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TRICOMI PROBLEM FOR AN ELLIPTIC-HYPERBOLIC EQUATION OF THE SECOND KIND

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Key words: Tricomi problem, elliptic-hyperbolic equation of the second type, generalized solutions of the class R_2.

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Abstract. Unique solvability of the Tricomi problem for an elliptic-hyperbolic equation of the second kind is proved with the help of the representation of the generalized solution to a hyperbolic equation with strong degeneration.

1 Introduction

The line $y = 0$ is a line of the parabolic degeneration for the equations

$$y^m U_{xx} + U_{yy} + aU_x + bU_y + cU = 0, \quad m > 0 \tag{1.1}$$

$$U_{xx} + y^m U_{yy} + aU_x + bU_y + cU = 0, \quad m > 0 \tag{1.2}$$

where the coefficients are given continuous functions in the upper half-plane of the variables x, y [1].

Let D be a domain bounded by the segment Γ_0 of the axis $y = 0$ and the arc Γ_1 in the half-plane $y > 0$. In the domain D, both (1.1) and (1.2) are equations of the elliptic type. For equation (1.2) in the domain D, the Dirichlet problem in usual setting (i.e. when we look for a solution of equation (1.2), regular in D, continuous in the closed domain $\overline{D}$ and taking given values on the boundary $\Gamma = \Gamma_0 + \Gamma_1$) is not always well-posed. If one assumes $c \leqslant 0$, this problem is well-posed in the following cases: a) $m < 1$; b) $m = 1$ if $b(x, 0) < 1$; c) $1 < m < 2$ if $b(x, 0) \leqslant 0$; d) $m \geqslant 2$ if $b(x, 0) < 0$.

If the condition $c \leqslant 0$ is valid, and the following additional assumptions hold: a) $b(x, 0) \geqslant 1$ for $m = 1$; b) $b(x, 0) > 0$ for $1 < m < 2$; c) $b(x, 0) \geqslant 0$ for $m \geqslant 2$ the following problem is uniquely solvable: find a solution of (1.2), regular in the domain D, continuous in the closed domain $\overline{D}$ and taking given values only on the arc Γ_1 (M.V. Keldysh), see [2].

Also the problem on existence and uniqueness of a solution of (1.2) bounded in D and satisfying on Γ_1 the condition

$$\frac{\partial U}{\partial N} + AU = \varphi,$$

where N is a given direction forming an acute angle with the interior normal to Γ_1, and , φ are given functions (O.A. Oleinik [12]).

If the Dirichlet problem is ill-posed for equation (1.2) in the domain D, it is natural to replace the condition of boundedness of $\lim_{y\to 0} U(x,y)$ by the condition of boundedness of

$$\lim_{y\to 0} \varphi(x,y)U(x,y),$$

where $\varphi(x,y)$ is a known function, satisfying $\lim_{y\to 0} \varphi(x,y) = 0$. For the equation

$$y^m U_{xx} - U_{yy} + aU_x + bU_y + cU = 0, \qquad m > 0, \tag{1.3}$$

the coefficients of which are given continuous functions of variables x, y, the line of parabolic degeneration is the line $y = 0$. the equation (1.3) is an equation of hyperbolic type in the half-plane $y > 0$.

Let D_1 denote the domain in the half-plane $y > 0$ bounded by the segment of the degeneration line and by the characteristics and of equation (1.3). Many works are devoted to study of the Cauchy problem for equation (1.3) in the domain D_1 with the initial data on

$$U(x,0) = \tau(x), \qquad U_y(x,0) = \nu(x), \tag{1.4}$$

where $\tau(x)$ and $\nu(x)$ are given smooth functions (S. Gellerstedt [7], F.I. Frankl [5], M.H. Protter [14], G. Helwig [8] and others.). For $m > 2$, the Cauchy problem for equation (1.3) with initial data (1.4) is found, generally speaking, ill-posed (see [14]). However, if some additional conditions are valid, for example if the condition

$$\lim_{y\to 0} y^{1-\frac{m}{2}} a(x,y) = 0, \quad m > 0,$$

holds, this problem is well-posed (see [6]).

A.V. Bitsadze has established the ill-posedness of the Cauchy problem for equation (1.3) and for the equation

$$U_{xx} - y^m U_{yy} + aU_x + bU_y + cU = 0, \quad m > 0.$$

He proposed to investigate this problem under the following modified initial data

$$\lim_{y\to 0} \varphi(x,y)U(x,y) = \tau(x), \quad \lim_{y\to 0} \psi(x,y)U_y(x,y) = \nu(x)$$

$$\lim_{y\to 0} \varphi(x,y) = 0, \quad \lim_{y\to 0} \psi(x,y) = 0$$

and under incomplete initial data (i.e. under absence of one of conditions (1.4)). In [4], it is proved that the Cauchy - Goursat problem for the equation (1.3) has a unique solution. In [3], A.V. Bitsadze suggested to study the Cauchy problem with the modified initial data for the equation

$$y^m U_{yy} - U_{xx} + a(x,y)U_x + b(x,y)U_y + c(x,y)U = 0, \quad 0 < m < 2, \quad y > 0 \tag{1.5}$$

essentially different from (1.3).

The Cauchy problem with modified initial data is studied in details for equation (1.5) in works [9], [16]. In particular, in [16] the equation

$$L_\alpha U \equiv yU_{yy} + U_{xx} + \alpha U_y = 0, \tag{1.6}$$

is considered in the domain D_2 in the half-plane $y < 0$ bounded by characteristics of equation (1.6)

$$AC : x - 2\sqrt{-y} = 0, \qquad BC : x + 2\sqrt{-y} = 1, \quad AB : y = 0.$$

All negative values of α, except integers, are considered. In D_2, the following modified Cauchy problem with the initial data on the degeneration line is correct:

$$U_\alpha(x, 0) = \tau(x), \tag{1.7}$$

$$\lim_{y\to -0}(-y)^\alpha [U_\alpha - A_n^-(\tau)]'_y = \nu(x). \tag{1.8}$$

Here τ, ν are given functions, $A_n^-(\tau)-$ is a given operator (it will be defined below), moreover $\tau \in C^{(2(n+1))}[0;1]$, $\nu \in C^{(2)}[0;1]$. The solution of this problem is defined in the characteristic variables by the formula

$$U_\alpha(\xi,\eta) = \gamma_1 \sum_{k=0}^{n} N_k(\alpha, n, \delta)(\eta-\xi)^{-2\delta-1}4^{-2k}\int_\xi^\eta \tau^{(2k)}(\lambda)(\lambda-\xi)^{k+\delta}(\eta-\lambda)^{k+\delta}d\lambda$$

$$-(-1)^n\gamma_2 4^{2(\alpha-1)}\int_\xi^\eta \nu(\lambda)(\lambda-\xi)^{1/2-\alpha}(\eta-\lambda)^{1/2-\alpha}d\lambda$$

$$\equiv A_n^-(\tau) - (-1)^n\gamma_2 4^{2(\alpha-1)}\int_\xi^\eta \nu(\lambda)(\lambda-\xi)^{1/2-\alpha}(\eta-\lambda)^{1/2-\alpha}d\lambda, \tag{1.9}$$

where

$$\gamma_1 = \frac{\Gamma(2+2\delta)}{\Gamma^2(1+\delta)}, \quad \gamma_2 = \frac{\Gamma(1-2\alpha)}{(1-\alpha)\Gamma^2(1/2-\alpha)},$$

$$N_k(\alpha, n, \delta) = \frac{2^{2k}C_n^k\Gamma(1+\delta)}{\Gamma(1+\delta+k)\prod_{s=0}^{k-1}(\alpha+s)},$$

$$\delta = \alpha + n - \frac{3}{2}, \quad \alpha = -n+\alpha_0, \quad 0<\alpha_0<1/2, \ 1/2<\alpha_0<1, \ n = 0,1,2,...,$$

Definition 1.1. [*16*]. A function $U_\alpha(\xi,\eta)$ is said to be a generalized solution of the Cauchy problem for equation (6) of class R_2 in the domain D_2 if it can be represented in form (9) and

$$\tau(x) = \int_0^x (x-t)^{-2\beta}T(t)dt, \tag{1.10}$$

where ν and T are functions continuous on $[0;1]$, $-2\beta = 2n - 2\delta - 2$.

It should be noted, that in the case of $n = 0$, $0 < \alpha_0 < 1/2$ the representation of the generalized solution of class R_2 was introduced by M.M. Smirnov [16].

In the present work the Tricomi problem is solved with the help of a new representation of generalized solutions of class R_2 obtained by the author [11].

2 Representation of a generalized solution of class R_2

Consider equality (1.10). It follows immediately by it that

$$\tau^{(2k)}(x) = \prod_{l=0}^{2k-1} (2n - 2\delta - 2 - l) \int\limits_0^x T(t)(x-t)^{2n-2\delta-2-2k} dt. \tag{2.1}$$

Substituting (2.1) in (1.9) we have

$$U_\alpha(\xi, \eta) = \gamma_1 (\eta - \xi)^{-2\delta-1} J_1 - J_2, \tag{2.2}$$

where

$$J_1 = \int\limits_0^\xi I_1(\xi;\eta;\zeta)T(\zeta)d\zeta + \int\limits_\xi^\eta I_2(\xi;\eta;\zeta)T(\zeta)d\zeta,$$

$$J_2 = (-1)^n \gamma_2 4^{2(\alpha-1)} \int\limits_\xi^\eta \nu(t)(t-\xi)^{1/2-\alpha}(\eta-t)^{1/2-\alpha} dt,$$

and also

$$I_1(\xi;\eta;\zeta) = \sum_{k=0}^n N_k(\alpha;n;\delta) 4^{-2k} \prod_{l=0}^{2k-1}(2n-2\delta-2-l)$$
$$\times \int\limits_\xi^\eta (t-\xi)^{k+\delta}(\eta-t)^{k+\delta}(t-\zeta)^{2n-2\delta-2-2k} dt \tag{2.3}$$

$$I_2(\xi;\eta;\zeta) = \sum_{k=0}^n N_k(\alpha;n;\delta) 4^{-2k} \prod_{l=0}^{2k-1}(2n-2\delta-2-l)$$
$$\times \int\limits_\xi^\eta (t-\xi)^{k+\delta}(\eta-t)^{k+\delta}(t-\zeta)^{2n-2\delta-2-2k} dt. \tag{2.4}$$

Integrals in (2.3) and (2.4) could be expressed via the hypergeometric functions, namely

$$I_1(\xi;\eta;\zeta) = (\eta-\xi)^{2\delta+1}(\eta-\zeta)^{2n-2\delta-2} \sum_{k=0}^n N_k(\alpha;n;\delta)4^{-2k}(2n-2\delta-2k-1)_{2k}$$
$$\times \frac{\Gamma^2(k+\delta+1)}{\Gamma(2k+2\delta+2)} Z^{2k} F(k+\delta+1, 2\delta+2+2k-2n, 2\delta+2k+2; Z), \quad Z = \frac{\eta-\xi}{\eta-\zeta},$$

$$I_2(\xi;\eta;\zeta)=(\eta-\xi)^{n-\delta-1}(\eta-\zeta)^{n+\delta}\sum_{k=0}^{n}N_k(\alpha;n;\delta)4^{-2k}(2\beta)_{2k}$$
$$\times\frac{\Gamma(k+\delta+1)\Gamma(2n-2\delta-1-2k)}{\Gamma(2n-\delta-k)}$$
$$\times Z_1^{-k+n}F(k+\delta+1,-k-\delta,2n-\delta-k;Z_1),\quad Z_1=\frac{\eta-\zeta}{\eta-\xi},$$

where for $\gamma>0$ and natural m, $(\gamma)_m=\gamma(\gamma+1)...(\gamma+m-1)$.

Lemma 2.1. *The following identity holds:*

$$[2(s-k)+1][2(s-k)+3]...[2(s-k)+(2l-1)]$$
$$=\widehat{P}_l(s)+\widehat{P}_{l-1}(s)k+...+\widehat{P}_0(s)k(k-1)...(k-l+1),\tag{2.5}$$

where $\widehat{P}_i(s), i=\overline{0,l}$, *are certain polynomials of degree* i.

This lemma is easily verified by induction.

Theorem 2.1. *The following identities are valid:*

$$\sum_{k=0}^{n}\frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+2)}N_k(\alpha,n,\delta)4^{-2k}(2\beta)_{2k}Z^{2k}$$
$$\times F(k+\delta+1,2\delta+2+2k-2n,2\delta+2k+2;Z)=(1-Z)^{-\beta},\tag{2.6}$$
$$\sum_{k=0}^{n}\frac{\Gamma(-\delta)}{\Gamma^2(1+\delta)\Gamma(-2\delta-1)}N_k(\alpha,n,\delta)4^{-2k}(2\beta)_{2k}Z_1^{2k}\times$$
$$\times\frac{\Gamma(k+\delta+1)\Gamma(2n-2\delta-2k-1)}{\Gamma(-k-\delta+2n)}Z_1^{-k+n}F(k+\delta+1,-k-\delta,-k-\delta+2n;Z_1)$$
$$=(-1)^n(1-Z_1)^{-\beta},\tag{2.7}$$

Proof. We shall prove (2.6) in full detail. We write down the hypergeometric functions in the left-hand side of (2.6) in the form of series. First, we decompose the right-hand side of (2.6) in power series:

$$\sum_{k=0}^{n}\frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+2)}N_k(\alpha,n,\delta)4^{-2k}(2\beta)_{2k}Z^{2k}$$
$$\times\sum_{m=0}^{\infty}\frac{(k+\delta+1)_m(2\beta+2k)_m}{(2\beta+2n+2k)_m m!}Z^m$$
$$=1+\beta Z+\frac{\beta(\beta+1)}{2!}Z^2+...+\frac{\beta(\beta+1)...(\beta+l-1)}{l!}Z^l+...\tag{2.8}$$

It is not difficult to verify that the coefficients in the left and right members of power series (2.8) at the same degrees Z^l with $[l/2]\leqslant n$ are calculated by the formula

$$\sum_{k=0}^{[l/2]} \frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+2)} N_k(\alpha,n,\delta) 4^{-2k}(2\beta)_{2k} \times \frac{(k+\delta+1)_{l-2k}(2\beta+2k)_{l-2k}}{(2\beta+2n+2k)_{l-2k}(l-2k)!} = \frac{(\beta)_l}{l!} \quad (2.9)$$

Let us show that (2.9) is the identity for any natural n.

For this, we need to consider the following cases.

Case A. $l=2s$. After not complicated transformations (2.9) can be rewritten as

$$\sum_{k=0}^{s} 2^{-k} C_n^k \frac{(\beta)_s(\delta+1+k+s+1)_{s-2k-1}[2\beta+2k]_{s-k}}{[2\delta+2]_s(2s-2k)!} = \frac{(\beta)_{2s}}{(2s)!} \quad (2.10)$$

where $[a]_k=(a+1)(a+3)...(a+2k-1)$.

If, now, we transform (20) and substitute $\delta=\beta+n-1$, we obtain

$$P_{2s}(\beta) \equiv \sum_{k=0}^{s} 2^{-k} C_n^k \frac{(n+\beta+s)_{s-k}[2\beta+2k+1]_{s-k}}{(2s-2k)!} = \frac{(\beta+s)_s[2n+2\beta]_s}{(2s)!} \quad (2.11)$$

The left and right members of (2.11) are polynomials with respect to β. Two polynomials of the same order are equal if the coefficients at the highest degree of β are equal and the values of the polynomials coincide at various points, the number of which is equal to the order of polynomials [15]. The polynomial in the right member equals zero at those values of β for which one of the factors equals zero. Let us next show that for these β the left-hand side of (2.11) also becomes zero. We need to show that for $\beta=-s$

$$P_{2s}(-s) \equiv 0. \quad (2.12)$$

We shall use the following formula [16]

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a-m}{k} \binom{a}{k}^{-1} = 0, \quad m<n. \quad (2.13)$$

Rewrite (2.13) in the form

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a-k)!}{(a-m-k)!} = 0, \quad m<n. \quad (2.14)$$

Set in (2.13) $a=n+s-1, m=a-n$. Since $s-1<s$, the condition $m<n$ holds: $a-k=n+s-1-k$, $(a-k)!=(n+s-k-1)!$, $(a-k-m)!=(n-k)!$ (2.14) follows from (2.12). We need to show that at $\beta=-s-1$

$$P_{2s}(-s-1) \equiv 0. \quad (2.15)$$

Let us show that

$$\sum_{k=0}^{s} (-1)^k C_n^k \frac{(n+s-k-2)!}{(n-k)!} = 0 \quad (2.16)$$

and

$$\sum_{k=0}^{s}(-1)^k C_n^k \frac{(n+s-k-2)!}{(s-k)!}k = 0 \tag{2.17}$$

We obtain (2.15) from (2.18) and (2.19). Set in (2.14) $a = n+s-2$, $m = s-2$. Then $(a-k)! = (n+s-k-2)!$, $(a-m-k)! = (n-k)!$. Since $s-2 < s$, the condition $m < n$ holds and (2.14) implies (2.16). Substitute the summation variable in the left member of (2.16) by the formula $k' = k-1$ and denote $s-1 = s'$. Then (2.17) follows from (2.14) where it suffices 0to set $a = n+s'-2$, $m = s'-1$. It is necessary to show that $P_{2s}(-s-2) \equiv 0$. Represent $P_{2s}(-s-2)$ in the form

$$P_{2s}(-s-2) = \frac{(-1)^s n!}{3(n-3)!2^s}\sum_{k=0}^{s}(-1)^k \frac{(n+s-k-3)![2(s-k)+1][2(s-k)+3]}{k!(n-k)!(s-k)!}$$

We have on the base of Lemma 2.1

$$(-1)^s \frac{3(n-3)!2^s}{n!} P_{2s}(-s-2) = \widehat{P_2}(s)\sigma_{21} + \widehat{P_1}(s)\sigma_{22} + \widehat{P_0}(s)\sigma_{23}$$

where

$$\widehat{P_2}(s) = (2s+1)(2s+3), \widehat{P_1}(s) = -8s-4, \widehat{P_0}(s) = 4,$$

$$\sigma_{21} = \sum_{k=0}^{s}(-1)^k \frac{(n+s-k-3)!}{k!(n-k)!(s-k)!}, \sigma_{22} = \sum_{k=0}^{s}(-1)^k \frac{(n+s-k-3)!}{k!(n-k)!(s-k)!}k,$$

$$\sigma_{23} = \sum_{k=0}^{s}(-1)^k \frac{(n+s-k-3)!}{k!(n-k)!(s-k)!}k(k-1)$$

It is not difficult to show that $\sigma_{21} = \sigma_{22} = \sigma_{23} \equiv 0$. This means that $P_{2s}(-s-2) \equiv 0$. Formula (2.6) can be analogously proved for $\beta = -s-3$, $\beta = -s-4$... The proof for $\beta = -2s+1$ is reduced by application of Lemma 2.1 to the proof of identical vanishing of all coefficients of the polynomials $\widehat{P}_{2s-i}(s), i = \overline{3, 2s}$. We have

$$\sum_{k=0}^{s}(-1)^k \frac{1}{k!(s-k)!} = \frac{1}{s!}\sum_{k=0}^{s}(-1)^k C_s^k = 0,$$

$$\sum_{k=0}^{s}(-1)^k \frac{k}{k!(s-k)!} = -\frac{1}{s'!}\sum_{k=0}^{s'}(-1)^{k'} C_{s'}^{k'} = 0, \quad \sum_{k=0}^{s}(-1)^k \frac{k(k-1)}{k!(s-k)!} = 0, ...,$$

$$\sum_{k=0}^{s}(-1)^k \frac{k(k-1)...(k-s+2)}{k!(s-k)!} = \sum_{k=s-1}^{s}(-1)^k \frac{1}{(k-s+1)!(s-k)!}$$

$$= (-1)^{s-1}\frac{1}{0!(s-s+1)!} + (-1)^s \frac{1}{1!0!} = 0.$$

Let us show that both the right member and the left one of (21) vanish if $2\beta + 2n + 2s - 1 = 0$. Vanishing of the right-hand side is obvious, and the left member takes the form

$$\sum_{k=0}^{s} C_n^k 2^{-k} \frac{\frac{1}{2}\frac{3}{2}...\frac{2s-2k-1}{2}(-2n)(-2n-2)...(-2n-2s+2k+2)}{(2s-2k)!} \equiv 2^s P_{2s}(-s) \equiv 0.$$

We can prove analogously that the left-hand and right-hand sides of (21) vanish identically if $2n+2\beta+2s-3, ..., 2\beta+2n+1=0$.

Case B. Consider now the case of odd $l=2s+1$. Then it is necessary to prove the following identity

$$\sum_{k=0}^{s} \frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+2)} N_k(\alpha,n,\delta) 4^{-2k} \prod_{l=0}^{2k-1}(-2\beta-2s-1)$$
$$\times \frac{(k+\delta+1)_{2s-2k+1}(2\beta+2k)_{2s-2k+1}}{(2\beta+2n+2k)_{2s-2k+1}(2s-2k+1)!} \equiv \frac{\beta(\beta+1)...(\beta+2s)}{(2s+1)!}$$

or

$$\sum_{k=0}^{n} C_n^k \frac{2^{-k}(n+\beta+s+1)_{s-2k+1}[2\beta+2k+1]_{2s-2k-2}}{(2s-2k+1)!}$$
$$= \frac{(\beta+s+1)_{s-1}(2\beta+2n+1)_{2s-2}}{(2s+1)!} \tag{2.18}$$

Formula (2.18) can be proved analogously to (2.11). Investigation of coefficients at the degrees of Z^l with $[l/2]>n$ in (2.3) is reduced to the proof of the identity

$$\sum_{k=0}^{n} \frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+1)} N_k(\alpha,n,\delta) 4^{-2k} \prod_{l=0}^{2k-1}(-2\beta-l)$$
$$\times \frac{(\delta+k+1)_{l-2k}(2\beta+2k)_{l-2k}}{(2\beta+2k+2n)_{l-2k}(l-2k)!} = \frac{(\beta)_l}{l!}. \tag{2.19}$$

Proof of (2.19) is divided into two cases: l even ($l=2s$) and l odd ($l=2s+1$). We use (35) for the proof. For choosing m, one should swap s and n, the further proof does not differ from the case of $[l/2]\leqslant n$. Equality (2.7) is proved by the same method as (2.6). The role of equality (2.14) is played now by the formula [15]

$$\sum_{k=0}^{n}(-1)^k C_n^k C_{a+bk}^m = 0, \quad 0<m<n.$$

□

Some special cases of Theorem 2.1 were obtained in [11]. On the base of Theorem 2.1, let us write the expressions $I_1(\xi,\eta,\zeta)$ and $I_2(\xi,\eta,\zeta)$ in the following forms:

$$I_1(\xi,\eta,\zeta) = \frac{\Gamma^2(\delta+1)}{\Gamma(2\delta+1)}(\eta-\xi)^{2\delta+1}(\eta-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}, \tag{2.20}$$

$$I_2(\xi,\eta,\zeta)=\frac{\Gamma(\delta+1)\Gamma(-2\delta-1)}{\Gamma(-\delta)}(-1)^n(\eta-\xi)^{2\delta+1}(\eta-\zeta)^{-\beta}(\zeta-\xi)^{-\beta}. \tag{2.21}$$

Substituting (2.20), (2.21) in (2.3),(2.4) we obtain the representation of a generalized solution of class R_2 [10]:

$$U(\xi,\eta)=\int_0^{\xi}(\eta-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}T(\zeta)d\zeta+\int_{\xi}^{\eta}(\eta-\zeta)^{-\beta}(\zeta-\xi)^{-\beta}N(\zeta)d\zeta, \tag{2.22}$$

where

$$N(\zeta)=\frac{1}{2\cos\pi\beta}T(\zeta)-(-1)^n 2^{4\beta-2}\gamma_2\nu(\zeta).$$

3 The modified Tricomi problem for an elliptic-hyperbolic equation of the second kind

Let $\Omega=D\cup D_2\cup AB$.

Problem T_n. To find in the domain Ω a function $U(x,y)$ satisfying the following conditions:

a) $U(x,y)\in C(\overline{\Omega})$.

b) $U(x,y)$ is a generalized solution of class R_2 defined below in D_2, twice continuously differentiable and satisfying the equation

$$L(U)\equiv yU_{yy}+U_{xx}+\alpha U_y=0 \tag{3.1}$$

in the domain D;

c) on the degeneration line of equation (3.1), the following sewing condition holds:

$$(-1)^n\lim_{y\to -0}(-y)^{\alpha}[U-A_n^-(\tau)]'_y=\lim_{y\to +0}y^{\alpha}[U'_y+A_n^+(U)]=\nu(x),$$

where $U(x,-0)=U(x,+0)=\tau(x)$ follows from condition a),

$$A_n^-(\tau)=\gamma_1\sum_{k=0}^{n}N_k(n,\alpha,\delta)(-y)^k\int_0^1\tau^{(2k)}(\lambda)[t(1-t)]^{k+\delta}dt,$$

$$\lambda=x-2\sqrt{-y}(1-2t),\quad \gamma_1=\frac{\Gamma(2+2\delta)}{\Gamma^2(1+\delta)},$$

$$\delta=\begin{cases}\alpha_0-3/2 \text{ for } 1/2<\alpha_0<1,\\ \alpha_0-1/2 \text{ for } \ 0<\alpha_0<1/2\end{cases}$$

$$N_k(n,\alpha,\delta)=\frac{2^{2k}C_n^k\Gamma(1+\delta)}{\Gamma(1+\delta+k)\prod\limits_{s=0}^{k-1}(\alpha+s)},$$

$$A_n^+(U) = \sum_{i=1}^{n} \delta_i y^{i-1} \frac{\partial^{2i} U}{\partial x^{2i}}, \quad \delta_i - \text{const};$$

d) $U(x, y)$ satisfies the boundary conditions

$$U\mid_\sigma = f(s), \quad 0 \le s \le l, \tag{3.2}$$

$$U\mid_{AC} = \varphi(x), \quad 0 \le x \le 1/2, \tag{3.3}$$

where f and φ are given continuous functions.

Definition 3.1. A function $U_\alpha(x, y)$, defined by formula (1.9), is called a generalized solution of equation (3.1) of class R_2 in the domain D_2, if $\nu \in C[0, 1]$ and the function τ is represented as

$$\tau(x) = \int_0^x (x - t)^{-2\beta} T(t) dt, \tag{3.4}$$

where $\beta = \left[\frac{3}{2} - \alpha\right]$ and T is some continuous function on [0, 1].

Theorem 3.1. *Problem T_n has a unique solution*

Proof. The uniqueness of the solution can be proved with the help of the maximum principle analogously to work [10].

The existence of the solution will be proved by reducing problem T_n to a singular integral equation.

The equation conjugate to equation (3.1) has the form

$$L^*(v) \equiv y v_{yy} + v_{xx} + (2 - \alpha) v_y = 0. \tag{3.5}$$

The following property of the Green function $G(\xi, t; x, y)$ of Problem T_n were proved in work [10]:

$$G(\xi, t; x, y) = q(\xi, t; x, y) - (4R^2)^{-\beta} q(\xi - 1/2, t; \overline{x}, \overline{y}), \tag{3.6}$$

where

$$\beta = \alpha - 1/2, \quad 4R^2 = (2x - 1)^2 + 16y$$

$$q(\xi, t; x, y) = k t^{\alpha - 1} r_1^{-2\beta} F\left(\beta, \beta, 2\beta; \frac{16\sqrt{yt}}{r_1^2}\right)$$

$$r_1^2 = (\xi - x)^2 + 4(\sqrt{t} - \sqrt{y})^2 \quad \overline{x} = \frac{x - 0.5}{4R^2}, \quad \overline{y} = \frac{y}{(4R^2)^2}.$$

Let $\sigma_0 : x(1 - x) = 4y$. We note also the following properties of the Green function $G(\xi, t; x, y)$:

1) $L_{(x,y)}(G) = 0$ and $L^*_{\xi,t}(G) = 0$ for $(x, y) \ne (\xi, t)$;
2) $G(\xi, t; x, y) = 0$ if $(x, y) \in \sigma_0$ or $(\xi, t) \in \sigma_0$;
3) $\lim\limits_{t \to +0} \left[t\overline{G}_t + (1 - \alpha)\overline{G} + \sum_{i=1}^{n} \delta_i t^i \frac{\partial^{2i} \overline{G}}{\partial \xi^{2i}} \right] = 0,$

where

$$\overline{G}(\xi, t; x) = G(\xi, t; x, 0).$$

Let D_h denote the part of domain D for $y \geq h > 0 : D_h = D \cap \{y \geq h > 0\}$.

The following identity is valid

$$\iint\limits_{D_h} [vL(U) - UL^*(v)]d\xi dt$$

$$= \int\limits_{\partial D_h} [vU_\xi - Uv_\xi]dt - [tvU_t - tUv_t - (1-\alpha)Uv]d\xi. \qquad (3.7)$$

Let us suppose that in (3.7) U is a solution of equation (3.1) in D_h. In the capacity of v we take the Green function for $y = 0$, i. e.

$$v = \overline{G}(\xi, t; x) = kt^{\alpha-1}\left[(\xi - x)^2 + 4t\right]^{-\beta}$$

$$-k(2x-1)^{-2\beta}t^{\alpha-1}\left[\left(\xi - \frac{1}{2} - \frac{1}{2(2x-1)^2}\right)^2 + 4t\right]^{-\beta}.$$

Then (3.7) can be rewritten as

$$\int\limits_{A'B'} U\left[t\overline{G}_t + (1-\alpha)\overline{G} + \sum_{i=1}^{n}\delta_i t^i \frac{\partial^{2i}\overline{G}}{\partial\xi^{2i}}\right]d\xi -$$

$$-\int\limits_{A'B'} t^{1-\alpha}\overline{G}t^\alpha[U_t + A_n^+(U)]d\xi -$$

$$-\sum_{i=1}^{n}\delta_i t^i \sum_{k=0}^{2i-1}(-1)^k \frac{\partial^k U}{\partial\xi^k}\frac{\partial^{2i-1-k}\overline{G}}{\partial\xi^{2i-1-k}}\Bigg|_{A'}^{B'} +$$

$$+\int\limits_{\sigma_0} U\left\{[t\overline{G}_t + (1-\alpha)\overline{G}]d\xi - \overline{G}_\xi dt\right\} = 0. \qquad (3.8)$$

Suppose that

$$\frac{\partial^k U}{\partial\xi^k}\mid_{(0,0)} = \frac{\partial^k U}{\partial\xi^k}\mid_{(1,0)} = 0, \quad k = \overline{0, 2n-1},$$

and

$$f(s) = f(\xi) = [\xi(1-\xi)]^{2n-1}f_1(\xi), \quad f_1 \in C[0,1]. \qquad (41)$$

By passing to the limit as $h \to 0$, we get the main relation between $\tau(x)$ and $\nu(x)$ on D

$$\tau(x) = k\int\limits_0^1 \nu(t)[|t-x|^{-2\beta} - (x + t - 2xt)^{-2\beta}]dt + F(x), \qquad (3.9)$$

where

$$F(x) = 4^{1-\alpha}k\beta x(1-x)\int\limits_0^1 f_1(\xi)[\xi(1-\xi)]^{n+\alpha_0-2}[x + \xi - 2\xi x]^{-\beta-1}d\xi.$$

In the domain D_2 we have to use representation (2.22) of the generalized solution of class R_2.

The generalized solution of equation (3.1) of class R_2 in D_2, as was proved above, is represented in the form

$$U(\xi,\eta)=\int\limits_0^{\xi}(\eta-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}T(\zeta)d\zeta+\int\limits_{\xi}^{\eta}(\eta-\zeta)^{-\beta}(\zeta-\xi)^{-\beta}N(\zeta)d\zeta, \qquad (3.10)$$

where

$$N(\zeta)=\frac{1}{2\cos\pi\beta}T(\zeta)-(-1)^n 2^{4\beta-2}\gamma_2\nu(\zeta).$$

Let $U(x,y)\in R_2(D_2)$. Using boundary conditions (3.3), we find another relation between $\tau(x)$ and $\nu(x)$ on D_2

$$\tau(x)=\gamma_3\int\limits_0^{x}\nu(t)(x-t)^{-2\beta}dt+\Phi(x,\varphi), \qquad (3.11)$$

where

$$\Phi(x,\varphi)=\frac{2\Gamma(1+\beta)}{\Gamma(1+2\beta)}D_{0x}^{2\beta-1}\left[x^{\beta}\varphi(x)\right],$$

$$\gamma_3=(-1)^n 2\cos\beta\pi\gamma_2.$$

Excluding the function τ from (3.10) and (3.12) we get for the function $\nu\in H(\delta)$ the following singular integral equation, which is equivalent to the problem under consideration:

$$\nu(x)-\lambda\int\limits_0^{1}\left(\frac{t}{x}\right)^{1-2\beta}\left[\frac{1}{t-x}-\frac{1}{x+t-2xt}\right]\nu(t)dt=\Psi(x,f,\varphi), \qquad (3.12)$$

here λ is a constant and

$$\Psi(x,f,\varphi)=\frac{1}{\gamma_3\Gamma(1-2\beta)}D_{x0}^{1-2\beta}[F-\Phi].$$

Taking into account conditions (3.9) for the function $\varphi(x)=x^{2n+\varepsilon}\varphi_1(x)$, where $\varepsilon>0$, $\varphi_1\in C[0,\frac{1}{2}]$, it can be shown that $\Psi(x,f,\varphi)\in H(\theta)$, with

$$\theta=\begin{cases}1-2\alpha_0, & 0<\alpha_0<\dfrac{1}{2},\\ 2-2\alpha_0, & \dfrac{1}{2}<\alpha<1.\end{cases}$$

As was shown in [13], for the $n=0$, singular integral equation (3.13) , has a unique solution for $\nu\in H(\theta)$. □

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