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NORMAL EXTENSIONS OF A SINGULAR DIFFERENTIAL OPERATOR ON THE RIGHT SEMI-AXIS

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Abstract. In this work, based on the method of Everitt-Zettl and using the Calkin-Gorbachuk method, all normal extensions of the minimal operator generated by a linear singular formally normal differential-operator expression of the first order in Hilbert spaces of vector-functions on the right semi-axis in terms of boundary values are described. Furthermore, the structure of the spectrum of these extensions is investigated.

1 Introduction

Many problems arising in the modeling of processes of multi-particle quantum mechanics, in quantum field theory, in the physics of rigid bodies, in boundary value problems for differential equations Hardy-Morrey etc. support studying singular normal operators in Hilbert spaces (see [1], [20]). It is known that the general theory of selfadjoint extensions of symmetric operators in a Hilbert space and their spectral theory have been investigated by many mathematicians (for example, see [5], [19], [18], [20] and references therein). On the other hand, note that in the multi-interval linear symmetric ordinary differential expression case the deficiency indices may be different for each interval, but equal for the direct sum of Hilbert spaces on different intervals. The selfadjoint extension theory for the case of an ordinary linear differential expression of arbitrary order is known due to the famous work of W.N. Everitt and A. Zettl [6] for any number of intervals of real-axis, finite or infinite. This theory is based on the Glazman-Krein-Naimark Theorem. Deep information on the selfadjoint extensions and direct and complete characterizations for the Sturm-Liouville differential expression on a finite or infinite interval with interior points or endpoints singularities can be found in the significant monograph of A. Zettl [20]. In special cases, the selfadjoint extension problems for multi-interval linear symmetric ordinary differential-operator expressions of the first order in a Hilbert space of vector-functions have been investigated in [2], [8], [9].

It is known that a densely defined closed operator N in any Hilbert space is called formally normal if $D(N) \subset D(N^*)$ and $\|Nf\| = \|N^*f\|$ for all $f \in D(N)$.

If a formally normal operator has no formally normal extension, then it is called maximal a formally normal operator. If a formally normal operator N satisfies the condition $D(N) = D(N^*)$, then it is called a normal operator [4].

Generalization of J. von Neumann's theory to the theory of normal extensions of formally normal operators in Hilbert space has been done by E.A. Coddington in work [4]. Some applications of this theory to two-point regular type differential operators in Hilbert spaces can be found in [10]-[17].

In this work, in the second section all normal extensions of the minimal formally normal operator generated by a linear differential expression in Hilbert spaces of vector-functions defined in right half-infinite interval are described. Furthermore, the spectrum of such extensions is investigated.

2 Description of normal extensions

Let H be a separable Hilbert space with a scalar product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. In the Hilbert space $L^2(H,(a,+\infty))$, $a \in \mathbb{R}$ consider a linear differential-operator expression of first order in the form

$$l(u) = Ju' + Au, \tag{2.1}$$

where E is the identity operator in H, $J^* = J$, $J^2 = E$, $A^* = A \in L(H)$, $A \geq 0$, $A^{\frac{1}{2}}J = JA^{\frac{1}{2}}$.

It is clear that the formally adjoint expression in $L^2(H,(a,+\infty))$ has the form

$$l^+(v) = -Jv' + Av. \tag{2.2}$$

On the set D_0' of continuously differentiable vector-functions $u(t)$ on $(a,+\infty)$ with compact support such that $l(u) \in L^2(H,(a,+\infty))$ consider the operator

$$L_0 : L_0'u := l(u), \quad u \in D_0'.$$

The closure of the L_0' in $L^2(H,(a,+\infty))$ is called the minimal operator generated by differential-operator expression (2.1) and it is denoted by L_0.

In a similar way the minimal operator L_0^+ in $L^2(H,(a,+\infty))$ generated by differential-operator expression (2.2) may be defined. The adjoint operator of L_0^+ (L_0) in the space $L^2(H,(a,+\infty))$ is called the maximal operator generated by (2.1) ((2.2)) and is denoted by L (L^+).

It is clear that

$$L_0 \subset L, \quad L_0^+ \subset L^+, \quad D(L_0) = \overset{0}{W}{}_2^1(H,(a,+\infty)), \quad D(L) = W_2^1(H,(a,+\infty)).$$

In the case when $J = E$, then it is easily shown that the minimal operator L_0 is a maximal formally normal in $L^2(H,(a,+\infty))$ and therefore has no normal extensions [4].

Theorem 2.1. *Let*

$$A^{\frac{1}{2}}W_2^1(H,(a,+\infty)) \subset W_2^1(H,(a,+\infty))$$

For every normal extension L_n, $L_0 \subset L_n \subset L$, of the minimal operator L_0 in $L^2(H,(a,+\infty))$, there exist a hyper maximal J-neutral subspace M such that the extension L_n is generated by differential-operator expression (2.1) and the boundary condition

$$u(a) \in M, \quad A^{\frac{1}{2}}u(a) \in M. \tag{2.3}$$

The subspace M is determined uniquely by the extension L_n, i.e. $L_n = L_M$.

To the contrary, the restriction of the maximal operator L_0 to the linear manifold of vector-functions $u(t) \in W_2^1(H,(a,+\infty))$ that satisfies (2.3) with any hyper maximal J-neutral subspaces M is a normal extension of a minimal operator L_0 in $L^2(H,(a,+\infty))$.

Proof. If L_n is any normal extension of L_0, then

$$Re(L_n) = \overline{A \otimes E}, \quad Re(L_n) : D(L_n) \to L^2(H,(a,+\infty)),$$

$$Im(L_n) = \overline{E \otimes (-iJ\frac{d}{dt})}, \quad Im(L_n) : D(L_n) \to L^2(H,(a,+\infty)),$$

where the symbol $\otimes$ denotes a tensor product, are selfadjoint extensions of $Re(L_0)$ and $Im(L_0)$ in $L^2(H,(a,+\infty))$ respectively. The operator $Im(L_0)$ is generated by the differential expression $-iJ\frac{d}{dt}$ and the condition $u(a) \in M$, where M is a hyper maximal J-neutral subspace in H such that it determined uniquely by the extension L_n, i.e. $Im(L_n) = Im(L_M)$ [7].

On the other hand since the extension L_n is a normal operator, then for every $u \in D(L_n)$ the following equality holds

$$(Re(L_n)u, Im(L_n)u)_{L^2} = (Im(L_n)u, Re(L_n)u)_{L^2}.$$

In other words, for every $u \in D(L_n)$ the equality

$$(Ju^{'}, Au)_{L^2} + (Au, Ju^{'})_{L^2} = 0$$

is satisfied. From last relation and the condition

$$A^{\frac{1}{2}}W_2^1(H,(a,+\infty)) \subset W_2^1(H,(a,+\infty))$$

we have

$$(Ju, Au)^{'}_{L^2} = -(Ju(a), Au(a))_H = 0.$$

That is for every $u \in D(L_n)$ holds the following second condition of normality

$$(Ju(a), Au(a))_H = 0.$$

On the other hand since the operator A is a selfadjoint positive operator in H and $A^{\frac{1}{2}}J = JA^{\frac{1}{2}}$, from the last condition it follows that

$$(JA^{\frac{1}{2}}u(a), A^{\frac{1}{2}}u(a))_H = 0.$$

This means that $A^{\frac{1}{2}}u(a) \in M$. It is clear that the hyper maximal J-neutral subspace M is determined uniquely by the extension L_n [7].

To the contrary, now let L_M be an operator generated by differential-operator expression (2.1) and boundary condition (2.3) in $L^2(H,(a,+\infty))$, that is

$$L_M u = l(u), \quad D(L_M) = \{u \in W_2^1(H,(a,+\infty)) : u(a) \in M, \quad A^{\frac{1}{2}}u(a) \in M\},$$

$$L_M : D(L_M) \subset L^2(H,(a,+\infty)) \to L^2(H,(a,+\infty)).$$

In this case for the adjoint operator L_M^* we have

$$L_M^* = (ReL_M + iImL_M)^* = ReL_M - iImL_M.$$

Hence

$$L_M^* v := -Jv' + Av,$$

$$D(L_M^*) = D(ReL_M \cap ImL_M) = D(ImL_M) = D(L_M).$$

On the other hand it is easy to check that the other condition of normality of the extension L_M is satisfied. This completes the proof of the theorem. □

3 Spectrum of normal extensions

Here the spectrum of the normal extension L_M of the minimal operator L_0 generated by linear differential-operator extension (2.1) and boundary condition (2.3) in $L^2(H,(a,+\infty))$ will be investigated.

Theorem 3.1. *The point spectrum of any normal extension L_M of the minimal operator L_0 in the Hilbert space $L^2(H,(a,+\infty))$ is empty, i.e. $\sigma_p(L_M) = \phi$.*

Proof. Consider the following problem for the point spectrum of the normal extension L_M in $L^2(H,(a,+\infty))$, $L_M u = \lambda u$, $\lambda = \lambda_r + i\lambda_i \in \mathbb{C}$, $u \in W_2^1(H,(a,+\infty))$, that is,

$$\begin{cases} Ju' + Au = \lambda u, \quad u \in W_2^1(H,(a,+\infty)) \\ u(a) \in M, \quad A^{\frac{1}{2}}u(a) \in M, \quad \lambda \in \mathbb{C} \end{cases}$$

If $\bar{\lambda} \in \sigma_p(L_M^*)$, then $-Ju' + Au = \bar{\lambda}u$.

In this situation from the above two equations we have

$$\begin{cases} Ju' = i\lambda_i u, \\ Au = \lambda_r u, \quad \lambda_i, \lambda_r \in \mathbb{R}, \quad u(a) \in M, \quad A^{\frac{1}{2}}u(a) \in M, \quad u \in W_2^1(H,(a,+\infty)). \end{cases}$$

It is clear that the general solution of the first of the above equations has the form

$$u_\lambda(t) = e^{i\lambda_i J(t-a)} f_\lambda, \quad f_\lambda \in M.$$

Therefore

$$\|u_\lambda(t)\|_{L^2}^2 = \int_a^{+\infty} \|e^{i\lambda_i J(t-a)} f_\lambda\|_H^2 dt = \int_a^{+\infty} (e^{i\lambda_i J(t-a)} f_\lambda, e^{i\lambda_i J(t-a)} f_\lambda)_H dt$$

$$= \int_a^{+\infty} (e^{i\lambda_i J(t-a) - i\lambda_i J(t-a)} f_\lambda, f_\lambda)_H dt = \int_a^{+\infty} \|f_\lambda\|_H^2 dt = (+\infty)\|f_\lambda\|_H.$$

In order that $u_\lambda \neq 0$ and $u_\lambda \in W_2^1(H, (a, +\infty))$ it is necessary and sufficient that $f_\lambda \neq 0$ and $f_\lambda \in M \subset H$. But in the case $f_\lambda \neq 0$, $u_\lambda(t) \notin W_2^1(H, (a, +\infty))$. This means that $\sigma_p(L_M) = \phi$. □

In general the following result is true.

Theorem 3.2. *For any normal extension L_M of the minimal operator L_0 in the Hilbert space $L^2(H, (a, +\infty))$ the following inclusion*

$$\sigma(L_M) \subset \{\lambda = \lambda_r + i\lambda_i \in \mathbb{C} : \quad 0 \leq \lambda_r \leq \|A\|, \quad \lambda_i \in \mathbb{R}\}$$

holds.

Proof. Indeed in this case for any for every $u \in D(L_M)$ we have

$$2Re(L_M u, u) = (L_M u, u) + (u, L_M u) = (Ju' + Au, u) + (u, Ju' + Au)$$

$$= (Ju, u)' + 2(Au, u) = 2(Au, u).$$

The last relation implies that

$$0 \leq Re(L_M u, u) \leq \|A\|(u, u).$$

Since the residual spectrum of any normal operators in any Hilbert space is empty, it suffices to investigate the continuous spectrum of the normal extensions L_M of the minimal operator L_0 in the Hilbert space $L^2(H, (a, +\infty))$. □

Theorem 3.3. *For any normal extension L_M of the minimal operator L_0 in the Hilbert space $L^2(H, (a, +\infty))$ the relation $\sigma_c(L_M) \subset \sigma(A) + i\mathbb{R}$ holds.*

Proof. It is known that for the spectrum of a normal operator the following relation is true

$$\sigma(L_M) \subset \sigma(ReL_M) + i\sigma(ImL_M),$$

where $ReL_M = \frac{1}{2}\overline{L_M + L_M^*}$ and $ImL_M = \overline{\frac{1}{2i}(L_M - L_M^*)}$ are selfadjoint operators. Now consider the selfadjoint operator

$$S : L^2(H, (a, +\infty)) \to L^2(H, (a, +\infty)), \quad Su(t) = Au(t).$$

In this case $\frac{1}{2}(L_M + L_M^*) \subset S$ and S is a bounded operator in $L^2(H, (a, +\infty))$. Hence the equality $ReL_M = S$ is obtained. Since the spectrum of S is $\sigma(A \otimes E)$, it follows that $\sigma(ReL_M) = \sigma(A)$ [3]. Consequently $\sigma_c(L_M) \subset \sigma(A) + i\mathbb{R}$. □

Example 1. *For any $\lambda \in \mathbb{C}$ the system of differential equations*

$$\begin{cases} v'(t, x) + e^{-2|x|} u(t, x) = \lambda u(t, x), \\ u'(t, x) + e^{-2|x|} v(t, x) = \lambda v(t, x) \end{cases}$$

with boundary condition

$$u(a, x) = iv(a, x), \quad x \in \mathbb{R}$$

does not have a non-zero solution (u,v) *in the space* $L^2(H,(a,+\infty))$, $H = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, $a \in \mathbb{R}$. *This is a result of Theorem 2.1 and Theorem 2.2 in the case when* $J : H \to H$, E *is a identity operator in* $L^2(\mathbb{R})$,

$$J := \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, A : H \to H, A := \begin{pmatrix} e^{-2|x|}E & 0 \\ 0 & e^{-2|x|}E \end{pmatrix}$$

and

$$M = \{(u(a,x), v(a,x)) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) : \quad u(a,x) = iv(a,x)\}.$$

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