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A NEW CHARACTERIZATION OF SPORADIC HIGMAN-SIMS AND HELD GROUPS

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Key words: element order, sporadic Higman-Sims group, sporadic Held group, Thompson's problem, number of elements of the same order.

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Abstract. Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\text{nse}(G) = \{s_k | k \in \omega(G)\}$. The projective special linear groups $L_3(4)$ and $L_3(5)$ are uniquely determined by nse. In this paper, we prove that if G is a group such that nse(G)=nse(M) where M is a sporadic Higman-Sims or Held group, then $G \cong M$.

1 Introduction

Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\text{nse}(G) = \{s_k | k \in \omega(G)\}$. Let $T(G) = \{(n, s_n) | n \in \omega(G) \text{ and } s_n \in \text{nse}(G)\}$, where s_n is the number of elements with order n. Thompson in 1987 put forward a very interesting problem with respect to algebraic number fields as follows (see [18]).

Thompson's Problem. Suppose that groups G and H are such that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

We see that if G and H are of the same order type, then

$$\text{nse}(G) = \text{nse}(H) \text{ and } |G| = |H|.$$

Let G be a group and M some simple K_i-group, $i = 3, 4$, then $G \cong M$ if and only if $|G| = |M|$ and nse(G)=nse(M) (see [16, 15]). Also the group A_{12} is characterizable by its order and nse (see [12]). Recently it was proved that all sporadic simple groups and projective special linear groups $L_2(2^m)$ with $2^m - 1$ prime or $2^m + 1$ prime, are characterizable by nse and their orders (see [2], [14] respectively).

Related to **Thompson's Problem** is the following one: can nse characterize finite simple groups? Up to now it is known that the projective special linear groups $L_2(q)$, where $q \in \{7, 8, 9, 11, 13, 16\}$ and projective general groups $PGL(2, p)$ can be characterized by only the set nse(G) (see [8, 17, 20], [1] respectively).

In this paper, it is shown that the sporadic Higman-Sims group HS and sporadic Held group He are determined by nse.

We introduce some notations which will be used in the proof of the main theorems. Let $a.b$ denote the products of an integer a by an integer b. Let r be a prime. Then we denote the number of the Sylow r-subgroups P_r of G by n_r or $n_r(G)$. Let $L_n(q)$ and $U_n(q)$ denote the projective special linear and unitary group of degree n over finite fields of order q. Let $S_n(q)$ and $O_n(q)$ denote the projective symplectic and symmetric group, respectively. The group $G_2(q)$ is the algebraic group G_2 over the finite field of order q. The other notations are standard (see [3], for instance).

2 Some lemmas

Lemma 2.1. [4] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \big| g^m = 1\}$, then $m \big| |L_m(G)|$.*

Lemma 2.2. [13] *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Lemma 2.3. [17] *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Lemma 2.4. [6, Theorem 9.3.1] *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:*

(1) $q_i^{\beta_i} \equiv 1$ (mod p_j) for some p_j.

(2) The order of some chief factor of G is divided by $q_i^{\beta_i}$.

To prove $G \cong HS$ or He, we need the structure of simple K_i-groups with $i = 4, 5$.

Lemma 2.5. [19] *Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) A_7, A_8, A_9 or A_{10}.

(2) M_{11}, M_{12} or J_2.

(3) One of the following:

(a) $L_2(r)$, where r is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$, and v is a prime greater than 3.

(b) $L_2(2^m)$, where $2^m - 1 = u$, $2^m + 1 = 3t^b$ with $m \geq 2$, u, t are primes, $t > 3$, $b \geq 1$.

(c) $L_2(3^m)$, where $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$.

(4) One of the following 28 simple groups: $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ *or* ${}^2F_4(2)'$.

Lemma 2.6. [7] *Each simple* K_5*-group is isomorphic to one of the following simple groups:*

(1) $L_2(q)$ *with* $|\pi(q^2-1)| = 4$.

(2) $L_3(q)$ *with* $|\pi((q^2-1)(q^3-1))| = 4$.

(3) $U_3(q)$ *with q satisfies* $|\pi((q^2-1)(q^3+1))| = 4$.

(4) $O_5(q)$ *with* $|\pi(q^4-1)| = 4$.

(5) $Sz(2^{2m+1})$ *with* $|\pi((2^{2m+1}-1)(2^{4m+2}+1))| = 4$.

(6) $R(q)$ *where q is an odd power of 3 and* $|\pi(q^2-1)| = 3$ *and* $|\pi(q^2-q+1)| = 1$.

(7) The following 30 simple groups: A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $O_7(3)$, $O_9(2)$, $PSp_6(3)$, $PSp_8(2)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, $O_8^+(3)$, $O_8^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$ *or* $G_2(9)$.

Lemma 2.7. *Let G be a simple* K_i*-group with i=4, 5 and* $7 \mid |G| \mid 2^{10}.3^3.5^2.7^3.17$. *Then then* $G \cong He$.

Proof. We will prove the Lemma with the following two steps.

Step 1. G is a simple K_4-group.

By Lemma 2.5(1)(2), order consideration rules out this case. So we consider Lemma 2.5(3). We will deal with this with the following cases.

- Case 1. $G \cong L_2(r)$, where $r \in \{3, 5, 7, 17\}$.
  - Let $r = 3$, then $|\pi(r^2-1)| = 1$, which contradicts $|\pi(r^2-1)| = 3$.
  - Let $r = 5, 7, 17$, then $|\pi(r^2-1)| = 2$, which contradicts $|\pi(r^2-1)| = 3$.
- Case 2. $G \cong L_2(2^m)$, where $u \in \{3, 5, 7, 17\}$.
  - Let $u = 3$. Then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in $\mathbb{N}$.
  - Let $u = 5, 17$. Then $2^m - 1 = u$. But the equation has no solution in $\mathbb{N}$.
  - Let $u = 7$. Then $m = 3$ and so $2^3 + 1 = 3t^b$, which means that $t = 3, b = 1$, a contradiction.
- Case 3. $G \cong L_2(3^m)$.

  We will consider the case by the following two subcases.

- Subcase 3.1. $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$.
  We can suppose that $t \in \{3, 5, 7, 17\}$.
  Let $t = 3, 5, 17$, the equation $3^m + 1 = 4t$ has no solution.
  Let $t = 7$, then $m = 3$. Therefore $3^3 - 1 = 2u^c$, we have $u = 13$ and $c = 1$. It follows that $13 \mid |G|$, a contradiction.
- Subcase 3.2. $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.
  We can suppose that $u \in \{3, 5, 7, 17\}$
  Let $u = 3, 5, 7, 11$, then the equation $3^m - 1 = 2u$ has no solution in $\mathbb{N}$, a contradiction.

In review of Lemma 2.5(4), order consideration rules out this case.

Step 2. G is a simple K_5-group.

By Lemma 2.6, we can assume that $q = 2^m$, 3, 3^2, 3^3, 5, 5^2, 7, 7^2, 7^3 or 17.

Let $G \cong L_2(q)$.

- If $q = 2, 3$, then $|\pi(q^2 - 1)| = 1$, a contradiction.
- If $q = 4, 8, 16, 9, 7, 17$, then $|\pi(q^2 - 1)| = 2$, a contradiction.
- If $q = 11, 25, 125, 256$, then $|\pi(q^2 - 1)| = 3$, a contradiction.
- If $q = 32, 128, 27, 49$, then $|\pi(q^2 - 1)| = 3$, a contradiction.
- If $q = 64, 256, 512$, then $|\pi(q^2 - 1)| = 4$. Then $G \cong L_2(64)$, but $13 \mid |L_2(64)|$, a contradiction.
- If $q = 256$, then $|\pi(q^2 - 1)| = 4$. Then $G \cong L_2(256)$, but $257 \mid |L_2(256)|$, a contradiction.
- If $q = 512$, then $|\pi(q^2 - 1)| = 4$. Then $G \cong L_2(512)$, but $59 \mid |L_2(512)|$, a contradiction.
- If $q = 1024$, then $|\pi(q^2 - 1)| = 5$, a contradiction.

Similarly as the proof of "$G \cong L_2(q)$", we can rule out the other cases except for Lemma 2.6(7).

In view of Lemma 2.6(7), by using order consideration, $G \cong He$. □

3 Main theorem and its proof

Let G be a group and s_n be the number of elements of order n. By Lemma 2.3 we have that G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$\begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d \mid m} s_d \end{cases} \tag{3.1}$$

We divide the proof of Main theorem into the following two lemmas.

Lemma 3.1. *Let G be a group with nse(G)=nse(HS)=$\{1$, 21175, 123200, 877800, 2010624, 2956800, 3080000, 3696000, 4435200, 6336000, 8064000, 8316000$\}$. Then $G\cong HS$.*

Proof. We prove the theorem by first proving that $\pi(G) \subseteqq \{2, 3, 5, 7, 11\}$, second showing that $|G| = |HS|$, and so $G \cong HS$.

By (3.1), $\pi(G) \subseteqq \{2, 3, 5, 7, 11, 17, 29, 3696001, 8316001\}$. If $m > 2$, then $\phi(m)$ is even, then s_2=21175, $2 \in \pi(G)$.

If $2.17 \in \omega(G)$, then by Lemma 2.1, $2.17 \mid 1 + s_2 + s_{17} + s_{2.17}$ and so $s_{2.17} \notin$nse(G), a contradiction. Therefore $2.17 \notin \omega(G)$. It follows that the Sylow 17-subgroup of G acts fixed point freely on the set of elements of order 2 and so $17 \mid s_2$, a contradiction. Hence $17 \notin \pi(G)$. Similarly we can prove that $2.29, 2.31, 2.3696001, 2.8316001 \notin \omega(G)$ and $29, 31, 3696001, 8316001 \notin \pi(G)$.

Hence $\pi(G) \subseteqq \{2, 3, 5, 7, 11\}$. By (3.1), $s_3 = 123200$ or 3080000, $s_5 = 2010624$, $s_7 = 6336000$ and $s_{11} = 8064000$.

By (3.1), $\phi(2^m) = 2^{m-1} \mid s_{2^m}$ and so $0 \le m \le 11$. By Lemma 2.1, $|P_2| \mid 1 + s_2 + s_{2^2} + s_{2^i}$ with $i = 2, 3, ...11$, and so $|P_2| \mid 2^{11}$.

If $3^a \in \omega(G)$, then $1 \le a \le 4$.

Let $s_3 = 123200$.

- Let $\exp(P_3) = 3$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3$ and $|P_3| \mid 3^6$.
- Let $\exp(P_3) = 3^2$. Then $|P_3| \mid 1 + s_3 + s_{3^2}$ and $|P_3| \mid 3^3$ (when $s_{3^2} = 8316000$).
- Let $\exp(P_3) = 3^3$. Then $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3}$ and $|P_3| \mid 3^3$.
- Let $\exp(P_3) = 3^4$. Then $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3} + s_{3^4}$ and $|P_3| \mid 3^4$.

Let $s_3 = 3080000$.

- Let $\exp(P_3) = 3$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3$ and $|P_3| \mid 3$.
- Let $\exp(P_3) = 3^2$. Then $|P_3| \mid 1 + s_3 + s_{3^2}$ and $|P_3| \mid 3^3$ (when $s_{3^2} = 3696000$).
- Let $\exp(P_3) = 3^3$. Then $|P_3| \mid 1+s_3+s_{3^2}+s_{3^3}$ and $|P_3| \mid 3^5$ (when $s_{3^2} = 3696000$, $s_{3^3} = 8316000$).
- Let $\exp(P_3) = 3^4$. Then $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3} + s_{3^4}$ and $|P_3| \mid 3^4$.

Therefore $|P_3| \mid 3^6$.

If $2^a.3^b \in \omega(G)$, then $1 \le a \le 10$ and $1 \le b \le 4$.

If $5^a \in \omega(G)$, then $1 \le a \le 5$.

- Let $\exp(P_5) = 5$. Then by Lemma 2.1, $|P_5| \mid 1 + s_5$ and $|P_5| \mid 5^4$.
- Let $\exp(P_5) = 5^2$. Then $|P_5| \mid 1 + s_5 + s_{5^2}$ and $|P_5| \mid 5^5$ (when $s_{5^2} = 3080000$).
- Let $\exp(P_5) = 5^3$. Then $|P_5| \mid 1 + s_5 + s_{5^2} + s_{5^3}$ and $|P_5| \mid 5^5$.
- Let $\exp(P_5) = 5^4$. Then $|P_5| \mid 1 + s_5 + s_{5^2} + s_{5^3} + s_{5^4}$ and $|P_5| \mid 5^5$.

- Let $\exp(P_5) = 5^5$. Then $5^5 \mid 1 + s_5 + s_{5^2} + s_{5^3} + s_{5^4} + s_{5^5}$, a contradiction since $s_{5^5} \in$nse(G).

Therefore $|P_5| \mid 5^5$.

- If $2^a.5 \in \omega(G)$, then $1 \leq a \leq 9$.
- If $3^2.5 \in \omega(G)$, then $\exp(P_3) = 3^2$ and $\exp(P_5) = 5$. Then $|P_3| \mid 3^3$.

  If $|P_3| = 3^2$, then by Lemma 2.3 of [14], $s_{3^2.5} = 4.s_{3^2}.t$ for some integer t. But the equation has no solution.

  If $|P_3| = 3^3$, then $s_{3^2} = 3^3.s$ for some $s \in$nse(G). So we have $s_{3^2} = 8316000, 3696000$. On the other hand, there is an element of order $3^2.5$ which has a centralizer of order larger than or equal to $3^2.5$, then $3 \nmid s_{3^2.5}$ or $3|||s_{3^2.5}$, and hence $s_{3^2.5} = 123200, 877800, 2010624, 2956800, 3080000, 3696000$. But by Lemma 2.1, $3^2.5 \mid 1 + s_3 + s_{3^2} + s_5 + s_{3.5} + s_{3^2.5}$ by some computations, a contradiction.

  Hence $3^2.5 \notin \omega(G)$.

- If $2^a.3^b.5^c$, then $1 \leq a \leq 8$, $1 \leq b \leq 2$ and $1 \leq c \leq 2$.

If $7^a \in \omega(G)$, then $1 \leq a \leq 2$. If $7^2 \in \omega(G)$, then by Lemma 2.1, $7^2 \mid 1 + s_7 + s_{7^2}$ and so $s_{7^2} \notin$nse(G). So $a = 1$. By Lemma 2.1, $|P_7| \mid 1 + s_7$ and so $|P_7| = 7$.

If $5.7 \in \omega(G)$, then by Lemma 2.3 of [14], $s_{5.7} = \phi(5).s_7.t$ for some integer t. But the equation has no solution since $s_{5.7} \in$nse(G).

If $11^a \in \omega(G)$, then $1 \leq a \leq 3$. If $11^3 \in \omega(G)$, then by Lemma 2.1, $11^3 \mid 1 + s_{11} + s_{11^2} + s_{11^3}$ and so $s_{11^3} \notin$nse(G). So $1 \leq a \leq 2$.

- Let $\exp(P_{11})$=11. Then by Lemma 2.1, $|P_{11}| \mid 1 + s_{11}$ and so $|P_{11}|$=11.
- Let $\exp(P_{11})$=121. Then by Lemma 2.1, $|P_{11}| \mid 1 + s_{11} + s_{11^2}$ and so $|P_{11}| \mid 1 + s_{11} + s_{11^2}$ and so $|P_{11}| = 121$.

If $7 \in \pi(G)$, then $\exp(P_7) = 7$. Since $n_7 = s_7/\phi(7)$, then $3, 5, 11 \in \pi(G)$.

If $3.7 \in \omega(G)$, then by Lemma 2.3 of [14], $s_{3.7} = 2.s_7.t$ for some integer t. But the equation has no solution since $s_{3.7} \in$nse(G). Therefore $3.7 \notin \omega(G)$. Similarly $s_{2.7} = s_7$. But by Lemma 2.1, $2.7 \mid 1 + s_2 + s_7 + s_{2.7}$(=12693176), a contradiction. Hence $2.7 \notin \omega(G)$.

If $11 \in \pi(G)$, then $\exp(P_{11})$=11, 121.

- Let $\exp(P_{11})$=11. Since $n_{11} = s_{11}/\phi(11)$, then $3, 5, 7 \in \pi(G)$.
- Let $\exp(P_{11})$=121. Since $n_{11} = s_{11^2}/\phi(11^2)$, then $3, 5, 7 \in \pi(G)$.

If $2.11 \in \omega(G)$, then $s_{2.11} = s_{11}$. By Lemma 2.1, $2.11 \mid 1+s_2+s_{11}+s_{2.11}$(=16149176), a contradiction. Hence $2.11 \notin \omega(G)$. Similarly, $3.11 \notin \omega(G)$, $5.11 \notin \omega(G)$ and $7.11 \notin \omega(G)$.

Therefore if $7 \in \pi(G)$, then $3, 5, 11 \in \pi(G)$; If $11 \in \pi(G)$, then $3, 5, 7 \in \pi(G)$.

In what follows, we prove that $\pi(G)$ could not be $\{2,\}$, $\{2,3\}$, $\{2,5\}$ and $\{2,3,5\}$, and hence $\pi(G)$ must be $\{2,3,5,7,11\}$.

Case a. $\pi(G)=\{2\}$.

Since $|\omega(G)|=12$ and $|\mathrm{nse}(G)|=12$, then this case can be ruled out.

Case b. $\pi(G)=\{2,3\}$.

Let exp(P_3)=3.

By Lemma 2.1, $|P_3| \mid 1+s_3$.

- If $s_3=123200$, then $|P_3| \mid 3^6$.
  - If $|P_3|=3$, then since $n_3=s_3/\phi(3)$, $5,7,11,19 \in \pi(G)$, a contradiction.
  - If $|P_3|=9$, then Therefore $44352000+123200k_1+877800k_2+2010624k_3+2956800k_4+3080000k_5+3696000k_6+4435200k_7+6336000k_8+8064000k_9+8316000k_{10}=2^m.3^2$ where k_1, ..., k_{10} and m are non-negative integers and $0 \le \sum_{i=1}^{10} s_i \le 10$. Since $44352000 \le |G|=2^m.3^2 \le 44352000+10.8316000$. The equation has no solution.
  - Similarly for $|P_3|=27,81,243,729$, we also can rule out these.
- If $s_3=3080000$, then $|P_3| \mid 3$. Since $n_3=s_3/\phi(3)$, then $5,7,11 \in \pi(G)$, a contradiction.

Let exp(P_3)=9.

By Lemma 2.1, $|P_3| \mid 1+s_3+s_9$ and so $|P_3| \mid 3^3$.

- Let $s_3=123200$.
  - If $|P_3|=9$, then since $n_3=s_9/\phi(9)$, 5, 7, or 11$\in \pi(G)$, a contradiction.
  - If $|P_3|=27$, then Therefore $44352000+123200k_1+877800k_2+2010624k_3+2956800k_4+3080000k_5+3696000k_6+4435200k_7+6336000k_8+8064000k_9+8316000k_{10}=2^m.3^3$ where k_1, ..., k_{10} and m are non-negative integers and $0 \le \sum_{i=1}^{10} s_i \le 21$. Since $44352000 \le |G|=2^m.3^3 \le 44352000+21.8316000$. The equation has no solution.
- Let $s_3=3080000$.
  - If $|P_3|=9$, then since $n_3=s_9/\phi(9)=3080000/6$, 5, 7, or 11$\in \pi(G)$, a contradiction.
  - If $|P_3|=27$, then similarly as "$s_3=123200$ and $|P_3|=3^3$", we can rule out this case.

Let exp(P_3)=27.

- Let $s_3=123200$. Then $|P_3| \mid 3^3$. Since $n_3=s_{3^3}/\phi(3^3)$, $5,7,11 \in \pi(G)$, a contradiction.
- Let $s_3=3080000$. Then $|P_3| \mid 3^5$.

- If $|P_3| = 3^3$, then since $n_3 = s_{3^3}/\phi(3^3) = 3080000/6$, 5, 7, 11$\in \pi(G)$, a contradiction.
- If $|P_3| = 3^4$, then Therefore $44352000 + 123200k_1 + 877800k_2 + 2010624k_3 + 2956800k_4 + 3080000k_5 + 3696000k_6 + 4435200k_7 + 6336000k_8 + 8064000k_9 + 8316000k_{10} = 2^m.3^4$ where k_1, ..., k_{10} and m are non-negative integers and $0 \leq \sum_{i=1}^{10} s_i \leq 31$. Since $44352000 \leq |G| = 2^m.3^3 \leq 44352000 + 31.8316000$. The equation has no solution.
- Similarly we can rule out the case when $|P_3| = 3^5$ as the case "$s_3 = 3080000$ and $|P_3| = 3^4$"

Let $\exp(P_3) = 81$.

We know that $|P_3| \mid 3^4$. Since $s_{3^4} = 8316000$ and $n_3 = s_{3^4}/\phi(3^4)$, then $5, 7, 11 \in \pi(G)$, a contradiction.

Case c. $\pi(G) = \{2, 5\}$.

Let $\exp(P_5)$=5. Then $|P_5| \mid 5^4$.

- If $|P_5| = 5$, then since $n_5 = s_5/\phi(5)$, $3, 7, 11 \in \pi(G)$, a contradiction.
- If $|P_5| = 25$, then $44352000 + 123200k_1 + 877800k_2 + 2010624k_3 + 2956800k_4 + 3080000k_5 + 3696000k_6 + 4435200k_7 + 6336000k_8 + 8064000k_9 + 8316000k_{10} = 2^m.5^2$ where k_1, ..., k_{10}, m are non-negative integers and $0 \leq \sum_{i=1}^{10} s_i \leq 12$. The equation has no solution in $\mathbb{N}$.
- If $|P_5| > 25$, similarly, we can rule out these cases as the case "$\exp(P_5) = 5$ and $|P_5| = 5^2$".

Let $\exp(P_5)$=25. Then $|P_5| \mid 5^5$.

- If $|P_5| = 25$, then since $n_5 = s_{25}/\phi(25)$, 3, 7 or 11$\in \pi(G)$, a contradiction.
- If $|P_5| = 125$, then $44352000 + 123200k_1 + 877800k_2 + 2010624k_3 + 2956800k_4 + 3080000k_5 + 3696000k_6 + 4435200k_7 + 6336000k_8 + 8064000k_9 + 8316000k_{10} = 2^m.3^n.5^3$ where k_1, ..., k_{10}, m and n are non-negative integers and $0 \leq \sum_{i=1}^{10} s_i \leq 21$. The equation has no solution in N.

Let $\exp(P_5)$=125. Then since $n_5 = s_{125}/\phi(125)$, 3, 7 or 11$\in \pi(G)$, a contradiction.

Case d. $\pi(G) = \{2, 3, 5\}$.

Similarly as the case "$\pi(G) = \{2, 5\}$", we also can rule out this case.

Case e. $\pi(G) = \{2, 3, 5, 7, 11\}$.

If $7.11 \in \omega(G)$, then by Lemma 2.3 of [14], $s_{7.11} = 6.s_{11}.t$ for some integer t. But the equation has no solution since $s_{7.11} \in$nse(G). So $7.11 \notin \omega(G)$, it follows that the Sylow 7-subgroup of G acts fixed point freely on the set of elements of order 11 and so $|P_7| \mid s_{11}$. Hence $|P_7| = 7$, and $|P_{11}| = 11$. Similarly $2.7 \notin \omega(G)$ and $|P_2| \mid 2^9$; $3.7 \notin \omega(G)$ and $|P_3| \mid 3^2$; $5.7 \notin \omega(G)$ and $|P_5| \mid 5^3$.

So we can assume that $|G| = 2^a.3^b.5^c.7.11$. Since $2^9.3^2.5^3.7.11 \leq |G| = 2^a.3^b.5^c.7.11$, then we have that $|G| = 2^9.3^2.5^3.7.11$. By assumption nse(G)=nse(HS), then by [2], $G \cong HS$. □

Lemma 3.2. *Let G be a group with nse(G)=nse(He)={1, 212415, 8529920, 13434624, 47980800, 93024000, 201519360, 223910400, 251899200, 268692480, 287884800, 474163200, 671731200, 719712000, 767692800}. Then $G\cong He$.*

Proof. We prove the theorem by first showing that $\pi(G) \subsetneqq \{2, 3, 5, 7, 17\}$, then proving that $G \cong He$.

By (3.1), $\pi(G) \subsetneqq \{2, 3, 5, 7, 11, 13, 17, 19, 151, 257, 281, 251899201, 767692801\}$. If $2.13 \in \omega(G)$, then by Lemma 2.1, $2.13 \mid 1+s_2+s_{13}+s_{2.13}$ and so we get a contradiction. So $2.13 \notin \omega(G)$. It follows that the Sylow 13-subgroup of G acts fixed point freely on the set of order 2 and so $|P_{13}| \mid s_2$, a contradiction. Thus $13 \notin \pi(G)$. Similarly 257, 281, 251899201, 767692801$\notin \pi(G)$.

If $2^a \in \omega(G)$, then by Lemma 2.1, $\phi(2^a) = 2^{a-1} \mid s_{2^a}$ and so $0 \leq a \leq 13$. By Lemma 2.1, $|P_2| \mid 1 + s_2 + s_{2^2} + ... + s_{2^i}$ with $i = 2, 3, ..., 13$, and $|P_2| \mid 2^{13}$.

If $3^a \in \omega(G)$, then $1 \leq a \leq 4$.

- Let $\exp(P_3) = 3$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3$ and $|P_3| \mid 3^3$.
- Let $\exp(P_3) = 3^2$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_{3^2}$ and $|P_3| \mid 3^4$.
- Let $\exp(P_3) = 3^3$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3}$ and $|P_3| \mid 3^5$ (when either $s_9 = 13434624$, $s_{27} = 671731200$ or $s_9 = 93024000$, $s_{27} = 767692800$ or $s_9 = s_{27} = 251899200$).
- Let $\exp(P_3) = 3^4$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3} + s_{3^4}$ and $|P_3| \mid 3^7$ (when $s_9 = 474163200$, $s_{27} = s_{81} = 719712000$).

Therefore $|P_3| \mid 3^7$.

If $2^a.3^b \in \omega(G)$, then $1 \leq a \leq 12$ and $1 \leq b \leq 4$.

If $5^a \in \omega(G)$, then $1 \leq a \leq 4$.

- Let $\exp(P_5) = 5$. Then by Lemma 2.1, $|P_5| \mid 1 + s_5$ and $|P_5| \mid 5^3$.
- Let $\exp(P_5) = 5^2$. Then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{5^2}$ and $|P_5| \mid 5^3$ (when $s_{5^2} = 93024000$).
- Let $\exp(P_5) = 5^3$. Then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{5^2} + s_{5^3}$ and $|P_5| \mid 5^4$ (when $s_{5^2} = 93024000$, $s_{5^3} = 719712000$, or $s_{5^2} = 287884800$, $s_{5^3} = 671731200$, or $s_{5^2} = 474163200$ $J\neg$ $s_{5^3} = 767692800$).
- Let $\exp(P_5) = 5^4$. Then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{5^2} + s_{5^3} + s_{5^4}$ and $|P_5| \mid 5^5$ (when s_{5^2} =47980800, s_{5^3} =474163200, s_{5^4} =719712000).

Therefore $|P_5| \mid 5^5$.

If $2^a.5^b$, then $1 \leq a \leq 11$ and $1 \leq b \leq 4$. If $3^a.5^b \in \omega(G)$, then $1 \leq a \leq 4$ and $1 \leq b \leq 4$. If $2^a.3^b.5^c$, then $1 \leq a \leq 10$, $1 \leq b \leq 4$ and $1 \leq c \leq 4$.

If $7^a \in \omega(G)$, then $1 \leq a \leq 4$.

- Let $\exp(P_7) = 7$. Then by Lemma 2.1, $|P_7| \mid 1 + s_7$ and $|P_7| \mid 7^3$.
- Let $\exp(P_7) = 7^2$. Then by Lemma 2.1, $|P_7| \mid 1 + s_7 + s_{7^2}$ and $|P_7| \mid 7^4$ (when $s_{7^2} = 223910400$).

- Let $\exp(P_7) = 7^3$. Then by Lemma 2.1, $|P_7| \mid 1+s_7+s_{7^2}+s_{7^3}$ and $|P_7| \mid 7^4$ (when either $s_{7^2} = 13434624$, $s_{7^3} = 474163200$ or $s_{7^2} = 201519360$, $s_{7^3} = 474163200$ or $s_{7^2} = 474163200$, $s_{7^3} = 671731200$).

- Let $\exp(P_7) = 7^4$. Then by Lemma 2.1, $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3}$ and $|P_7| \mid 7^5$ (when $s_{7^2} = s_{7^3} = 201519360$, $s_{7^4} = 251899200$).

If $5.7 \in \omega(G)$, then by Lemma 2.1, $5.7 \mid 1 + s_5 + s_7 + s_{5.7}$, a contradiction. So $5.7 \notin \omega(G)$. If $2^a.7^b \in \omega(G)$, then $1 \leq a \leq 12$ and $1 \leq b \leq 4$. If $3^2.7 \in \omega(G)$, then by Lemma 2.1, $3^2.7 \mid 1 + s_3 + s_{3^2} + s_7 + s_{3.7} + s_{3^2.7}(=1112088321)$, a contradiction. Therefore $3^2.7 \notin \omega(G)$

If $11^a \in \omega(G)$, then $a = 1$. By Lemma 2.1, $|P_{11}| \mid 1 + s_{11}$ and $|P_{11}| \mid 11$.

If $17^a \in \omega(G)$, then $1 \leq a \leq 3$. If $a = 2$, then by (3.1), $17^2 \mid 1 + s_{17} + s_{17^2}$, a contradiction. So $a = 1$, by Lemma 2.1, $|P_{17}| \mid 1 + s_{17}$ and $|P_{17}| \mid 17$.

If $19^a \in \omega(G)$, then $1 \leq a \leq 2$. If $a = 2$, then by (3.1), $19^2 \mid 1 + s_{19} + s_{19^2}$, a contradiction. So $a = 1$, by Lemma 2.1, $|P_{19}| \mid 1 + s_{19}$ and $|P_{19}| \mid 19$.

If $151^a \in \omega(G)$, then $a = 1$. By Lemma 2.1, $|P_{151}| \mid 1 + s_{151}$ and $|P_{151}| \mid 151$.

To remove the primes 11, 19 and 151, we prove that $17 \in \pi(G)$.

Suppose that $17 \notin \pi(G)$.

If $3, 5, 7, 11, 17, 19, 151 \notin \pi(G)$, then G is a 2-group. Since $|\omega(G)| = 14$ and $|\mathrm{nse}(G)| = 15$, then the equation has no solution.

Let $151 \in \pi(G)$. Since $n_{151} = s_{151}/\phi(151)$, then $17 \in \pi(G)$, a contradiction.

Let $19 \in \pi(G)$. Since $n_{19} = s_{19}/\phi(19)$, then $17 \in \pi(G)$, a contradiction.

Let $11 \in \pi(G)$. Since $n_{11} = s_{11}/\phi(11)$, then $17 \in \pi(G)$, a contradiction.

Let $7 \in \pi(G)$. Then we know that $\exp(P_7) = 7, 7^2, 7^3, 7^4$.

Let $\exp(P_7) = 7$. Then $|P_7| \mid 1 + s_7$ and so $|P_7| \mid 7^3$.

- If $|P_7| = 7$, then $n_7 = s_7/\phi(7)$, and so $17 \in \pi(G)$, a contradiction.

- If $|P_7| = 7^2$, then from above $151, 19, 17, 11 \notin \pi(G)$ and since $\pi(G) \subseteqq \{2, 3, 5, 7, 11, 17, 19, 151\}$, we can assume that $\pi(G) \subseteqq \{2, 3, 5, 7\}$. So $4030387200+8529920k_1+13434624k_2+47980800k_3+93024000k_4+201519360k_5+223910400k_6 + 251899200k_7 + 268692480k_8 + 287884800k_9 + 474163200k_{10} + 671731200k_{11} + 719712000k_{12} + 767692800k_{13} = 2^m.3^n.5^p.7^2$, where k_1, ..., k_{13}, m, n, p are non-negative integers and $0 \leq \sum_{i=1}^{13} k_i \leq 310$. Since $4030387200 \leq |G| \leq 4030387200 + 310.767692800$, then the equation has no solution in $\mathbb{N}$.

- If $|P_7| = 7^3$, then we similarly have rule out this case as "$\exp(P_7) = 7$ and $|P_7| = 7^2$".

Let $\exp(P_7) = 7^2$. Then $|P_7| \mid 7^4$.

- If $|P_7| = 7^2$ or 7^3, we can rule out these cases as "$\exp(P_7) = 7$ and $|P_7| = 7^2$".

- If $|P_7 = 7^4|$, then $4030387200 + 8529920k_1 + 13434624k_2 + 47980800k_3 + 93024000k_4 + 201519360k_5 + 223910400k_6 + 251899200k_7 + 268692480k_8 + 287884800k_9 + 474163200k_{10} + 671731200k_{11} + 719712000k_{12} + 767692800k_{13} =$

$2^m.3^n.5^p.7^4$, where k_1, ..., k_{13}, m, n, p are non-negative integers and $0 \le \sum_{i=1}^{13} k_i \le 322$. Since $4030387200 \le |G| \le 4030387200 + 322.767692800$, then we have that (m, n, p)=(9, 3, 3), (10, 3, 3), (11, 3, 3), (11, 3, 3), (12, 3, 3), (13, 3, 3), (12, 3, 2), (13, 3, 2), (11, 2, 3), (12, 2, 3), (13, 2, 3) or (13, 1, 3). We also can rule out this case. For example. Let $|G| = 2^9.3^3.5^3.7^4$, then using the programme of [8], we can rule out.

Let $\exp(P_7) = 7^3$. Then $|P_7| \mid 7^4$.

- If $|P_7| = 7^3$, then the equation " $4030387200 + 8529920k_1 + 13434624k_2 + 47980800k_3 + 93024000k_4 + 201519360k_5 + 223910400k_6 + 251899200k_7 + 268692480k_8 + 287884800k_9 + 474163200k_{10} + 671731200k_{11} + 719712000k_{12} + 767692800k_{13} = 2^m.3^n.5^p.7^4$ " has no solution from case "$\exp(P_7) = 7$ and $|P_7| = 7^3$".
- If $|P_7| = 7^4$, then $4030387200 + 8529920k_1 + 13434624k_2 + 47980800k_3 + 93024000k_4 + 201519360k_5 + 223910400k_6 + 251899200k_7 + 268692480k_8 + 287884800k_9 + 474163200k_{10} + 671731200k_{11} + 719712000k_{12} + 767692800k_{13} = 2^m.3^n.5^p.7^4$, where k_1, ..., k_{13}, m, n, p are non-negative integers and $0 \le \sum_{i=1}^{13} k_i \le 334$. Since $4030387200 \le |G| \le 4030387200 + 334.767692800$, then we can rule out this case as "$\exp(P_7) = 7^2$ and $|P_7| = 7^4$".

Let $\exp(P_7) = 7^4$. Then $|P_7| \mid 7^5$.

- If $|P_7| = 7^4$, then $4030387200 + 8529920k_1 + 13434624k_2 + 47980800k_3 + 93024000k_4 + 201519360k_5 + 223910400k_6 + 251899200k_7 + 268692480k_8 + 287884800k_9 + 474163200k_{10} + 671731200k_{11} + 719712000k_{12} + 767692800k_{13} = 2^m.3^n.5^p.7^4$, where k_1, ..., k_{13}, m, n, p are non-negative integers and $0 \le \sum_{i=1}^{13} k_i \le 346$. Since $4030387200 \le |G| \le 4030387200 + 346.767692800$, then we can rule out this case as "$\exp(P_7) = 7^2$ and $|P_7| = 7^4$".
- If $|P_7| = 7^5$, then by Lemma 2.2, $s_{7^4} = 7^4.t$ for some integer t. But the equation has no solution since $s_{7^4} \in$nse(G).

Let $5 \in \pi(G)$. We know that $\exp(P_5) = 5, 5^2, 5^3, 5^4$.
Let $\exp(P_5) = 5$. Then $|P_5| \mid 5^3$.

- If $|P_5| = 5$, then since $n_5 = s_5/\phi(5)$, $17 \in \pi(G)$, a contradiction.
- If $|P_5| = 5^2$, then $4030387200 + 8529920k_1 + 13434624k_2 + 47980800k_3 + 93024000k_4 + 201519360k_5 + 223910400k_6 + 251899200k_7 + 268692480k_8 + 287884800k_9 + 474163200k_{10} + 671731200k_{11} + 719712000k_{12} + 767692800k_{13} = 2^m.3^n.5^p.7^4$, where k_1, ..., k_{13}, m, n, p are non-negative integers and $0 \le \sum_{i=1}^{13} k_i \le 107$. Since $4030387200 \le |G| \le 4030387200 + 107.767692800$, then the equation has no solution.

- If $|P_5| = 5^3$, then similarly as the case "$\exp(P_5) = 5$ and $|P_5| = 5^2$", we can rule out this case.

Let $\exp(P_5) = 5^2$. Then $|P_5| \mid 5^3$.
Similarly we can rule out this case as the case "$\exp(P_5) = 5$ and $|P_5| = 5^2$".
Let $\exp(P_5) = 5^3$. Then $|P_5| \mid 5^4$.
Similarly we can rule out this case as the case "$\exp(P_5) = 5$ and $|P_5| = 5^2$".
Let $\exp(P_5) = 5^4$. Then $|P_5| \mid 5^5$.
If $|P_5| = 5^4$, we can rule out this case as "$\exp(P_7) = 7^2$ and $|P_7| = 7^4$".
If $|P_5| = 5^5$, then Lemma 2.2, $s_{5^4} = 5^4.t$ for some integer t, but the equation has no solution since $s_{5^4} \in$nse(G).

Let $3 \in \pi(G)$. We know that $2 \in \pi(G)$.

Therefore $4030387200 + 8529920k_1 + 13434624k_2 + 47980800k_3 + 93024000k_4 + 201519360k_5 + 223910400k_6 + 251899200k_7 + 268692480k_8 + 287884800k_9 + 474163200k_{10} + 671731200k_{11} + 719712000k_{12} + 767692800k_{13} = 2^m.3^n$, where k_1, ..., k_{13}, m, n are non-negative integers and $0 \leq \sum_{i=1}^{13} k_i \leq 51$. Since $4030387200 \leq |G| \leq 4030387200 + 51.767692800$, then the equation has no solution since m and n are at most 13 and 7 respectively.

Therefore $17 \in \pi(G)$.

If $11.17 \in \omega(G)$, then by Lemma 2.3 of [14], $s_{11.17} = 10.s_{17}.t$ for some integer t. But the equation has no solution since $s_{11.17} \in$nse(G). Hence $11.17 \notin \omega(G)$. It following that the Sylow 11-subgroup of G acts fixed point freely on the set of elements of order 17 and $|P_{11}| \mid s_{17}$, a contradiction. Similarly we can prove that $19.17, 151.17 \notin \omega(G)$ and $19, 151 \notin \pi(G)$.

Therefore $\{2, 17\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 17\}$. So we consider the following cases.

Case a. $\pi(G) = \{2, 17\}$.

Since $|P_{17}| = 17$, then $n_{17} = s_{17}/\phi(17) = 2^7.3^3.5^2.7^3$, $3, 5, 7 \in \pi(G)$, a contradiction.

Similarly we can rule out these cases: $\pi(G) = \{2, 3, 17\}, \{2, 5, 17\}, \{2, 7, 17\}, \{2, 3, 5, 17\}, \{2, 3, 7, 17\}, \{2, 5, 7, 17\}$.

Case b. $\pi(G) = \{2, 3, 5, 7, 17\}$.

If $2.17 \in \omega(G)$, then by Lemma 2.3 of [14], $s_{2.17} = s_{17}.t$ for some integer t and $s_{2.17} = s_{17}$. By Lemma 2.1, $2.17 \mid 1 + s_2 + s_{17} + s_{2.17} (=948538816)$, a contradiction. So $2.17 \in \omega(G)$, it follows that the Sylow 2-subgroup of G acts fixed freely on the set of elements of order 17 and $|P_2| \mid s_{17}$. Therefore $|P_2| \mid 2^{11}$. Similarly $3.17 \notin \omega(G)$ and $|P_3| \mid 3^3$; $5.17 \notin \omega(G)$ and $|P_5| \mid 5^2$; $7.17 \notin \omega(G)$ and $|P_7| \mid 7^3$.

We can assume that $|G| = 2^m.3^n.5^p.7^q.17$. Since $2^{10}.3^3.5^2.7^3.17 \leq |G| = 2^m.3^n.5^p.7^q.17$, then $|G| = 2^{10}.3^3.5^2.7^3.17$ or $|G| = 2^{11}.3^3.5^2.7^3.17$.

In the following, we first prove that there is no group such that $|G| = 2^{10}.3^3.5^2.7^3.17$ and nse(G)=nse(He), then by [8], get the desired result.

G is insoluble. Assume that G is soluble. Since $s_{17} = 474163200$ and $|P_{17}| = 17$, then $n_{17} = s_{17}/\phi(17) = 2^7.3^3.5^2.7^3$. By Lemma 2.4, $5^2 \equiv 1 \pmod{17}$, a contradiction. Therefore G is insoluble.

Therefore there is a normal series $1 \trianglelefteq K \trianglelefteq L \trianglelefteq G$ such that L/K is a simple K_i-group with $i = 3, 4, 5$.

If L/K is isomorphic to a simple K_3-group, from [5], $L/K \cong A_5$, A_6, A_7, A_8, $L_2(7)$, $L_2(8)$, $L_2(49)$, $U_3(3)$, $L_3(4)$ or J_2. We can prove L/K is not a simple K_3-group. For example, $L/K \cong L_2(7)$. Then $|G/L| \mid 2^8.3^2.5^2.7^2.17$. Let $A/K := C_{G/K}(L/K)$. Then $A/K \cap L/K = 1$. It is easy to see that $(G/K)/(A/K) \lesssim \mathrm{Aut}(L/K) = SL(2,7)$ and so $G/A \lesssim SL(2,7)$. Since $A/K, L/K \lhd G/K$, $A/K \times L/K \leq G/K$. Therefore $|L/K| \mid |G/A|$ and so $G/K \cong L_2(7)$ or $SL(2,7)$. i.e., $|A| = 2^8.3^2.5^2.7^2.17$ or $2^7.3^2.5^2.7^2.17$. By Sylow's theorem, $n_{17}(A)$=1, 18, 35, 120, 256, 392, 630, 800, 1225, 1344, 4200, 7056, 8960, 14400, 22050, 47040, 313600. Since $A \lhd G$, we have that $n_{17}(A) = n_{17}(G)$, and so $s_{17}(G) = 16.n_{17}(A)$=16, 288, 560, 1920, 4096, 6272, 10080, 12800, 19600, 21504, 67200, 112896, 143360, 230400, 352800, 752640, 5017600, but none of which belongs to nse(G).

Hence G is isomorphic to a simple K_i-group with $i = 4, 5$, then by Lemma 2.7, So $G/A \leq \mathrm{Aut}(He)$. Therefore $G/A \cong He$, or $G/A \cong 2.He$.

If $G/A \cong He$, then order consideration $|A| = 2$ and $A = Z(G)$. So there exists an element of order 2.17, which is a contradiction.

If $G/A \cong 2.He$, then $A = 1$. But nse($2.He$) $\neq$nse(G), a contradiction.

Therefore $|G| = 2^{10}.3^3.5^2.7^3.17 = |He|$. By assumption, nse($G$)=nse($He$), so by [2], $G \cong He$. □

Theorem 3.1. *Let G be a group. Then $G \cong M$ if and only if nse(G)=nse(M) with $M = HS$ or He.*

Proof. By Lemma 3.1 and Lemma 3.2, we have the desired result. □

4 Some applications

We know that if the two groups G and H are of the same order type, then

$$\mathrm{nse}(G)=\mathrm{nse}(H) \text{ and } |G| = |H|.$$

Whether can the conditions "nse(G)=nse(H) and $|G| = |H|$" characterize some finite simple groups. Recently, some simple groups for instance, simple K_n-groups with $n = 3, 4$[16, 15], projective linear groups $L_3(4)$[10] and $L_3(5)$[9] and $L_2(2^m)$ with 2^m+1 prime or $2^m - 1$ prime[14], Projective special unitary group $U_3(5)$[11], are characterizable by nse and their orders. Hence we have the following corollary.

Corollary 4.1. *Let G be a group. Then $G \cong M$ if and only if nse(G)=nse(M) and nse(G)=nse(M) with $M = HS$ or He.*

Proof. See [15]. □

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