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THE PREDUAL SPACE OF A JBW*-TRIPLE

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Abstract. In the paper JB*-triples which are dual Banach spaces are considered as analogues of JBW*-algebras. For JBW*-triples an analogue of the classical theorem of Vitali-Hahn-Saks on convergent sequences of measures and a theorem on weak compactness of a set of normal functionals are proved.

It is known that normed Jordan triple systems have been studied because of their connection to bounded symmetric domains in Banach spaces and to C*-algebras. JB*-triples generalize C*-algebras: any norm-closed subspace of the space of all bounded linear operators on a complex Hilbert space which is also closed under the Jordan triple product $\{xy^*z\} = \frac{1}{2}(xy^*z + zy^*x)$ is a JB*-triple. Let us remind that a map $x \to x^*$ is called an *involution* if it satisfies the following conditions: $(x^*)^* = x$, $(\lambda x)^* = \overline{\lambda} x^*$, $(x+y)^* = x^*+y^*$, $(xy)^* = y^*x^*$, where $\lambda \in \mathbb{C}$. In this paper, we consider JBW*-triples, i.e., JB*-triples which are dual Banach spaces. On the basis of works [3, 1, 6, 2], for JBW*-triples there are proved an analogue of the classical theorem of Vitali-Hahn-Saks on convergent sequences of measures and a theorem of weak compactness of a set of normal functionals. They are generalizations of the results of works [1, 6], where JBW-algebras have been considered.

A *Jordan *-triple* is a complex vector space U with a sesquilinear map $(x, y) \to x\Box y^*$ from $U \times U$ to the space $L(U)$ of all linear operators on U such that

1) the triple product $\{xy^*z\} = x\Box y^*(z)$ is symmetric in x and z;

2) $\{uv^*\{xy^*z\}\} = \{\{uv^*x\}y^*z\} - \{x\{vu^*y\}^*z\} + \{xy^*\{uv^*z\}\}, \ \forall u, v, x, y, z \in U.$

A Jordan *-triple U is called *abelian* if $\{xy^*\{uv^*z\}\} = \{\{xy^*u\}v^*z\}$ for all $u, v, x, y, z \in U$. An element $e(\neq 0) \in U$ is called *tripotent*, if $\{ee^*e\} = e$. A non-zero tripotent e induces a decomposition of a Jordan *-triple U into the eigenspaces of $e\Box e^*$, the *Peirce decomposition* $U = U_1(e) \oplus U_{1/2}(e) \oplus U_0(e)$, where $U_k = U_k(e) = \{z \in U : \{ee^*z\} = kz\}$, for $k = 0, 1/2, 1$. For U_k the following multiplication rules hold

$\{U_1U_0^*U\} = \{U_0U_1^*U\} = 0$ and $\{U_iU_j^*U_k\} \subset U_{i-j+k}$, where $i,j,k \in \{0,1/2,1\}$ and $U_l = 0$ for $l \neq 0,1/2,1$. Put

$$P_k^e(z) = \begin{cases} z, & \text{if } z \in U_k, \\ 0, & \text{if } z \in U_j, j \neq k. \end{cases}$$

Tripotents e and f are called *compatible*, if $P_j^eP_k^f = P_k^fP_j^e$ (i.e. P_j^e and P_k^f commute), for all $j,k \in \{0,1/2,1\}$; *orthogonal* (denote by $e \perp f$), if $e \in U_0(f)$.

A JB*-*triple* is a Jordan *-triple U endowed with a complete norm such that the triple product is jointly continuous, $z\Box z^*$ is a hermitian operator with pozitive spectrum and $\|\{zz^*z\}\| = \|z\|^3$, for all $z \in U$. A JB*-triple U is called a JBW*-*triple* if U (as a Banach space) has a predual U_* such that the triple product is separately $\sigma(U,U_*)$-continuous. Any JB*-algebra (respectively, JBW*-algebra) is a JB*-triple (respectively, JBW*-triple) with respect to the product $\{xy^*z\} = (x \circ y^*) \circ z - (x \circ z) \circ y^* + (y^* \circ z) \circ x$. Conversely, if e is a tripotent in a JB*-triple (respectively, JBW*-triple) U, then $U_1(e)$ is a JB*-algebra (respectively, JBW*-algebra) with product $xy = \{xe^*y\}$ and involution $x \to \{ex^*e\}$.

Theorem 1. (Proposition 3.19 [4]). *Let U be a JBW*-triple with predual U_* and let $f \in U^*$. Then the following conditions are equivalent*

(i) $f \in U_*$;

(ii) $f(\sum_{i\in I} e_i) = \sum_{i\in I} f(e_i)$, *for every orthogonal family $(e_i)_{i\in I}$ of tripotents.*

Corollary. *Let A be a JBW*-algebra with predual A_* and let $f \in A^*$. Then the following conditions are equivalent*

(i) $f \in A_*$;

(ii) $f(\sum_{i\in I} e_i) = \sum_{i\in I} f(e_i)$, *for every orthogonal family $(e_i)_{i\in I}$ of projections.*

A functional $f \in A^*$ is called a *normal* if one of equivalent conditions of Theorem 1 is satisfied.

Theorem 2. *Let U be a JBW*-triple. Let $\{\varphi_n\}$ be a sequence of functionals in U_* and let $\varphi_n(x) \to \varphi(x)$, for all $x \in U$. Then $\varphi \in U_*$.*

Proof. Let $(e_j)_{j\in J}$ be an arbitrary orthogonal family of tripotents in U and let $e = \sum_{i\in J} e_j$. By Corollary 3.13 [4] a tripotent e_j is a e-projection for all $j \in J$, i.e. e_j is projection on JBW*-algebra $A = U_1(e)$ for all $j \in J$. Since $\{\varphi_n\} \in U_*$ by Lemma 3.18 [3] $\varphi_n |_A \in A_*$ for all n. Then by Proposition 3.18 [4] and $\varphi_n(x) \to \varphi(x)$ (for all $x \in U$) we have $\varphi |_A \in A_*$.

By Corollary we obtain $\varphi(\sum_{j\in J} e_j) = \sum_{j\in J} \varphi(e_i)$. Then Theorem 1 implies that $\varphi \in U_*$. □

Theorem 3. *Let M be a norm bounded subset of U_*. Then the following conditions are equivalent*

(1) *M is weakly relatively compact;*

(2) *a restriction of M on each maximal abelian subtriple of U is weakly relatively compact;*

(3) *for any sequence of orthogonal tripotents (e_n) the convergence $\lim_{n\to\infty} f(e_n) = 0$ is uniform for all $f \in M$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Further, since any orthogonal family of tripotents is compatible ([3, 1.18]), then the subtriple generated by this family is abelian. By the Zorn Lemma it is contained in a maximal abelian subtriple V. Then from $\sigma(U, U_*)$-continuity of each component of the triple product it follows that a subtriple V is $\sigma(U, U_*)$-closed. Therefore by Lemma 3.13 [3] a subtriple V is isometrically isomorphic to an abelian W*-algebra. Then the implications (2) $\Leftrightarrow$ (3) follow by Grothendieck's results ([2, Theorem 4, Corollary 1]). Hence it suffices to prove that (2) $\Rightarrow$ (1). We consider the space U_* as the subspace of U^*. Since M is norm-bounded in U^* by Banach-Anaouglu's Theorem its $\sigma(U^*, U)$-closure $\overline{M}$ in U^* is $\sigma(U^*, U)$-compact.

The assertion will be proved, if we show that, every element $f \in \overline{M} \subset U^*$ belongs $U_* \subset U^*$, since the weak topology in U_* is the restriction of the weak topology in U^*. Let $f \in \overline{M}$. If V is any maximal abelian subtriple, then by assumption (2) the restriction $M \mid_V$ is weakly relatively compact in U_*, and therefore $f \mid_V \in \overline{M \mid_V} \subset U_*$. Hence the restriction of f on any maximal abelian subtriple is normal. Since any orthogonal family of tripotents is compatible the map f is a completely additive functional on the tripotents. Then Theorem 1 implies that $f \in U_*$. □

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