \overline{f(\zeta)}\sum_{k=0}^{n-1}\left(\frac{1}{\bar\zeta - z\omega^{2k}} - \frac{\bar\zeta - \text{Re}\left[i^{1/n}\omega^{2k}\right]}{\bar\zeta^2 - 2\bar\zeta\text{Re}\left[i^{1/n}\omega^{2k}\right] + 1}\right)\right\} d\xi d\eta. \end{aligned} \tag{3.20}$$

Proof. If w is a solution to the problem (3.19), then the function $\Phi = w - T[f]$ satisfies $\Phi_{\bar z} = 0$ in Ω and Re $\Phi = \varphi - \text{Re } T[f]$ on $\partial\Omega$. By Theorem 3.1 we find $\Phi(z) = S_\Omega[\varphi - \text{Re } T[f]](z) + ic_0$, where S_Ω is the Schwarz-type operator, defined by (3.15), T is the Pompeiu operator, and $c_0 \in \mathbb{R}$ has to be determined by the side condition $\text{Im } w\left(i^{1/n})\right) = c$. We calculate $S_\Omega[\text{Re } T[f]](z)$ as the following. From

$$\begin{aligned} S_\Omega[T[f]](z) &= \frac{1}{\pi i}\int_{\partial\Omega} T[f](\zeta)\sum_{k=0}^{n-1}\frac{1}{\zeta - z\omega^{2k}} d\zeta \\ &= \frac{1}{\pi}\int_\Omega f(\tilde\zeta)\sum_{k=0}^{n-1}\frac{1}{\pi i}\int_{\partial\Omega}\frac{d\zeta}{(\zeta - \tilde\zeta)(\zeta - z\omega^{2k})} d\tilde\xi d\tilde\eta \\ &= \frac{2}{\pi}\int_\Omega f(\tilde\zeta)\sum_{k=1}^{n-1}\frac{1}{\tilde\zeta - z\omega^{2k}} d\tilde\xi d\tilde\eta, \end{aligned}$$

and similarly,

$$S_\Omega\left[\overline{T[f]}\right](z) = -\frac{2}{\pi}\int_\Omega \overline{f(\tilde\zeta)}\sum_{k=0}^{n-1}\frac{1}{\bar{\tilde\zeta} - z\omega^{2k}} d\tilde\xi d\tilde\eta,$$

follows

$$S_\Omega[\text{Re } T[f]](z) = \frac{1}{\pi}\int_\Omega\left\{ f(\tilde\zeta)\sum_{k=1}^{n-1}\frac{1}{\tilde\zeta - z\omega^{2k}} - \overline{f(\tilde\zeta)}\sum_{k=0}^{n-1}\frac{1}{\bar{\tilde\zeta} - z\omega^{2k}}\right\} d\tilde\xi d\tilde\eta.$$

Hence,

$$\begin{aligned} w(z) &= \Phi(z) + T[f](z) = ic_0 + S_\Omega[\varphi](z) \\ &- \frac{1}{\pi}\int_\Omega\left\{ f(\tilde\zeta)\sum_{k=0}^{n-1}\frac{1}{\tilde\zeta - z\omega^{2k}} - \overline{f(\tilde\zeta)}\sum_{k=0}^{n-1}\frac{1}{\bar{\tilde\zeta} - z\omega^{2k}}\right\} d\tilde\xi d\tilde\eta, \end{aligned}$$

that is,

$$w(z) = ic_0 + S_\Omega[\varphi](z) + T_\Omega[f](z). \tag{3.21}$$

From the given side condition, we get

$$c = \text{Im}\ w\left(i^{1/n}\right) = c_0 + \text{Im}\ S_\Omega[\varphi]\left(i^{1/n}\right) + \text{Im}\ T_\Omega[f]\left(i^{1/n}\right),$$

from which one finds

$$\begin{aligned} c_0 = c &+ \frac{1}{\pi}\int_{\partial\Omega}\varphi(\zeta)\sum_{k=0}^{n-1}\frac{\zeta - \text{Re}\left[i^{1/n}\omega^{2k}\right]}{\zeta^2 - 2\zeta\text{Re}\left[i^{1/n}\omega^{2k}\right] + 1}d\zeta \\ &+ \frac{1}{\pi i}\int_{\Omega}\left\{f(\zeta)\sum_{k=0}^{n-1}\frac{\zeta - \text{Re}\left[i^{1/n}\omega^{2k}\right]}{\zeta^2 - 2\zeta\text{Re}\left[i^{1/n}\omega^{2k}\right] + 1}\right. \\ &\qquad \left. - \overline{f(\zeta)}\sum_{k=0}^{n-1}\frac{\overline{\zeta} - \text{Re}\left[i^{1/n}\omega^{2k}\right]}{\overline{\zeta}^2 - 2\overline{\zeta}\text{Re}\left[i^{1/n}\omega^{2k}\right] + 1}\right\}d\xi d\eta. \end{aligned} \tag{3.22}$$

Therefore (3.20) follows. To prove that (3.20) provides a solution to problem (3.19) observe

$$w_{\bar{z}}(z) = \partial_{\bar{z}}T_\Omega[f](z) = f(z),$$

and for $\zeta \in \partial\Omega$, according to Theorems 3.1 and 3.2,

$$\lim_{z\to\zeta}\text{Re}\ \{w(z)\} = \lim_{z\to\zeta}\text{Re}\left\{S_\Omega[\varphi](z) + T_\Omega[f](z)\right\} = \varphi(\zeta),$$

and $\text{Im}\left\{w\left(i^{1/n}\right)\right\} = c$. □

Remark 2. Theorem 17 [20], p.75 is a special case of Theorem 3.3.

4 Schwarz problem for the generalized Beltrami equation

In this section, we deal with the Schwarz problem

$$w_{\bar{z}} = \mathcal{F}\left(z,\ w,\ w_z\right),\ \text{ in } \Omega, \tag{4.1}$$

$$\begin{aligned} &\text{Re}\ w(\zeta) = \varphi(\zeta)\ \text{ on } \partial\Omega, \\ &\text{Im}\ w\left(i^{1/n}\right) = c_0,\ c_0 \in \mathbb{R}, \end{aligned} \tag{4.2}$$

where

$$\mathcal{F}\left(z,\ w,\ w_z\right) = \mu_1 w_z + \mu_2\overline{w_z} + aw + b\overline{w} + c, \tag{4.3}$$

with $\mu_1,\ \mu_2,\ a,\ b,\ c \in L_{p,2}\left(\overline{\Omega};\mathbb{C}\right)$, $p > 2$ such that

$$|\mu_1(z)| + |\mu_2(z)| \le \mu_0 < 1,$$

$$|\mu_1(z)| + |\mu_2(z)| = O\left(|z|^{-\epsilon}\right),\ \text{ as } z \to \infty,$$

and $\varphi \in L_2\left(\partial\Omega;\mathbb{R}\right) \cap C^\alpha\left(\partial\Omega;\mathbb{R}\right)$ for some $0 < \alpha < 1$, $0 < \epsilon$.
In this section we reduce the Schwarz problem (4.1), (4.2) to a fixed-point problem. The existence of a unique solution will be proved by applying the fixed point theorem under sufficient conditions on the coefficients of Eq.(4.1). Firstly, we pay more attention to the properties of the operators S_Ω, T_Ω, defined by (3.15) and (3.17), respectively.

Lemma 4.1. *Let $\varphi \in L_2(\partial\Omega; \mathbb{C}) \cap C^\alpha(\partial\Omega; \mathbb{C})$ satisfy*

$$|\varphi(z_1) - \varphi(z_2)| \leq H_1|z_1 - z_2|^\alpha,\ z_1,\ z_2 \in \partial\Omega.$$

Then $S_\Omega[\varphi] \in C^\alpha\left(\overline{\Omega}\right)$ and

$$|S_\Omega[\varphi](z_1) - S_\Omega[\varphi](z_2)| \leq k_1(\alpha)H_1|z_1 - z_2|^\alpha,\ z_1, z_2 \in \overline{\Omega} \cap \mathbb{C},$$

$$|S_\Omega[\varphi](z)| \leq M(\alpha)H_1 + \sup_{\partial\Omega}|\varphi|.$$

Proof. Let $\varphi_m \in C_0^\infty(\partial\Omega; \mathbb{R})$, $m \in \mathbb{N}$ such that $\lim_{m\to\infty}\|\varphi_m - \varphi\|_2 = 0$. For supp$\varphi_m$, [30], we have

$$|S_\Omega[\varphi_m](z_1) - S_\Omega[\varphi_m](z_2)| \leq k_1(\alpha)H_1|z_1 - z_2|^\alpha,\ z_1, z_2 \in \overline{\Omega},$$

$$|S_\Omega[\varphi_m](z)| \leq M(\alpha)H_1 + \sup_{\partial\Omega}|\varphi_m|.$$

Since

$$|S_\Omega[\varphi_m - \varphi](z)| \leq M(\alpha)\|\varphi_m - \varphi\|_2$$

which becomes small enough, for fixed φ, when m is large, this proves the assertion. □

Similarly, the following result can be proved.

Lemma 4.2. *If φ is a Hölder continuously differentiable function on $\partial\Omega$ such that*

$$|\varphi'(z_1) - \varphi'(z_2)| \leq H_2|z_1 - z_2|^\alpha,\ z_1,\ z_2 \in \partial\Omega,$$

then $S_\Omega[\varphi] \in C^{1,\alpha}\left(\overline{\Omega}\right)$ and

$$|S'_\Omega[\varphi](z_1) - S'_\Omega[\varphi](z_2)| \leq k_2(\alpha)H_2|z_1 - z_2|^\alpha,\ z_1, z_2 \in \overline{\Omega}.$$

The operator T_Ω, defined by (3.17), has similar properties as the classical Pompeiu operator.

Lemma 4.3. *If $f \in L_{p,2}\left(\overline{\Omega}; \mathbb{C}\right)$, $p > 2$, $\alpha = \frac{p-2}{p}$, then*

$$|T_\Omega[f](z)| \leq M(p, n)\|f\|_{p,2},\ z \in \overline{\Omega},$$

$$|T_\Omega[f](z)| \leq M(p, n, R)\|f\|_{p,2}|z|^{-\alpha},\ 1 < R \leq |z|,$$

$$|T_\Omega[f](z_1) - T_\Omega[f](z_2)| \leq M(p, n,)\|f\|_{p,2}|z_1 - z_2|^\alpha,\ z_1, z_2 \in \overline{\Omega}$$

where T_Ω is defined by (3.17).

Proof. Let

$$\Omega_1 = \{z \in \Omega;\ |z| < 1\}$$

Then $T_\Omega[f](z)$ can be rewritten as

$$T_\Omega[f](z) = -\frac{1}{\pi}\int_{\Omega_1}\sum_{k=0}^{n-1}\left(\frac{f(\zeta)}{\zeta - z\omega^{2k}} - \frac{\overline{f(\zeta)}}{\overline{\zeta} - z\omega^{2k}}\right)d\xi d\eta$$
$$-\frac{1}{\pi}\int_{\Omega_1}\sum_{k=0}^{n-1}\left(\frac{f(\frac{1}{\zeta})}{\frac{1}{\zeta} - z\omega^{2k}} - \frac{\overline{f(\frac{1}{\zeta})}}{\frac{1}{\overline{\zeta}} - z\omega^{2k}}\right)|\zeta|^{-4}d\xi d\eta.$$

Since

$$\frac{1}{\zeta_*\left(1-z\zeta_*\omega^{2k}\right)}=\frac{z\omega^{2k}}{1-z\zeta_*\omega^{2k}}+\frac{1}{\zeta_*},\ \zeta_*\in\{\zeta,\bar{\zeta}\},$$

then, we get

$$T_\Omega[f](z)=g_1(z)+g_2(z)-g_2\left(\frac{1}{z}\right),$$

$$g_1(z)=-\frac{1}{\pi}\int_{\Omega_1}\sum_{k=0}^{n-1}\left(\frac{f(\zeta)}{\zeta-z\omega^{2k}}-\frac{\overline{f(\zeta)}}{\bar{\zeta}-z\omega^{2k}}\right)d\xi d\eta,$$

$$g_2(z)=-\frac{1}{\pi}\int_{\Omega_1}\sum_{k=0}^{n-1}\left(\frac{\bar{\zeta}^{-2}f(\zeta)}{\zeta-z\omega^{2k}}-\frac{\zeta^{-2}\overline{f(\zeta)}}{\bar{\zeta}-z\omega^{2k}}\right)d\xi d\eta.$$

Therefore, by theorem 23 [5] p.74, we obtain

$$|T_\Omega[f](z)|\le M(p,n)\left\{\|f\|_{p,\Omega_1}+\left\|\bar{\zeta}^{-2}f\left(\frac{1}{\zeta}\right)\right\|_{p,\Omega_1}\right\}\le M(p,n)\|f\|_{p,2},$$

$$|g_1(z_1)-g_1(z_2)|\le M(p,n)\|f\|_{p,2}|z_1-z_2|^\alpha,$$

$$\left|g_2(\frac{1}{z_1})-g_2(\frac{1}{z_2})\right|\le M(p,n)\|f\|_{p,2}|z_1-z_2|^\alpha,$$

$$|g_1(z)|\le M(p,n)\|f\|_{p,2}\frac{1}{|z|-1},\ \ 1<|z|,$$

$$\left|g_2(0)-g_2(\frac{1}{z})\right|\le M(p,n)\|f\|_{p,2}\,|z|^{-\alpha}.$$

Thus,

$$|T_\Omega[f](z)|\le M(p,n)\|f\|_{p,2}\left(\frac{1}{|z|-1}+|z|^{-\alpha}\right)\le M(p,n)\|f\|_{p,2}|z|^{-\alpha},$$

because for $1<R<|z|$, $0<\alpha<1$

$$\frac{|z|^\alpha}{|z|-1}\le\frac{|z|}{|z|-1}\le\frac{R}{R-1}.$$

The last estimate follows from those for the functions g_1 and g_2. □

Next we state the boundedness property of the Ahlfors-Beurling-type operator, defined by

$$\Pi_\Omega[f](z)=-\frac{1}{\pi}\int_\Omega\sum_{k=0}^{n-1}\left\{\frac{f(\zeta)}{\left(\zeta-z\omega^{2k}\right)^2}-\frac{\overline{f(\zeta)}}{\left(\bar{\zeta}-z\omega^{2k}\right)^2}\right\}\omega^{2k}d\xi d\eta. \tag{4.4}$$

Lemma 4.4. *If* $f\in L_{p,2}(\Omega;\mathbb{C})$, $p>2$, *then* $\Pi_\Omega[f]\in L_p(\Omega;\mathbb{C})$ *and*

$$\|\Pi_\Omega[f]\|_p\le\Lambda_p\|f\|_{p,2}.$$

For the proof we refer to [27].

Remark 3. *The smallest constant* Λ_p, for which the above inequality holds, it said to be the L_p-norm of the operator Π_Ω, *i.e.*

$$\|\Pi_\Omega\|_p = \Lambda_p.$$

Return to the main problem (4.1), (4.2). The following theorem reduces the Schwarz problem (4.1), (4.2) to an equivalent integral equation.

Theorem 4.1. *A function* $w \in C^{1+\alpha}\left(\overline{\Omega};\mathbb{C}\right)$ *is a solution to the Schwarz problem* (4.1), (4.2) *if and only if* w *solves the integral equation*

$$w(z) = \Phi(z) + T_\Omega[\mathcal{F}\left(z, w, w_z\right)](z), \tag{4.5}$$

where T_Ω *is defined by* (3.17) *and* $\Phi \in C^{\alpha}\left(\overline{\Omega};\mathbb{C}\right)$ *is an analytic function in the sector* Ω*, satisfying the Schwarz problem*

$$\begin{aligned} &\text{Re } \Phi(\zeta) = \varphi, \ \text{ on } \partial\Omega, \\ &\text{Im } \Phi\left(i^{1/n}\right) = c_0 - \text{Im } T_\Omega[\mathcal{F}]\left(i^{1/n}\right). \end{aligned} \tag{4.6}$$

Proof. Assume that $w \in C^{1+\alpha}\left(\overline{\Omega} : \mathbb{C}\right)$ is a solution to the Schwarz problem (4.1), (4.2). Define a function Φ by

$$\Phi(z) = w(z) - T_\Omega[\mathcal{F}\left(z, w, w_z\right)](z).$$

Differentiating with respect to $\bar{z}$, gives

$$\Phi_{\bar{z}}(z) = w_{\bar{z}}(z) - \mathcal{F}\left(z, w, w_z\right)(z) = 0,$$

in the Sobolev sense [5, 27]. Then Φ is analytic in Ω and the conditions (4.2) lead to

$$\begin{aligned} &\text{Re } \Phi(\zeta) = \varphi - \text{Re } T_\Omega[\mathcal{F}](\zeta) = \varphi, \ \text{ on } \partial\Omega, \\ &\text{Im } \Phi\left(i^{1/n}\right) = c_0 - \text{Im } T_\Omega[\mathcal{F}]\left(i^{1/n}\right), \end{aligned}$$

because, by Theorem 3.2,

$$\text{Re } T_\Omega[\mathcal{F}](\zeta) = 0, \ \ \zeta \in \partial\Omega.$$

It is given that $\varphi \in C^{\alpha}\left(\partial\Omega;\mathbb{R}\right)$, then $\Phi \in C^{\alpha}\left(\overline{\Omega};\mathbb{C}\right)$. Hence, a solution w to the Schwarz problem (4.1), (4.2) solves the integral equation (4.5), with Φ analytic in Ω satisfying (4.6). Conversely, if w is a solution to the integral equation (4.5) where Φ is analytic in Ω satisfying the conditions (4.6). Differentiating equation (4.5) with respect to $\bar{z}$, one gets

$$w_{\bar{z}} = \mathcal{F}\left(z, w, w_z\right)(z), \ \text{ in } \Omega.$$

Furthermore, Re $w = \varphi$ on $\partial\Omega$ and Im $w\left(i^{1/n}\right) = c_0$. This proves that the function $w \in C^{1+\alpha}\left(\overline{\Omega};\mathbb{C}\right)$, defined by (4.5), solves the Schwarz problem (4.1), (4.2). $\square$

Remark 4. By Theorem 3.3 the Schwarz problem (4.6) is uniquely solvable by

$$\Phi(z) = ic_* + S_\Omega[\varphi](z), \ z \in \Omega,$$

where S_Ω is defined by (3.15) and c_* is a real constant given by (3.22) for $f(\zeta) = \mathcal{F}(z, w, w_z)(\zeta)$ *i.e.*

$$\begin{aligned} c_* = c_0 &+ \frac{1}{\pi}\int_{\partial\Omega}\varphi(\zeta)\sum_{k=0}^{n-1}\frac{\zeta - \mathrm{Re}\left[i^{1/n}\omega^{2k}\right]}{\zeta^2 - 2\zeta\mathrm{Re}\left[i^{1/n}\omega^{2k}\right]+1}d\zeta \\ &+\frac{1}{\pi i}\int_\Omega\left\{\mathcal{F}(\zeta, w, w_\zeta)\sum_{k=0}^{n-1}\frac{\zeta - \mathrm{Re}\left[i^{1/n}\omega^{2k}\right]}{\zeta^2 - 2\zeta\mathrm{Re}\left[i^{1/n}\omega^{2k}\right]+1}\right. \\ &\left.-\overline{\mathcal{F}(\zeta, w, w_\zeta)}\sum_{k=0}^{n-1}\frac{\overline{\zeta} - \mathrm{Re}\left[i^{1/n}\omega^{2k}\right]}{\overline{\zeta}^2 - 2\overline{\zeta}\mathrm{Re}\left[i^{1/n}\omega^{2k}\right]+1}\right\}d\xi d\eta. \end{aligned}$$

On basis of Theorem 4.1 we can reduce the Schwarz problem (4.1), (4.2) to a fixed-point problem. Differentiating Equation (4.5) with respect to z, we get

$$w_z = \Phi' + \Pi_\Omega[\mathcal{F}(z, w, w_z)].$$

Denote w_z by h. Thus we have the system of two integral equations

$$w = \Phi + T_\Omega[\mathcal{F}(z, w, h)], \tag{4.7}$$
$$h = \Phi' + \Pi_\Omega[\mathcal{F}(z, w, h)], \tag{4.8}$$

where Π_Ω is defined by (4.4).
The system of integral equations (4.7), (4.8) defines an operator $\mathcal{P}$, by

$$\left.\begin{aligned} &\mathcal{P}:(w,h)\to(\mathcal{W},\mathcal{H}) \\ &\mathcal{W} = \Phi_{(w,h)} + T_\Omega[\mathcal{F}(., w, h)], \\ &\mathcal{H} = \Phi'_{(w,h)} + \Pi_\Omega[\mathcal{F}(., w, h)], \end{aligned}\right. \tag{4.9}$$

where $\Phi_{(w,h)}$ is an analytic function in Ω satisfying the conditions

$$\mathrm{Re}\ \Phi_{(w,h)} = \varphi, \ \text{ on } \partial\Omega,$$

$$\mathrm{Im}\ \Phi_{(w,h)}\left(i^{1/n}\right) = c_0 - \mathrm{Im}\ T_\Omega[\mathcal{F}(., w, h)]\left(i^{1/n}\right).$$

The pair (w, h) turns out to be a fixed point to the operator $\mathcal{P}$. The system of integral equations (4.7), (4.8) can be used for solving the Schwarz problem (4.1), (4.2). Let (w, h) be a fixed point to the operator $\mathcal{P}$, *i.e.*

$$w = \Phi_{(w,h)} + T_\Omega[\mathcal{F}(., w, h)], \tag{4.10}$$
$$h = \Phi'_{(w,h)} + \Pi_\Omega[\mathcal{F}(., w, h)], \tag{4.11}$$

Differentiating the first equation (4.10) with respect to z, we get

$$w_z = \Phi'_{(w,h)} + \Pi_\Omega[\mathcal{F}(., w, h)].$$

Comparing this result with (4.11), implies that $w_z = h$ holds for a fixed point (w, h) to the operator $\mathcal{P}$. Hence, Equation (4.10) can be rewritten as

$$w = \Phi_{(w,w_z)} + T_\Omega[\mathcal{F}\,(., w, w_z)]$$

Differentiating this equation with respect to $\bar{z}$, obtains Equation (4.1). Further, $\mathrm{Re}\, w = \mathrm{Re}\ \Phi_{(w,w_z)} = \varphi$ on $\partial\Omega$, $\mathrm{Im}\ w\left(i^{1/n}\right) = c_0$. This completes the proof of the following Lemma.

Lemma 4.5. *Let (w, h) be a fixed point of the operator $\mathcal{P}$. Then its first component function w is a solution to the Schwarz problem* (4.1), (4.2), *and the second component h equals to the derivative w_z of the solution $w = w(z)$.*

Now we are going to solve this fixed point problem. Consider the space

$$\mathcal{B} = \left\{(w, h);\ w,\, h \in C^\alpha\left(\overline{\Omega}; \mathbb{C}\right)\right\},$$

equipped with the norm

$$\|(w, h)\|_* = \max\left\{\|w\|_\alpha,\ \|h\|_\alpha\right\},\quad (w, h) \in \mathcal{B},$$

where $\|\cdot\|_\alpha$ is the usual Hölder norm in $C^\alpha\left(\overline{\Omega}\right)$. Since $C^\alpha\left(\overline{\Omega}\right)$ is a Banach space, then $\mathcal{B}$ is a Banach space.

Next we obtain sufficient conditions on the coefficients of $\mathcal{F}$ under which the operator $\mathcal{P}$ is contractive. For (w_1, h_1) and (w_2, h_2) in $\mathcal{B}$, let $\mathcal{P}(w_1, h_1) = (\mathcal{W}_1, \mathcal{H}_1)$ and $\mathcal{P}(w_2, h_2) = (\mathcal{W}_2, \mathcal{H}_2)$. Then

$$\mathcal{W}_1 - \mathcal{W}_2 = \Phi_{(w_1,h_1)} - \Phi_{(w_2,h_2)} + T_\Omega[\mathcal{G}],\ \ \mathcal{H}_1 - \mathcal{H}_2 = \Pi_\Omega[\mathcal{G}],$$

where $\Phi_{(w_1,h_1)} - \Phi_{(w_2,h_2)}$ is an imaginary constant given by

$$\Phi_{(w_1,h_1)} - \Phi_{(w_2,h_2)} = -i\mathrm{Im}\ T_\Omega[\mathcal{G}]\left(i^{1/n}\right),$$

where $\mathcal{G} := \mathcal{F}(., w_1, h_1) - \mathcal{F}(., w_2, h_2)$.

Let

$$\Lambda_1 := \|T_\Omega\|_\alpha\left(L_1\|h_1 - h_2\|_\alpha + L_2\|w_1 - w_2\|_\alpha\right),$$

and

$$\Lambda_2 := \|\Pi_\Omega\|_\alpha\left(L_1\|h_1 - h_2\|_\alpha + L_2\|w_1 - w_2\|_\alpha\right),$$

where $L_1 := \|\mu_1\|_p + \|\mu_2\|_p$ and $L_2 := \|a\|_p + \|b\|_p$.

Since T_Ω and Π_Ω from $L_p\left(\overline{\Omega}\right)$, $p > 2$, to $C^\alpha\left(\overline{\Omega}\right)$, $\alpha = (p-2)/p$, are bounded operators, then

$$\|T_\Omega[\mathcal{G}]\|_\alpha \le \|T_\Omega\|_\alpha\|\mathcal{G}\|_\alpha \le \Lambda_1,\ \ \|\Pi_\Omega[\mathcal{G}]\|_\alpha \le \|\Pi_\Omega\|_\alpha\|\mathcal{G}\|_\alpha \le \Lambda_2. \tag{4.12}$$

Introducing the distance

$$d\left((w_1, h_1),\ (w_2, h_2)\right) = \max\left\{\|w_1 - w_2\|_\alpha, \|h_1 - h_2\|_\alpha\right\},$$

in the space $\mathcal{B}$, then

$$\begin{aligned} d\left(\mathcal{P}\left(w_1,h_1\right),\, \mathcal{P}\left(w_2,h_2\right)\right) &= \max\left\{\|\mathcal{W}_1-\mathcal{W}_2\|_\alpha,\|\mathcal{H}_1-\mathcal{H}_2\|_\alpha\right\} \\ &\le \max\left\{|\Phi_{(w_1,h_1)}-\Phi_{(w_2,h_2)}|+\|T_\Omega[\mathcal{G}]\|_\alpha,\|\Pi_\Omega[\mathcal{G}]\|_\alpha\right\} \\ &\le \max\left\{k_1\Lambda_1,\Lambda_2\right\} \\ &\le \max\left\{k_1\|T_\Omega\|_\alpha,\|\Pi_\Omega\|_\alpha\right\}\left(L_1+L_2\right)d\left(\left(w_1,h_1\right),\,\left(w_2,h_2\right)\right). \end{aligned}$$

Therefore, the operator $\mathcal{P}$ is contractive if

$$\|\mu_1\|_p+\|\mu_2\|_p+\|a\|_p+\|b\|_p<\frac{1}{\max\left\{k_1\|T_\Omega\|_\alpha,\|\Pi_\Omega\|_\alpha\right\}}, \tag{4.13}$$

where k_1 is the Hölder constant of $S_\Omega[\varphi](z)$, given in Lemma 4.1. This proves the following result.

Proposition 4.1. *The operator $\mathcal{P}$, defined by* (4.9) *is contractive if* (4.13) *holds.*

Hence the following statement can be proved.

Theorem 4.2. *The Schwarz problem*

$$w_{\bar z}=\mu_1 w_z+\mu_2\overline{w_z}+aw+b\overline{w}+c,\ \text{in } \Omega,$$

$$\operatorname{Re}\, w(\zeta)=\varphi(\zeta)\ \text{on } \partial\Omega,$$
$$\operatorname{Im}\, w\left(i^{1/n}\right)=c_0,\ c_o\in\mathbb{R},$$

where $\mu_1,\ \mu_2,\ a,\ b,\ c\in L_{p,2}\left(\overline{\Omega};\mathbb{C}\right),\ p>2$, such that

$$|\mu_1(z)|+|\mu_2(z)|\le\mu_0<1,$$

$$|\mu_1(z)|+|\mu_2(z)|=O\left(|z|^{-\epsilon}\right),\ \text{as}\, z\to\infty,$$

and $\varphi\in L_2\left(\partial\Omega;\mathbb{R}\right)\cap C^\alpha\left(\partial\Omega;\mathbb{R}\right)$ for some $0<\alpha<1,\ 0<\epsilon$,
is uniquely solvable if the coefficients $\mu_1,\ \mu_2,\ a,\ b,\ c$ satisfy (4.13).

Proof. If (4.13) is satisfied, the operator $\mathcal{P}$ is contractive. Hence, by the contraction mapping principle, $\mathcal{P}$ has a unique fixed point (w,h) satisfying (4.7), (4.8). By Lemma 4.5 the first component w is the unique solution of the boundary value problem under consideration. □

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