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INJECTIVE BANACH LATTICES: A SURVEY

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Key words: AL-space, AM-space, injective Banach lattice, Boolean-valued model, Boolean-valued transfer principle, tensor product, homogeneous Banach lattice.

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Abstract. The work is aimed to survey recent results on injective Banach lattices, outline the Boolean-valued approach, and pose some open problems. The central idea of the investigation is the Boolean-valued transfer principle from AL-spaces to injective Banach lattices: Every injective Banach lattice is embedded into an appropriate Boolean-valued model, becoming an AL-space. To illustrate the method, a concrete description of injective Banach lattices similar to that of AL-spaces is presented and the isometric classification of injective Banach lattices is accomplished.

1 Introduction

The aim of this work is to survey recent results on injective Banach lattices obtained in [38, 39, 40, 41, 42], outline a Boolean-valued approach, and pose some open problems. The central idea of the investigation is a *Boolean-valued transfer principle* from AL-spaces to injective Banach lattices: It was announced in [40] and proved in [38] that every injective Banach lattice is embedded into an appropriate Boolean-valued model, becoming an AL-space. According to this fact and fundamental principles of Boolean-valued models, each theorem about the AL-space within Zermelo–Fraenkel set theory has its counterpart for the original injective Banach lattice interpreted as a Boolean-valued AL-space. To illustrate the method we present a concrete description of injective Banach lattices similar to that of AL-spaces which relays upon Maharam's representation of measure algebras [18, 66] and accomplish the isometric classification of injective Banach lattices.

The first four sections contain preliminary background of Banach lattice theory including the definition and geometric characterizations of injective Banach lattices. In Sections 5–7 we collect some Boolean-valued concepts: Boolean-valued universe, Boolean-valued truth values, principles and functors of Boolean-valued analysis. In Section 8 Gordon's theorem stating that the Boolean-valued interpretation of the reals is a universally complete vector lattice is presented. It is demonstrated in Sections 9 and 10 that an injective Banach lattice admits a Boolean-valued representation which is an AL-space. In Sections 11 and 12 two ways of constructing new injectives are presented: formation of direct sums and tensor products of injective Banach lattices.

Section 13 is dedicated to a well known concrete description of AL-spaces based on Maharam's representation of measure algebras and Kakutani's representation of AL-spaces. Boolean-valued interpretation of such description leads to main representation and classification results presented in Sections 14–16. In Section 17 some open problems are stated.

We let $:=$ denote the assignment by definition, while $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ symbolize the naturals, the reals, and the complexes.

2 Banach lattices

A *Banach lattice* is a Banach space over the reals that is equipped with a partial order $\le$ for which the *supremum* $x \vee y$ and the *infimum* $x \wedge y$ exist for all vectors $x, y \in X$, and such that the positive cone $X_+ := \{x \in X : 0 \le x\}$ is closed under addition and multiplication by non-negative real numbers and the order is connected with the norm by the condition that $|x| \le |y| \Longrightarrow \|x\| \le \|y\|$, where the *absolute value* is defined by $|x| := x \vee (-x)$. A subset $U \subset X$ is called *order bounded* if U is contained in an *order interval* $[a, b] := \{x \in X : a \le x \le b\}$ for some $a, b \in X$. A Banach lattice X is said to be *Dedekind complete* if every non-empty order bounded set in X has the least upper bound $\sup(U) \in X$ and the greatest lower bound $\inf(U) \in X$.

Most of the Banach spaces that appear naturally in analysis (L^p, l^p, $C(K)$, c, c_0 etc.) are Banach lattices. Banach spaces of continuously differentiable functions are not Banach lattices. The Banach lattice $C([0,1])$ is not Dedekind complete.

Assume that a measure space (Ω, Σ, μ) is semi-finite, that is, if $A \in \Sigma$ and $\mu(A) = \infty$ then there exists $B \in \Sigma$ with $B \subset A$ and $0 < \mu(A) < \infty$. The vector lattice $L^0(\Omega, \Sigma, \mu)$ (of μ-equivalence classes) of μ-measurable functions on Ω is Dedekind complete if and only if (Ω, Σ, μ) is localizable. In this event $L^p(\Omega, \Sigma, \mu)$ is also Dedekind complete. (A measure space (Ω, Σ, μ) is *localizable* or *Maharam* if it is semi-finite and, for every $\mathcal{A} \subset \Sigma$, there exists a $B \in \Sigma$ such that (i) $A \setminus B$ is negligible for every $A \in \mathcal{A}$; (ii) if $C \in \Sigma$ and $A \setminus C$ is negligible for every $A \in \mathcal{A}$, then $B \setminus C$ is negligible, see [17, 241G].)

A *band* in a Banach lattice X is a subset of the form $B := A^\perp := \{x \in X : (\forall\, a \in A)\, |x| \wedge |a| = 0\}$ for a nonempty $A \subset X$. Clearly, $B \cap B^\perp = \{0\}$. A band B in X whith satisfies $X = B \oplus B^\perp$ is referred to as a *projection band,* while the associated projection (onto B parallel to $B^\perp$) is called a *band projection.* Let $\mathbb{P}(X)$ stand for the complete Boolean algebra of all band projections in X. Observe that $\mathbb{P}(L^0(\Omega, \Sigma, \mu)) \simeq \Sigma/\mu^{-1}(0)$.

A linear mapping T from a Banach lattice X to a Banach lattice Y is called *positive* if it sends positive vectors to positive vectors, i.e., $T(X_+) \subset Y_+$. If a linear operator preserves lattice operations, it is called a *lattice homomorphism.* A lattice homomorphism is positive. A one-to-one surjective lattice homomorphism is called a *lattice isomorphism.* A *lattice isometry* is a lattice isomorphism which is also an isometry.

Two classes of Banach lattices play a significant role in the Banach lattice theory.

Definition 2.1. *A Banach lattice X is said to be an AL-space (respectively, AM-space) if $\|x + y\| = \|x\| + \|y\|$ (respectively, $\|x \vee y\| = \max\{\|x\|, \|y\|\}$) whenever $|x| \wedge |y| = 0$. An AM-space has a (strong order) unit $u \ge 0$ if the order interval $[-u, u]$ is the unit ball of X.*

Kakutani Representation Theorem. *An arbitrary AL-space is lattice isometric to* $L^1(\Omega, \Sigma, \mu)$ *for some localizable measure space* (Ω, Σ, μ).

Kakutani–Kreĭns Representation Theorem. *An AM-space is lattice isometric to a sublattice of C(K) for some compact Hausdorff space K. Moreover, if the AM-space has a strong order unit then it is lattice isometric to* $C(K)$ *itself.*

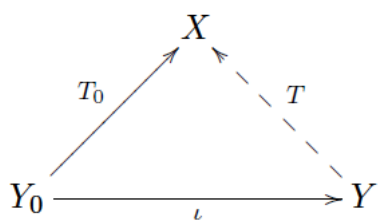
Nakano–Stone Completeness Theorem. *Let K be a compact Hausdorff topological space. The vector lattice* $C(K)$ *is Dedekind complete if and only if K is extremally disconnected* ($\equiv$ *the closure of any open set in K is open*).

Remark 3. Banach lattices were first considered by Kantorovich in [30]. For an extensive treatment of Banach lattices see [5, 51, 53, 61, 77, 81]. More information can be found in [2, 18, 31, 36, 47, 63]. Complex Banach lattice is defined by *complexification*, see [61]. The above three theorems see in [28], [29, 35], and [54, 68], respectively.

3 Injective Banach lattices

Definition 3.1. *A real Banach lattice X is said to be injective if, for every Banach lattice Y, every closed vector sublattice* $Y_0 \subset Y$, *and every positive linear operator* $T_0 : Y_0 \to X$ *there exists a positive linear extension* $T : Y \to X$ *with* $\|T_0\| = \|T\|$.

This definition is illustrated by the commutative ($T_0 = T \circ \iota$) diagram:



Equivalently, X is an injective Banach lattice if, whenever X is lattice isometrically imbedded into a Banach lattice Y, there exists a positive contractive projection from Y onto X. One more equivalence definition states that every positive operator from X to any Banach lattice admits a norm preserving positive extension to any Banach lattice containing X as a vector sublattice.

Remark 4. Thus, the injective Banach lattices are the injective objects in the category of Banach lattices with the positive contractions as morphisms. Arendt [6, Theorem 2.2] proved that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms.

Lotz [50] was the first who introduced this concept and proved among other things the following two results. But the first example of injective Banach lattice was indicated by Abramovich [1].

Theorem 3.1. (Abramovich, [1]; Lotz, [50]) *A Dedekind complete AM-space with unit is an injective Banach lattice.*

Taking into account the Kakutani–Kreĭns Representation Theorem one can state Theorem 3.1 equivalently: The Banach lattice of continuous function $C(K)$ is injective, whenever K is an extremally disconnected Hausdorff compact topological space.

Theorem 3.2. (Lotz, [50]) *Every AL-space is an injective Banach lattice.*

Remark 5. Theorem 3.2 shows that there is an essential difference between injective Banach lattices and injective Banach spaces, since $C(K)$ with extremally disconnected compactum K is the only (up to isomorphism) injective object in the category of Banach spaces and linear contractions (see Goodner [20], Kelley [33], Nachbin [55]).

Remark 6. A separable Banach lattice X is said to be *separably-injective* if for every pair of separable Banach lattices $Y \subset Z$ and every positive (continuous) linear map from Y to X, there exists a norm preserving positive linear extension from Z to Y. In [11, Theorem 3] Buskes made the observation that every separably-injective Banach lattice is injective. More details concerning injective Banach lattices see in Cartwright [12], Gierz [19], Haydon [26], Lotz [50], Schaefer [62], Wickstead [79].

Remark 7. A Banach lattice X is called λ-*injective* if $\|T\| \leq \lambda\|T_0\|$ in Definition 2. In what follows injective means 1-injective; λ-injective Banach lattices ($\lambda > 1$) are not considered. For λ-injective Banach lattices ($\lambda > 1$) see [48, 49, 52].

4 Characterization of injective Banach lattices

A geometric property which enables us to characterize injective Banach lattices was discovered by Cartwright [12].

Definition 4.1. (1) *A Banach lattice X has the splitting property if, given $x_1, x_2, y \in X_+$ with $\|x_1\| \leq 1$, $\|x_2\| \leq 1$, and $\|x_1 + x_2 + y\| \leq 2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leq 1$, and $\|x_2 + y_2\| \leq 1$.*

(2) A Banach lattice X has the *Cartwright property* if, given $x_1, x_2, y \in X_+$ and $0 < r_1, r_2 \in \mathbb{R}$ with $\|x_1\| \leq r_1$, $\|x_2\| \leq r_2$, and $\|x_1 + x_2 + y\| \leq r_1 + r_2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leq r_1$, and $\|x_2 + y_2\| \leq r_2$.

(3) A Banach lattice X has the *finite order intersection property* if, given $z \in X_+$, finite collections $x_1, \dots, x_n \in X_+$, $y_1, \dots, y_m \in X_+$, and strictly positive reals $r_1, \dots, r_n \in \mathbb{R}_+$, $s_1, \dots, s_m \in \mathbb{R}_+$ such that $\|x_i\| \leq r_i$, $\|y_j\| \leq s_j$, and $\|x_i + y_j + z\| \leq r_i + s_j$ for all $i := 1, \dots, n$ and $j := 1, \dots, m$, there exist $u, v \in X_+$ with $z = u + v$, $\|x_i + u\| \leq r_i$, and $\|y_j + v\| \leq s_j$ for all $i := 1, \dots, n$ and $j := 1, \dots, m$.

Theorem 4.1. (Cartwright, [12]) *For a Banach lattice the splitting property, the Cartwright property, and the finite order intersection property are equivalent.*

Theorem 4.2. (Cartwright, [12]) *A Banach lattice has the splitting property if and only if its second dual is injective.*

Remark 8. Gierz in [19, Corollaries 3.3 and 3.4] proved that every Banach lattice with the splitting property (and hence every injective Banach lattice) has the approximation property.

Definition 4.2. *A Banach lattice X is said to have: the property* (P) *if there exists a positive contractive projection in* X'' *onto* X [53, p. 47]*; the Levi property if* $0 \leq x_\alpha \uparrow$ *and* $\|x_\alpha\| \leq 1$ *imply that* $\sup_\alpha x_\alpha$ *exists in* X [3, Definition 7 (2)]*; the Fatou property if* $0 \leq x_\alpha \uparrow x$ *implies* $\|x_\alpha\| \uparrow \|x\|$ [3, Definition 7 (3)]. *A Banach lattice with the Levi* (*Fatou*) *property is also called order semicontinuous* (*resp. monotonically complete*) [53].

A Dedekind complete Banach lattice X with a separating order continuous dual has property (P) if and only if it has the Levi and Fatou properties [63, Propositions 7.6 and 7.10]. Cartwright [12, Corollary 3.8] proved that a Banach lattice is injective if and only if it has the Cartwright property and the property (P). Haydon demonstrated that the property (P) may be replaced with the intrinsic 'completeness' property.

Theorem 4.3. (Haydon, [26]) *A Banach lattice is injective if and only if it has the Cartwright, Fatou, and Levi properties.*

Definition 4.3. *A band projection* π *in a Banach lattice* X *is called an* M*-projection if* $\|x\| = \max\{\|\pi x\|, \|\pi^\perp x\|\}$ *for all* $x \in X$*, where* $\pi^\perp := I_X - \pi$*. The set* $\mathbb{M}(X)$ *of all* M*-projections in* X *forms a Boolean subalgebra of* $\mathbb{P}(X)$*. The* f*-subalgebra of the center* $\mathcal{Z}(X)$ *generated by* $\mathbb{M}(X)$ *is called the* M*-center of* X *and denoted by* $\mathcal{Z}_m(X)$*.*

Observe that $\mathbb{M}(X)$ is an order closed subalgebra of $\mathbb{P}(X)$ whenever X has the Fatou and Levi properties. In this event the relations $\mathbb{B} \simeq \mathbb{M}(X)$ and $\Lambda(\mathbb{B}) \simeq \mathcal{Z}_m(X)$ are equivalent. Note also that if X is an AL-space and Y is an AM-space then $\mathbb{M}(X) = \{0, I_X\}$ and $\mathbb{M}(Y) = \mathbb{P}(Y)$.

Remark 9. Haydon proved three representation theorems for injective Banach lattices, see [26, Theorems 5C, 6H, and 7B]. These results may be also deduced from our representation theorem (see Theorems 10 and 11 below and [38, Corollary 6.10]).

Remark 10. The notion of an M-projection goes back to [4] and plays a crucial role in the theory of injective Banach lattices. In a wider context of a general Banach space theory the concept is presented in [7] and [25].

5 Boolean-valued analysis

In 1963 P. Cohen discovered his *method of 'forcing'* and also proved the independence of the Continuum Hypothesis. A comprehensive presentation of the Cohen forcing method gave rise to the *Boolean-valued models of set theory*, which were first introduced by D. Scott and R. Solovay (see D. Scott [65]) and P. Vopěnka [76]. A systematic account of the theory of Boolean-valued models and its applications to independence proofs can be found in [8, 67, 73]. A brief and nice overview see in [45].

D. Scott [65] forecasted in 1969: "We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good arguments."

The development of Boolean-valued analysis started at the end of the seventies have shown that D. Scott was perfectly right (see, for example, [43, 44]). The term *Boolean-valued analysis,* coined by G. Takeuti (see [70]), signifies the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras.

As these models, the classical Cantorian paradise in the shape of the von Neumann universe V and a specially-trimmed Boolean-valued universe $\mathrm{V}^{(\mathbb{B})}$ are usually taken. Comparative analysis is carried out by means of some interplay between V and $\mathrm{V}^{(\mathbb{B})}$.

Boolean-valued analysis starts with the fact that each internal field of reals of a Boolean-valued model descends into a universally complete vector lattice. This remarkable fact was discovered by E. Gordon in [7], see also [22, 23]. In the same period, two important particular cases were independently studied by G. Takeuti, who observed that the vector lattice of (equivalence classes of) measurable function and a commutative algebra of (unbounded) self-adjoint operators in Hilbert space can be considered as instances of Boolean-valued reals [69, 70, 71].

It should be also mentioned that Boolean-valued models are related to quantum mathematics stemmed from J. von Neumann's book on the mathematical foundations of quantum mechanics [56] and fuzzy mathematics emerged in the development of the theory of fuzzy sets after the Zadeh's work [82]. The birth of quantum logic was marked by the seminal joint paper of G. Birkhoff and J. von Neumann [9] in which a systematic attempt to propose a propositional calculus for quantum logic was made. The mathematical and logical investigation of various aspects of quantum mechanics is the topic of the "Handbook of Quantum Logic and Quantum Structures" [14]. Quantum set theory was introduced by G. Takeuti in [72] as the quantum counterpart of the Boolean valued set theory. Zhang [83, 84] and Weidner [78] gave models for a unified treatment of fuzzy set theory and Boolean-valued set theory. Takuti and Titani [74, 75] have introduced and investigated intuitionistic fuzzy logic and used similar models for studying intuitionistic fuzzy set theory. Significant attention has been paid to fuzzy algebraic structures, see, for example, [60].

In the next two section we sketch the machinery of Boolean-valued analysis. A detailed presentation can be found in [43, 44], see also [36].

6 The universe of Boolean-valued sets

Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with the operations meet $\wedge$, join $\vee$, and complementation $(\cdot)^*$, with distinguished elements unit $\mathbb{1}$ and zero $\mathbb{O}$. Given an ordinal α, put

$$\mathrm{V}_\alpha^{(\mathbb{B})} := \big\{\, x \,:\, x \text{ is a function, } (\exists\,\beta)\big(\beta < \alpha,\ \mathrm{dom}(x) \subset \mathrm{V}_\beta^{(\mathrm{B})},\ \mathrm{Im}(x) \subset \mathbb{B}\big)\big\}.$$

After this recursive definition the *Boolean-valued universe* $\mathrm{V}^{(\mathbb{B})}$ or, in other words, the *class of* $\mathbb{B}$*-sets* is introduced as (with On standing for the class of all ordinals):

$$\mathrm{V}^{(\mathbb{B})} := \bigcup_{\alpha\in\mathrm{On}} \mathrm{V}_\alpha^{(\mathbb{B})}.$$

In case of the two element Boolean algebra $2 := \{\mathbb{0}, \mathbb{1}\}$ this procedure yields a version of the classical *von Neumann universe* V (cp. [44, Theorem 4.2.8]).

Let φ be an arbitrary formula of ZFC, Zermelo–Fraenkel set theory with choice. The *Boolean truth value* $[\![\varphi]\!] \in \mathbb{B}$ is introduced by induction on the complexity of φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ $\big($for instance, $[\![\varphi_1 \vee \varphi_2]\!] := [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$ and $[\![\forall x \varphi(x)]\!] = \bigwedge\{[\![\varphi(u)]\!] : u \in \mathrm{V}^{(\mathbb{B})}\}\big)$ and taking into consideration the way in which a formula is built up from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$ $\big($with $x, y \in \mathrm{V}^{(\mathbb{B})}\big)$ are defined by means of the following recursion schema:

$$[\![x \in y]\!] = \bigvee_{t \in \mathrm{dom}(y)} \big(y(t) \wedge [\![t = x]\!]\big),$$

$$[\![x = y]\!] = \bigvee_{t \in \mathrm{dom}(x)} \big(x(t) \Rightarrow [\![t \in y]\!]\big) \wedge \bigvee_{t \in \mathrm{dom}(y)} \big(y(t) \Rightarrow [\![t \in x]\!]\big).$$

The sign $\Rightarrow$ symbolizes the implication in $\mathbb{B}$; i. e., $(a \Rightarrow b) := (a^* \vee b)$.

We say that the *statement* $\varphi(x_1, \dots, x_n)$ *is valid or the elements* $x_1, \dots, x_n$ *possess the property* φ *within (inside)* $\mathrm{V}^{(\mathbb{B})}$ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$. In this event, we write also

$$\mathrm{V}^{(\mathbb{B})} \vDash \varphi(x_1, \dots, x_n).$$

The universe $\mathrm{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following statement is fulfilled:

Transfer Principle. *For every theorem* φ *of ZFC, we have* $[\![\varphi]\!] = \mathbb{1}$ (*also in ZFC*)*; i. e.,* φ *is true inside the Boolean-valued universe* $\mathrm{V}^{(\mathbb{B})}$.

Maximum Principle. *Let* $\varphi(x)$ *be a formula of ZFC. Then* (*in ZFC*) *there is a* $\mathbb{B}$*-valued set* x_0 *satisfying* $[\![(\exists\, x)\varphi(x)]\!] = [\![\varphi(x_0)]\!]$.

Corollary 6.1. *If* $\mathrm{V}^{(\mathbb{B})} \models (\exists\, x)\varphi(x)$*, then* $\mathrm{V}^{(\mathbb{B})} \models \varphi(x_0)$ *for some* $x_0 \in \mathrm{V}^{(\mathbb{B})}$.

7 Ascending and descending

As was mentioned above, a smooth mathematical toolkit for revealing interplay between the interpretations of one and the same fact in the two models V and $\mathrm{V}^{(\mathbb{B})}$ is needed. The relevant *ascending-and-descending technique* rests on the functors of *canonical embedding* (or *standard name*), *descent*, and *ascent*, see [43, 44] for details.

The standard name functor. Given $X \in \mathrm{V}$, we denote by $X^\wedge \in \mathrm{V}^{(\mathbb{B})}$ the *standard name* of X. The standard name is an embedding of V into $\mathrm{V}^{(\mathbb{B})}$. Moreover, the standard name sends V onto $\mathrm{V}^{(2)}$, i. e., $\mathrm{V} \simeq \mathrm{V}^{(2)} \subset \mathrm{V}^{(\mathbb{B})}$, where $2 := \{\mathbb{0}, \mathbb{1}\} \subset \mathbb{B}$.

A formula is *restricted* provided that each bound variable in it is restricted by a bounded quantifier (i. e., each of its quantifier occurs in the form $(\forall\, x \in y)$ or $(\exists\, x \in y)$) or if it can be proved equialent in ZFC to a formula of this kind.

Restricted Transfer Principle. *Let* $\varphi(x_1, \dots, x_n)$ *be a restricted formula of ZFC. Then* (*in ZFC*) *for every collection* $x_1, \dots, x_n \in \mathrm{V}$ *we have*

$$\varphi(x_1, \dots, x_n) \iff \mathrm{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

The descent functor. Given an arbitrary element X of the Boolean-valued universe $\mathrm{V}^{(\mathbb{B})}$, we define the *descent* $X{\downarrow}$ of X as $X{\downarrow} := \{y \in \mathrm{V}^{(\mathbb{B})} :\ [\![y \in x]\!] = \mathbb{1}\}$. The class $X{\downarrow}$ is a set, i. e., $X{\downarrow} \in \mathrm{V}$ for all $X \in \mathrm{V}^{(\mathbb{B})}$. If $[\![X \neq \varnothing]\!] = \mathbb{1}$ then $X{\downarrow}$ is nonempty.

Suppose that $X, Y, f \in \mathrm{V}^{(\mathbb{B})}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$, i. e., f is a mapping from X into Y inside $\mathrm{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ into $Y{\downarrow}$ satisfying $[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1}$ for all $x \in X{\downarrow}$. The descent of a mapping is *extensional*:

$$[\![x_1 = x_2]\!] \leq [\![f(x_1) = f(x_2)]\!] \quad (x_1, x_2 \in X{\downarrow}).$$

Assume that P is an n-ary relation on X inside $\mathrm{V}^{(\mathbb{B})}$; i. e., $X, P \in \mathrm{V}^{(\mathbb{B})}$ and $[\![P \subset X^{n^\wedge}]\!] = 1$ $(n \in \mathbb{N})$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that

$$(x_1, \dots, x_n) \in P' \Longleftrightarrow [\![(x_1, \dots, x_n)^B \in P]\!] = \mathbb{1}.$$

We denote the relation P' by the same symbol $P{\downarrow}$ and call it the *descent* of P.

The ascent functor. Let $X \in \mathrm{V}$ and $X \subset \mathrm{V}^{(\mathbb{B})}$; i. e., let X be some set composed of $\mathbb{B}$-valued sets or, in other words, $X \in \mathscr{P}(\mathrm{V}^{(\mathbb{B})})$. There exists a unique $X{\uparrow} \in \mathrm{V}^{(\mathbb{B})}$ such that $[\![y \in X{\uparrow}]\!] = \bigvee\{[\![x = y]\!] :\ x \in X\}$ for all $y \in \mathrm{V}^{(\mathbb{B})}$. The element $X{\uparrow}$ is called the *ascent* of X. Observe that the ascent extend the standard name in the sense that $Y^\wedge$ is the ascent of $\{y^\wedge :\ y \in Y\}$ whenever $Y \in \mathrm{V}$.

Let $X, Y \in \mathscr{P}(\mathrm{V}^{(\mathbb{B})})$, and $f : X \to Y$. There exists a unique function $f{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ inside $\mathrm{V}^{(\mathbb{B})}$ such that $f{\uparrow}(A{\uparrow}) = f(A){\uparrow}$ is valid for every subset $A \subset X$ if and only if f is extensional.

8 Boolean-valued reals

Recall the well-known assertion of ZFC: *There exists a field of reals that is unique up to isomorphism.* Denote by $\mathbb{R}$ the field of reals (in the sense of V). Successively applying the transfer and maximum principles, we find an element $\mathscr{R} \in \mathrm{V}^{(\mathbb{B})}$ for which $[\![\,\mathscr{R}$ is a field of reals$\,]\!] = \mathbb{1}$. Moreover, if an arbitrary $\mathscr{R}\,' \in \mathrm{V}^{(\mathbb{B})}$ satisfies the condition $[\![\,\mathscr{R}\,'$ is a field of reals$\,]\!] = \mathbb{1}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}\,'$ are isomorphic$\,]\!] = \mathbb{1}$. In other words, there exists an internal field of reals $\mathscr{R} \in \mathrm{V}^{(\mathbb{B})}$ which is unique up to isomorphism. We call $\mathscr{R}$ the *internal reals* in $\mathrm{V}^{(\mathbb{B})}$.

Consider another well-known assertion of ZFC: *If $\mathbb{P}$ is an Archimedean ordered field then there is an isomorphic embedding h of the field $\mathbb{P}$ into $\mathbb{R}$ such that the image $h(\mathbb{P})$ is a subfield of $\mathbb{R}$ containing the subfield of rational numbers. In particular, $h(\mathbb{P})$ is dense in $\mathbb{R}$.*

Note also that $\varphi(\cdot)$, presenting the conjunction of the axioms of an Archimedean ordered field, is restricted; therefore, $[\![\,\varphi(\mathbb{R}^\wedge)\,]\!] = \mathbb{1}$ by the Restricted Transfer Principle, i. e., $[\![\,\mathbb{R}^\wedge$ is an Archimedean ordered field$\,]\!] = \mathbb{1}$. 'Pulling' the above assertion through the transfer principle, we conclude that $[\![\,\mathbb{R}^\wedge$ is isomorphic to a dense subfield of $\mathscr{R}\,]\!] = \mathbb{1}$. We further assume that $\mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}$. It is easy to see that the elements $0^\wedge$ and $1^\wedge$ are the zero and unity of $\mathscr{R}$.

Look now at the descent $\mathbf{R} := \mathscr{R}{\downarrow}$ of the algebraic structure $\mathscr{R} := (\mathrm{R}, \oplus, \odot, \leqq, 0, 1)$ within $\mathrm{V}^{(\mathbb{B})}$. In other words, $\mathbf{R} := (\mathrm{R}{\downarrow}, \oplus{\downarrow}, \odot{\downarrow}, \leqq{\downarrow}, 0, 1)$ is considered as the descent $\mathrm{R}{\downarrow}$ of the underlying set R together with the descended operations $\oplus{\downarrow}$ and $\odot{\downarrow}$ and order $\leqq\downarrow$ of the structure $\mathscr{R}$. For simplicity, we will denote the operations and order relation in $\mathscr{R}$ and $\mathscr{R}\downarrow$ by the conventional symbols $+$, $\cdot$, and $\leq$.

The fundamental result of Boolean-valued analysis is the Gordon Theorem which describes an interplay between $\mathbb{R}$, $\mathscr{R}$, and $\mathbf{R}$: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean-valued model.*

Theorem 8.1. (Gordon, [7]) *Let $\mathbb{B}$ be a complete Boolean algebra, $\mathscr{R}$ be a field of reals in $\mathrm{V}^{(\mathbb{B})}$, and $\mathbf{R} = \mathscr{R}\downarrow$. Then the following assertions hold:*

(1) *The algebraic structure $\mathbf{R}$ (with the descended operations and order) is an universally complete vector lattice.*

(2) *The internal field $\mathscr{R} \in \mathrm{V}^{(\mathbb{B})}$ can be chosen so that*

$$[\![\, \mathbb{R}^{\wedge} \text{ is a dense subfield of the field } \mathscr{R} \,]\!] = \mathbb{1}.$$

(3) *There is a Boolean isomorphism χ from $\mathbb{B}$ onto $\mathbb{P}(\mathbf{R})$ such that*

$$\begin{gathered}\chi(b)x = \chi(b)y \Longleftrightarrow b \leq [\![\, x = y \,]\!],\\ \chi(b)x \leq \chi(b)y \Longleftrightarrow b \leq [\![\, x \leq y \,]\!]\\ (x, y \in \mathbf{R};\ b \in \mathbb{B}).\end{gathered}$$

Let $\Lambda \subset \mathbf{R} = \mathscr{R}\downarrow$ be the order ideal in $\mathbf{R}$ generated by $1^{\wedge}$ equipped with the order-unit norm $\|\cdot\|_{\infty}$:

$$\begin{gathered}\Lambda := \{x \in \mathbf{R} :\ (\exists C \in \mathbb{R}) - C1^{\wedge} \leq x \leq C1^{\wedge}\};\\ \|x\|_{\infty} := \inf\{C > 0 :\ -C1^{\wedge} \leq x \leq C1^{\wedge}\} \quad (x \in \Lambda).\end{gathered}$$

Write $\Lambda = \Lambda(\mathbb{B})$, since Λ is uniquely defined by $\mathbb{B}$. Clearly, Λ is a Dedekind complete AM-space with unit $1^{\wedge}$. By Kakutani–Kreĭns Representation Theorem $\Lambda \simeq C(K)$ with K being an extremally disconnected compact Hausdorff space.

Remark 11. The version of the Gordon Theorem for complexes is also true: *Each complex universally complete vector lattice is an interpretation of the complexes in a Boolean-valued model.* If $\mathscr{C}$ denotes the complexes within $\mathrm{V}^{(\mathbb{B})}$, then $\mathbf{C} := \mathscr{C}\downarrow$ and $\mathbf{C} = \mathbf{R} \oplus i\mathbf{R}$, i. e., the complex vector lattice $\mathbf{C}$ is the complexification of $\mathbf{R}$.

9 Boolean-valued Banach lattices

What kind of category is produced by applying the descending procedure to the category of Banach lattices in $\mathrm{V}^{(\mathbb{B})}$? The answer is given in terms of $\mathbb{B}$-cyclicity.

Let $(\mathscr{X}, \|\cdot\|)$ be a Banach lattice in $\mathrm{V}^{(\mathbb{B})}$. Define the map $N : \mathscr{X}\downarrow \to \mathbf{R} := \mathscr{R}\downarrow$ as the descent $N(\cdot) := (\|\cdot\|)\downarrow$ of the norm $\|\cdot\|$. Then $\mathscr{X}\downarrow$ (with the descended operations and order) is a vector lattice and N is an $\mathbf{R}$-valued norm on $\mathscr{X}\downarrow$.

Definition 9.1. *The bounded descent $\mathscr{X}\Downarrow$ of $\mathscr{X}$ is defined as the set*

$$\mathscr{X}\Downarrow := \{x \in \mathscr{X}\downarrow :\ N(x) \in \Lambda\}$$

equipped with the descended operations and order equipped with the mixed norm:

$$|\!|\!| x |\!|\!| := \|N(x)\|_{\infty} \quad (x \in \mathscr{X}\Downarrow).$$

Definition 9.2. *Say that X is a Banach lattice with a Boolean algebra of band projections $\mathbb{B}$ if there is a Boolean isomorphism $\varphi : \mathbb{B} \to \mathbb{P}(X)$ and $\varphi(\mathbb{B})$ is a complete subalgebra in $\mathbb{P}(X)$. In this event $\mathbb{B}$ is identified with $\varphi(\mathbb{B})$ and one write $\mathbb{B} \subset \mathbb{P}(X)$. Let $\mathbb{B} \subset \mathbb{P}(X)$ and $\mathbb{B} \subset \mathbb{P}(Y)$. Say that X and Y are lattice $\mathbb{B}$-isometric and write $X \simeq_{\mathbb{B}} Y$ if there is a lattice isometry T from X onto Y with an additional property $b \circ T = T \circ b$ for all $b \in \mathbb{B}$.*

Definition 9.3. *A partition of unity in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$. The set of all partitions of unity in $\mathbb{B}$ is denoted by* $\mathrm{Prt}(\mathbb{B})$. *Let $(b_\xi)_{\xi\in\Xi} \in \mathbb{B}$ and $(x_\xi)_{\xi\in\Xi} \subset X$. The element $x \in X$ is called a mixture of (x_ξ) by (b_ξ) and is denoted by $x := \mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi)$, whenever $b_\xi x_\xi = b_\xi x$ for all $\xi \in \Xi$.*

Definition 9.4. *The Banach lattice X is said to be $\mathbb{B}$-cyclic if $\mathbb{B} \subset \mathbb{P}(X)$ and the closed unit ball B_X of X is* mix*-complete, i. e., has the property:*

$$(x_\xi) \subset B_X,\ (b_\xi) \in \mathrm{Prt}(\mathbb{B}) \implies \exists\ \mathrm{mix}_\xi(b_\xi x_\xi) \in B_X.$$

Theorem 9.1. ([38, Theorem 4.1]) *A bounded descent $\mathscr{X}{\Downarrow}$ of a Banach lattice $\mathscr{X}$ from the model $\mathrm{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-cyclic Banach lattice. Conversely, if X is a $\mathbb{B}$-cyclic Banach lattice, then in the model $\mathrm{V}^{(\mathbb{B})}$ there exists up to lattice isometry a unique Banach lattice $\mathscr{X}$ whose bounded descent $\mathscr{X}{\Downarrow}$ is lattice $\mathbb{B}$-isometric to X. Moreover, $\mathbb{B} \simeq \mathbb{M}(X)$ if and only if $[\![$there is no M-projection in $\mathscr{X}$ other than 0 and $I_{\mathscr{X}}]\!] = \mathbb{1}$, i. e.,*

$$\mathbb{B} \simeq \mathbb{M}(X) \iff [\![\mathbb{M}(\mathscr{X}) = \{0, I_{\mathscr{X}}\}]\!] = \mathbb{1}.$$

Definition 9.5. *The internal Banach lattice $\mathscr{X}$ in Theorem 9.1 is called the Boolean-valued representation of X.*

Remark 12. It follows from Theorem 9.1 that the descent of a category of Banach lattices and positive operators inside $\mathrm{V}^{(\mathbb{B})}$ is the category of $\mathbb{B}$-cyclic Banach lattices and positive $\mathbb{B}$-linear operators, see [38]. A detailed presentation of the descent of the category of Banach spaces and bounded linear operators see in [43] and [36].

10 Boolean-valued AL-spaces

Now, we describe a Boolean valued analysis approach to the theory of injective Banach lattices developed in [38, 39, 40, 41]. First we clarify what the Boolean-valued representation of injective Banach lattice is, see [38, Theorem 4.7].

Theorem 10.1. *Suppose that X is a $\mathbb{B}$-cyclic Banach lattice and $\mathscr{X} \in \mathrm{V}^{(\mathbb{B})}$ is its Boolean-valued representation. Then the following assertions hold:*

(1) *X is injective if and only if $[\![\mathscr{X}$ is injective$]\!] = \mathbb{1}$.*

(2) *X is injective and $\mathbb{B} \simeq \mathbb{M}(X)$ if and only if $[\![\mathscr{X}$ is injective and $\mathbb{M}(\mathscr{X}) = \{0, I_{\mathscr{X}}\}]\!] = \mathbb{1}$.*

Theorem 10.2. (Haydon, [26]) *Let X is an injective Banach space. Then X is an AL-space if and only if $\mathbb{M}(X) = \{0, I_X\}$.*

Now, putting together Theorems 9.1, 10.1, and 10.2 we arrive at our main representation theorem for injective Banach lattice, see [38, Theorem 5.1] and [40, Theorem 1].

Theorem 10.3. *A bounded descent $\mathscr{X}{\Downarrow}$ of an AL-space $\mathscr{X}$ from $\mathrm{V}^{(\mathbb{B})}$ is an injective Banach lattice with $\mathbb{B} \simeq \mathbb{M}(\mathscr{X}{\Downarrow})$. Conversely, if X is an injective Banach lattice and $\mathbb{B} \simeq \mathbb{M}(X)$, then there exist an AL-space $\mathscr{X}$ in $\mathrm{V}^{(\mathbb{B})}$ whose bounded descent is lattice $\mathbb{B}$-isometric to X; in symbols, $X \simeq_{\mathbb{B}} \mathscr{X}{\Downarrow}$.*

Remark 13. Theorem 10.3 implies the *transfer principles* from AL-spaces to injective Banach spaces which can be stated as follows:

(1) Every injective Banach lattice embeds into an appropriate Boolean-valued model, becoming an AL-space (Theorem 10.3).

(2) Each theorem about the AL-space within Zermelo–Fraenkel set theory with choice has its counterpart for the original injective Banach lattice interpreted as a Boolean-valued AL-space (Boolean-valued Transfer Principe, Section 5).

(3) Translation of theorems from AL-spaces to injective Banach lattices is carried out by general operations and principles of Boolean-valued analysis (outlined in Sections 6 and 7).

The following important representation result (see [38, Corollary 5.2] and [40, Theorem 2]) which do not involve the concept of Boolean-valued model can deduce immediately from Theorem 10.3. Before stating this result, recall some definitions. A positive operator $T : X \to Y$ between vector lattices is said to: 1) be a *Maharam operator* whenever it is an order continuous and order interval preserving, i. e., $T([0, x]) \subset [0, Tx]$ for all $x \in X_+$; 2) have the *Levi property* if $\sup x_\alpha$ exists in Y for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in Y; 3) be *strictly positive* if $Tx = 0$ implies $x = 0$ for all $x \in X_+$. If $Y = \Lambda$ and T is strictly positive then $L^1(T)$ denotes the domain of T endowed with the norm $|||x||| = \|T(|x|)\|_\infty$ $(x \in L^1(T))$, see Definition 9.1.

Theorem 10.4. *If T is a strictly positive Maharam operator with the Levi property taking values in a Dedekind complete AM-space Λ with unit, then $(L^1(\Phi), |||\cdot|||)$ is an injective Banach lattice with $\mathbb{M}(L^1(\Phi)) = \mathbb{P}(\Lambda)$.*

Conversely, any injective Banach lattice X is lattice $\mathbb{B}$-isometric to $(L^1(T), |||\cdot|||)$ for some strictly positive Maharam operator T with the Levi property taking values in a Dedekind complete AM-space Λ with unit, where $\mathbb{B} = \mathbb{M}(L^1(T)) \simeq \mathbb{P}(\Lambda)$.

11 Direct sums of injective Banach lattices

Every injective Banach lattice is lattice isometric to a direct sum of concrete injective Banach lattices. To see that we have first to study an appropriate notion of direct sum decomposition.

Denote by $\mathrm{Prt} := \mathrm{Prt}(\mathbb{B})$ and $\mathcal{P}_{\mathrm{fin}}(X)$ the set of all partitions of unity in $\mathbb{B}$ and the collection of all finite subsets of X, respectively. Let $\mathrm{Prt}_\sigma := \mathrm{Prt}_\sigma(\mathbb{B})$ stands for the part of $\mathrm{Prt}(\mathbb{B})$ consisting of countable families. Define $\mathbb{B}\langle X_0\rangle$ by

$$\mathbb{B}\langle X_0\rangle := \big\{x \in X : \ x = \mathrm{mix}_\xi(b_\xi x_\xi), \ (x_\xi) \subset X_0, \ (b_\xi) \in \mathrm{Prt}(\mathbb{B})\big\}.$$

The following result see in [39, Theorem 7.2] and [41, Theorem 2]).

Theorem 11.1. *Let* $(X_\alpha)_{\alpha\in\mathrm{A}}$ *be a family of injective Banach lattices. Assume that there is a complete Boolean algebra* $\mathbb{B}$ *and a family* $(b_\alpha)_{\alpha\in\mathrm{A}}$ *in* $\mathbb{B}$ *with* $\bigvee_{\alpha\in\mathrm{A}} b_\alpha = \mathbb{1}$ *and* $\mathbb{M}(X_\alpha) \simeq \mathbb{B}_\alpha = [\mathbb{O}, b_\alpha]$ *for all* $\alpha \in \mathrm{A}$*. Then there exists a unique up to a lattice* $\mathbb{B}$*-isometry injective Banach lattice* X *such that the following hold:*

(1) $\mathbb{B} \simeq \mathbb{M}(X)$.

(2) *For any* $\alpha \in \mathrm{A}$ *there is a lattice* $\mathbb{B}_\alpha$*-isometry* $\iota_\alpha : X_\alpha \to X$.

(3) $\big(\iota_\alpha(X_\alpha)\big)_{\alpha\in\mathrm{A}}$ *is a family of pair-wise disjoint bands in* X.

(4) $\mathbb{B}\big\langle\bigoplus_{\alpha\in\mathrm{A}} \iota_\alpha(X_\alpha)\big\rangle$ *is norm dense in* X.

(5) *For any* $\mathbf{x} = (x_\alpha)_{\alpha\in\mathrm{A}} \in X$ *we have*

$$\|\mathbf{x}\|_{\mathrm{ins}} = \sup_{\theta\in\mathcal{P}_{\mathrm{fin}}(\mathrm{A})} \inf_{(\pi_k)\in\mathrm{Prt}_\sigma} \sup_{k\in\mathbb{N}} \sum_{\alpha\in\theta} \|\pi_k x_\alpha\|.$$

Definition 11.1. *The Banach lattice* $(X, \|\cdot\|_{\mathrm{ins}})$ *defined in Theorem 11.1 is called the injective sum of injective Banach lattices. Denote* $\sum^{\boxplus}_{\alpha\in\mathrm{A}} X_\alpha := X$.

Remark 14. Theorem 11.1 may be proved by using the above representation result (Theorem 10.3): if $\mathscr{X}_\alpha$ and $\mathscr{X}$ are Boolean-valued representations of X_α and X, respectively, and an internal function $\beta : \mathrm{A}^\wedge \to \{0,1\}$ is defined by $\beta(\alpha^\wedge) = \chi(b_\alpha)$ for all $\alpha \in \mathrm{A}$ (see Theorem 8.1(3)), then $\mathscr{X} \simeq \sum^{\oplus}_{\alpha\in\mathrm{A}^\wedge} \beta(\alpha)\mathscr{X}_\alpha$. Alternatively, it can be deduced from Theorem 10.4.

12 Tensor products of injective Banach lattices

If one of the Banach lattices X or Y is an AL-space then the *projective tensor product* $X\hat{\otimes}_\pi Y$ is a Banach lattice. If one of the Banach lattices X or Y is a Dedekind complete AM-spaces with unit then the *injective tensor product* $X\check{\otimes}_\varepsilon Y$ is a Banach lattice. However, both $X\hat{\otimes}_\pi Y$ and $X\check{\otimes}_\varepsilon Y$ need not be Banach lattices, see [10, § 9].

In [15] Fremlin introduced for every two Archimedean vector lattices X and Y a new Archimedean vector lattice $X \bar{\otimes} Y$. The *Fremlin projective tensor product* $X\hat{\otimes}_{|\pi|}Y$ of Banach lattices X and Y is the completion of $X \bar{\otimes} Y$ with respect to the *positive projective norm* $\|\cdot\|_{|\pi|}$, see [16].

The *Wittstock injective tensor product* $X\check{\otimes}_{|\varepsilon|}Y$ of Banach lattices X and Y is the completion of $X \otimes Y$ with respect to the *positive injective norm* $\|\cdot\|_{|\varepsilon|}$, [80].

Let X and Y be injective Banach lattices. No one of the four tensor products $X \otimes_\epsilon Y$, $X \otimes_\pi Y$, $X\check{\otimes}_{|\epsilon|}Y$, $X\hat{\otimes}_{|\pi|}Y$ is in general an injective Banach lattice. But there exists a 'mixed' *positive injective-projective tensor product* $X \otimes_{\epsilon|\pi|} Y$ which is always injective. Denote by $\mathbb{B} := \mathbb{B}_1 \hat{\otimes} \mathbb{B}_2$ the Dedekind completion of the free product $\mathbb{B}_1 \otimes \mathbb{B}_2$ of Boolean algebras $\mathbb{B}_1$ and $\mathbb{B}_2$, see [34, §11]. Details concerning the following result can be found in [42, Theorem 4.4, Corollary 5.3] (see also [41, Theorem 2]).

Theorem 12.1. *Let* X_1 *and* X_2 *be arbitrary injective Banach lattices. Then there exist a unique up to isomorphism injective Banach lattice* $X_1 \hat{\otimes}_{\epsilon|\pi|} X_2$ *and a lattice bimorphism* $\bar{\otimes} : X_1 \times X_2 \to X_1 \hat{\otimes}_{\epsilon|\pi|} X_2$ *such that the following hold:*

(1) *$\bar{\otimes}$ induces an embedding ϕ of the Fremlin tensor product $X_1 \bar{\otimes} X_2$ into $X_1 \hat{\otimes}_{\epsilon|\pi|} X_2$.*

(2) *There is a Boolean isomorphism $\jmath$ of $\mathbb{M}(X_1) \hat{\otimes} \mathbb{M}(X_2)$ onto $\mathbb{M}(X_1 \hat{\otimes}_{\epsilon|\pi|} X_2)$ with $\jmath(\pi_1 \otimes \pi_2)(x_1 \bar{\otimes} x_2) = \pi_1 x_1 \bar{\otimes} \pi_2 x_2$ for all $\pi_i \in \mathbb{M}(X_i)$ and $x_i \in X_i$ $(i = 1, 2)$.*

(3) *$\|x_1 \otimes x_2\|_{\epsilon|\pi|} = \|x_1\| \cdot \|x_2\|$ for all $x_1 \in X_1$ and $x_2 \in X_2$.*

(4) *$\mathbb{B}\langle X_1 \bar{\otimes} X_2 \rangle$ is norm dense in $X_1 \hat{\otimes}_{\epsilon|\pi|} X_2$ with $\mathbb{B} := \mathbb{M}(X_1 \hat{\otimes}_{\epsilon|\pi|} X_2)$.*

(5) *$X_1 \hat{\otimes}_{\epsilon|\pi|} X_2 = X_0^{\downarrow\uparrow}$, where X_0 comprises all finite sums $\sum_{k=1}^n \pi_k \phi(u_k)$ with $\pi_k \in \mathbb{B}$ and $u_k \in X_1 \bar{\otimes} X_2$ $(k = 1, \dots, n \in \mathbb{N})$.*

(6) *For any $x \in X_1 \bar{\otimes} X_2$ we have the representation*

$$\|x\|_{\text{inj}} = \inf \left\{ \sup_{\xi \in \Xi} \sum_{i=1}^{n_\xi} \|\pi_\xi u_{i,\xi}\| \cdot \|\rho_\xi v_{i,\xi}\| \right\}$$

where infimum is taken over all $(\pi_\xi) \in \mathrm{Prt}(\mathbb{B}_1)$, $(\rho_\xi) \in \mathrm{Prt}(\mathbb{B}_2)$, and $0 \le u_{i,\xi} \in X_1$, $0 \le v_{i,\xi} \in X_2$ $(i \le n_\xi \in \mathbb{N})$ with

$$|x| \le \sum_{i=1}^{n_\xi} u_{i,\xi} \otimes v_{i,\xi} \quad (\xi \in \Xi).$$

Remark 15. In the particular case of $\mathbb{B} = \mathbb{B}_1 = \mathbb{B}_2 = \{\mathbb{O}, \mathbb{1}\}$ we obtain from Theorem 12.1 relationship between the L_1 space of a product measure and the L_1 spaces of its factors, see [17, Theorems 253F and 253G]. Thus, Theorem 12.1 can be deducsed either from Theorem 12.1 by interpreting the mentioned relationship within $\mathbb{V}^{\mathbb{B}}$, or from Theorem 10.4 by using standart tools.

13 Representation of AL-spaces

For every cardinal γ, there exists a canonical measure on the unit cube $[0, 1]^\gamma$ that is the γth power of Lebesgue's measure on $[0, 1]$. The associated measure algebra and the corresponding Banach lattice of integrable functions will be denoted by $\mathbb{I}^\gamma$ and $L_1([0, 1]^\gamma)$, respectively.

The famous Maharam theorem tells us that the measure algebras $\mathbb{I}^\gamma$ are the 'building blocks' for every Maharam algebra ($\equiv$ the measure algebra of a localizable measure space). More precisely, every atomless nonzero (finite) Maharam algebra is isomorphic to a direct sum of concrete measure algebras $\mathbb{I}^\gamma$, and it is uniquely determined by a family of cardinals γ, see [18, 321A] and [66, 17.5.3]. Transferring the structure theory of Maharam algebras, yields an important representation theorem for AL-spaces (Theorem 13.1 below, see [66, 26.4.7]).

Definition 13.1. *The density character of a subset S of a topological space is the smallest cardinal γ such that S contains a dense subset of cardinality γ.*

Definition 13.2. *A Banach lattice X is said to be γ-homogeneous if X is non-atomic and whenever $x, y \in X$ with $x \le y$ and $x \neq y$ the density character of $[x, y]$ is γ.*

The Banach lattice $L^1([0,1]^\gamma)$ is γ-homogeneous for every cardinal γ. If γ is finite or countable then $L^1([0,1]^\gamma)$ is lattice isometric to $L^1([0,1])$, but $L^1([0,1]^\beta)$ and $L^1([0,1]^\gamma)$ are not lattice isometric whenever β and γ are infinite cardinals with $\beta < \gamma$. Moreover, every γ-homogeneous AL-space is lattice isometric to $\delta L_1([0,1]^\gamma)$ for some cardinal δ, where δY denotes the l_1-direct sum of δ many copies of Y.

Every non-atomic Banach lattice can be decomposed into a direct sum of homogeneous Banach lattices. Thus, the Banach lattices $L_1([0,1]^\gamma)$ are the 'building blocks' for non-atomic AL-spaces. More precisely, for any nonatomic AL-space X there exist a family of cardinals $(\delta_\gamma)_{\gamma\in\Gamma}$ with Γ being a set of cardinals such that the lattice isometry holds: $X \simeq \big(\sum^{\oplus}_{\gamma\in\Gamma} \delta_\gamma L_1([0,1]^\gamma)\big)_{l_1}$. Recall also that an atomic AL-space is of the form $l^1(\gamma)$ for some cardinal γ. Gathering all this yields the following Kakutani–Maharam representation result, see [46, 66].

Theorem 13.1. *Let X be an AL-space. Then there exists a unique family of cardinals $(\delta_\gamma)_{\gamma\in\Gamma\cup\{0\}}$ with Γ being a set of infinite cardinals such that δ_γ is either equal to 1 or is uncountable for each $\gamma \in \Gamma$ and*

$$X \simeq l_1(\delta_0) \oplus \sum_{\gamma\in\Gamma}^{\oplus} \delta_\gamma L_1\big([0,1]^\gamma\big),$$

where $\simeq$ denotes lattice isometry, $\oplus$ and $\sum^{\oplus}$ denote l_1-joins, and δY stands for the l_1-join of δ copies of Y.

14 Representation of injective Banach lattices

Let Λ_γ be a Dedekind complete AM-space with unit and L_γ be an AL-space. Then $M_\gamma \hat{\otimes}_{\epsilon|\pi|} L_\gamma$ is an injective Banach lattice by Theorem 12.1. Moreover, in view of Theorem 11.1, $\sum^{\boxplus}_{\gamma\in\Gamma} M_\gamma \hat{\otimes}_{\epsilon|\pi|} L_\gamma$ is also an injective Banach lattice. Actually, every injective Banach lattice have a similar representation, so that Dedekind complete AM-spaces with unit and AL-spaces are the *'building blocks'* for any injective Banach lattice. This follows from Theorems 10.3 and 13.1. More precisely, using Theorem 10.3 and interpreting Theorem 13.1 in the appropriate Boolean-valued model leads to the following result.

Denote $\beta \diamond X := \sum^{\boxplus}_{\alpha\in\mathrm{A}} X_\alpha$, where $X_\alpha = X$ for all $\alpha \in \mathrm{A}$.

Definition 14.1. *Let γ is a cardinal. A complete Boolean algebra $\mathbb{B}$ is said to be γ-stable whenever $\mathrm{V}^{(\mathbb{B})} \models \gamma^\wedge = |\gamma^\wedge|$, i. e. $[\![\gamma^\wedge = |\gamma^\wedge|]\!] = \mathbb{1}$. An element $b \in \mathbb{B}$ is called γ-stable if the relative Boolean algebra $[\mathbb{O}, b]$ is γ-stable, see* [44, Definition 12.3.7 and Theorem 12.3.8].

Theorem 14.1. *Let X be an injective Banach lattice, $\Lambda = \mathscr{Z}_m(X)$ and $\mathbb{B} = \mathbb{M}(X)$. Then there exists a unique pair of families $(b_\alpha)_{\alpha\in\mathrm{A}}$ and $(b_{\beta\gamma})_{\beta\in\mathrm{B}(\gamma),\gamma\in\Gamma}$ in the Boolean algebra $\mathbb{B}$ such that the following assertions hold:*

(1) A *is a set of cardinals and $(b_\alpha)_{\alpha\in\mathrm{A}}$ is pair-wise disjoint;*

(2) Γ *is a set of infinite cardinals, each element of $\mathrm{B}(\gamma)$ is either equal to 1 or is uncountable for all $\gamma \in \Gamma$, the family $(b_{\beta\gamma})_{\beta\in\mathrm{B}(\gamma)}$ is pair-wise disjoint, and $\mathbb{1} = \bigvee_{\alpha\in\mathrm{A}} b_\alpha \vee \bigvee_{\gamma\in\Gamma} \bar{b}_\gamma$, where $\bar{b}_\gamma := \bigvee_{\beta\in\mathrm{B}(\gamma)} b_{\beta\gamma}$;*

(3) *b_α is α-stable for all $\alpha \in \mathrm{A}$, $b_{\beta\gamma}$ is β-stable for all $\beta \in \mathrm{B}(\gamma)$, $\gamma \in \Gamma$, and $\bar{b}_\gamma$ is γ-stable for all $\gamma \in \Gamma$;*

(4) *the representation holds with $\Lambda_\alpha = b_\alpha(\Lambda)$ and $\Lambda_{\beta\gamma} = b_{\beta\gamma}(\Lambda)$:*

$$X \simeq_{\mathbb{B}} \left(\sum_{\alpha\in\mathrm{A}} \Lambda_\alpha \otimes_{\varepsilon|\pi|} l^1(\alpha) \right)_\infty \boxplus \sum_{\gamma\in\Gamma}^{\boxplus} \left(\sum_{\beta\in\mathrm{B}(\gamma)} \beta \diamond \left(\Lambda_{\beta\gamma} \otimes_{\varepsilon|\pi|} L^1([0,1]^\gamma) \right) \right)_\infty .$$

Remark 16. As was mentioned above, according to Theorem 10.3, our representation theorem 14.1 is just an interpretation of Theorem 13.1 in the Boolean-valued model $\mathrm{V}^{(\mathbb{B})}$ with $\mathbb{B} \simeq \mathbb{M}(X)$: there exist a family $(\mathscr{X}_\gamma)_{\gamma\in\Gamma^\wedge\cup\{0\}}$ and a function $\sigma : \Gamma^\wedge \cup \{0\} \to \{0,1\}$ within $\mathrm{V}^{(\mathbb{B})}$ such that $\mathscr{X} \simeq \sigma(0)\mathscr{X}_0 \oplus \sum_{\gamma\in\Gamma^\wedge}^{\oplus} \sigma(\gamma)\mathscr{X}_\gamma$, where $\mathscr{X}_0$ is an atomic AL-space and $\mathscr{X}_\gamma$ is a $|\gamma|$-homogeneous AL-space with $|\gamma|$ being the cardinality of γ. Details can be found in [41].

15 Classification of injective Banach lattices

Theorem 14.1 enables us to give a complete isometric classification of injective Banach lattices, see [41]. This should be compared with the representation and classification of of Kaplansky–Hilbert modules and AW^*-algebras of [57, 58, 59], see also [36].

Definition 15.1. *A passport of an injective Banach lattice X is the pair of families $\mathbf{a} := \big((\pi_\alpha)_{\alpha\in\mathrm{A}}, (\pi_{\beta\gamma})_{\beta\in\mathrm{B}(\gamma),\gamma\in\Gamma}\big)$ in $\mathbb{B} = \mathbb{M}(X)$ satisfying $(1-4)$ from Theorem 14.1. This pair is called a cardinal series whenever only (1) and (2) are fulfilled and a decomposition series if, in addition, (4) is also satisfied.*

Definition 15.2. *Two cardinal series $\mathbf{a}$ and $\mathbf{b}$ of injective Banach lattices X and Y respectively are congruent, if there exists a Boolean isomorphism h from $\mathbb{B}$ onto $\mathbb{B}'$ such that $\mathbf{b}$ and $h(\mathbf{a})$ are $\mathbb{B}$-equivalent, where $h(\mathbf{a}) := \big((h(a_\alpha))_{\alpha\in\mathrm{A}}, (h(a_{\beta\gamma}))_{\beta\in\mathrm{B}(\gamma),\gamma\in\Gamma}\big)$. It can be easily seen that the passports $\Pi(X) := \mathbf{a}$ and $\Pi(Y) := \mathbf{b}$ are congruent if and only if $\mathrm{A} = \Lambda$, $\Gamma = \Delta$, $\mathrm{B}(\gamma) = \Theta(\gamma)$ for all $\gamma \in \Gamma$ and $b_\alpha = h(a_\alpha)$, $b_{\beta\gamma} = h(a_{\beta\gamma})$ for all $\alpha \in \mathrm{A}$, $\beta \in \mathrm{B}(\gamma)$, and $\gamma \in \Gamma$.*

Theorem 15.1. ([41, Theorem 5]) *Injective Banach lattices X and Y are lattice isometric if and only if $\mathbb{M}(X)$ and $\mathbb{M}(Y)$ are isomorphic and $\Pi(X)$ and $\Pi(Y)$ are congruent.*

Theorem 15.2. ([41, Theorem 6]) *A cardinal series in a Boolean algebra $\mathbb{M}(X)$ is a decomposition series of Banach lattice X if and only if it is congruent to the passport $\Pi(X)$. In particular, any two decomposition series of an injective Banach lattice are congruent.*

Remark 17. The representation of an injective Banach lattice in Theorem 14.1 is not unique in general, without assumption (3) of Theorem 14.1, cf. [36, Ch. 8]. The reason for this is the so-called *cardinal collapsing* phenomena in $\mathrm{V}^{(\mathbb{B})}$: it is possible for two infinite cardinals $\varkappa < \lambda$ to satisfy $\mathrm{V}^{(\mathbb{B})} \models |\varkappa^\wedge| = |\lambda^\wedge|$. In this event we say that λ has been *collapsed* to $\varkappa$ in $\mathrm{V}^{(\mathbb{B})}$, see [8, Ch. 5]. The following result specifies Theorem 15.2 which answers the question of how much the non-uniqueness in Theorem 14.1 may be.

Definition 15.3. *A subset $X_0 \subset X$ is said to be $\mathbb{B}$-dense in X whenever $\mathbb{B}\langle X_0\rangle$ is norm dense in X. The $\mathbb{B}$-density character of a subset S of a $\mathbb{B}$-cyclic Banach lattice is the smallest cardinal γ such that S contains a $\mathbb{B}$-dense subset of cardinality γ.*

Definition 15.4. *A $\mathbb{B}$-cyclic Banach lattice X is said to be $(\mathbb{B},\gamma)$-homogeneous if X has no $\mathbb{B}$-atom (see [38, Definition 8.1]) and whenever $x, y \in X$ with $x \leq y$ and $x \neq y$ the $\mathbb{B}$-density character of the order interval $[x, y]$ is γ, while X is $\mathbb{B}$-homogeneous, whenever X is $(\mathbb{B},\gamma)$-homogeneous for some γ.*

Theorem 15.3. *Given two infinite cardinals $\varkappa$ and λ with $\varkappa < \lambda$, there exist a complete Boolean algebra $\mathbb{B}$ and injective Banach lattices X and Y such that $\mathbb{M}(X)$ and $\mathbb{M}(Y)$ are isomorphic to $\mathbb{B}$, X and Y are lattice $\mathbb{B}$-isometric, X is $(\mathbb{B},\varkappa)$-homogeneous, and Y is $(\mathbb{B},\lambda)$-homogeneous.*

16 Functional representation of injective Banach lattices

Let K be an extremally disconnected compactum, while X be an arbitrary Banach space. Denote by $C_\#(K, X)$ the set of equivalence classes of norm bounded continuous vector-functions u acting from $\mathrm{dom}(u) \subset K$ to X with $K \setminus \mathrm{dom}(u)$ meager. (Vector-functions u and v are equivalent if they coincide on the intersection of their domains $\mathrm{dom}(u)$ and $\mathrm{dom}(v)$.)

The set $C_\#(K, X)$ may be naturally endowed with the structure of a $C(K)$-module. The continuous extension $N(u)$ of the point-wise norm $k \mapsto \|u(k)\|$ ($k \in \mathrm{dom}(u)$) to K determines a decomposable norm $u \mapsto N(u)$ on $C_\#(K, X)$ with values in $C(K)$. Moreover, $C_\#(K, X)$ is a Banach space with respect to the mixed norm defined as $u \mapsto \|N(u)\|_\infty$ and even a Banach lattice if so is X. (Details can be found in [36, 2.3.3, 7.1.1, 7.1.2].) If X is an AL-space then $C_\#(K, X)$ is an injective Banach lattice such that $\mathbb{M}(C_\#(K, X))$ is isomorphic to the Boolean algebra of clopen subsets of K and $\mathcal{Z}_m(C_\#(K, X)) \simeq C(K)$, see [40].

Theorem 16.1. ([41, Theorem 7]) *Let X be an injective Banach lattice and K be the Stone space of the Boolean algebra $\mathbb{M}(X)$. Denote by K_α and $K_{\beta\gamma}$ the clopen sets in K corresponding to b_α and $b_{\beta\gamma}$ in Theorem* 14.1. *Then the representation holds:*

$$X \simeq_{\mathbb{B}} \Bigg(\sum_{\alpha\in\mathrm{A}} C_\#\big(K_\alpha, l^1(\alpha)\big)\Bigg)_\infty \boxplus \sum_{\gamma\in\Gamma}{}^{\boxplus} \Bigg(\sum_{\beta\in\mathrm{B}(\gamma)} \beta \diamond C_\#\big(K_{\beta\gamma}, L^1([0,1]^\gamma)\big)\Bigg)_\infty .$$

If $\mathbb{M}(X)$ is isomorphic the complete Boolean algebra associated with a localizable measure space (Ω, Σ, μ), then $C_\#\big(K_\alpha, l^1(\alpha)\big)$ and $C_\#\big(K_{\beta\gamma}, L^1([0,1]^\gamma)\big)$ in Theorem 16.1 can be replaced respectively by $L^\infty\big(\Omega_\alpha, l^1(\alpha)\big)$ and $L^\infty\big(\Omega_{\beta\gamma}, L^1([0,1]^\gamma)\big)$ with disjoint $\Omega_\alpha, \Omega_{\beta\gamma} \in \Sigma$. This yields one more representation result.

Theorem 16.2. ([41, Corollary 2]) *Let X be an injective Banach lattice and $\mathbb{M}(X) \simeq \mathbb{B} := \mathbb{B}(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ). Then there exist two M-bands X_1 and X_2 of X, two families of measurable subsets $(\Omega_\alpha)_{\alpha\in\mathrm{A}}$ and $(\Omega_{\beta\gamma})_{\beta\in\mathrm{B}(\gamma),\gamma\in\Gamma}$ uniquely determined up to a null set such that $\Omega = \bigcup_{\alpha\in\mathrm{A}} \cup \bigcup_{\gamma\in\Gamma} \bigcup_{\beta\in\mathrm{B}(\gamma)}$ and*

$$X = X_1 \boxplus X_2,$$

$$X_1 \simeq_{\mathbb{B}} \Big(\sum_{\alpha\in \mathrm{A}} L^{\infty}\big(\Omega_{\alpha},\, l^1(\alpha)\big)\Big)_{\infty},$$

$$X_2 \simeq_{\mathbb{B}} \sum_{\gamma\in\Gamma}^{\boxplus}\Big(\sum_{\beta\in \mathrm{B}(\gamma)} \beta\diamond\Big(L^{\infty}\big(\Omega_{\beta\gamma},\, L^1([0,1]^{\gamma})\big)\Big)\Big)_{\infty},$$

where Γ *and* $\mathrm{B}(\gamma)$ *are the sets of infinite cardinals.*

Remark 18. If $\mathbb{B}$ satisfy the countable chain condition then $\mathrm{V}^{(\mathbb{B})} \models |\kappa^{\wedge}| = |\lambda^{\wedge}|$ implies $\kappa=\lambda$ for any pair of cardinals κ and λ. The Boolean algebra $\mathbb{B} := \Sigma/\mu^{-1}(0)$ associated with the measure algebra (Ω,Σ,μ), $\mu(\Omega)<\infty$, satisfies the countable chain condition. It follows from these comments that the representation in Theorem 16.2 is essentially unique, contrary to the general case, see Remark 15.

17 Open Problems

A real Banach lattice X is said to be λ-*injective*, if for every Banach lattice Y, closed sublattice $Y_0 \subset Y$, and positive $T_0 : Y_0 \to X$ there exists a positive extension $T : Y \to X$ with $\|T\| \leq \lambda\|T_0\|$. It was proved in [48] that every finite-dimensional λ-injective Banach lattice is lattice isomorphic to $\big(\sum_{j\leq k}^{\oplus} l_1(n_j)\big)_{l_\infty}$, while it was shown in [52] that every order continuous λ-injective Banach lattice is lattice isomorphic to $L_1(\mu)$ space. But the general question, as far as I know, is still open:

Problem 1. *Is every* λ*-injective Banach lattice order isomorphic to* 1*-injective Banach lattice?*

One of the intriguing problems, dating from the work [24], is the classification of the Banach space whose duals are isometric to an *AL*-space, see also [49]. I believe that the injective version of this problem deserves an independent study.

Problem 2. *Classify and characterize the Banach spaces whose duals are injective Banach lattices.*

As is seen from Theorem 10.3 an injective Banach lattice X has a mixed *LM*-structure. Thus, the dual X' should have, in a sense, an *ML*-structure. Hence a natural question arises:

Problem 3. *What kind of duality theory is there for injectives?*

Every Banach space has an injective envelope, see [13, 32]. What is the injective envelope in the category of Banach lattices and lattice isometries? Following [13] we can give the definition: An *injective envelope* of a Banach lattice X is a pair $({}^{\epsilon}X,\iota)$ with ${}^{\epsilon}X$ an injective Banach lattice and $\iota : X \to {}^{\epsilon}X$ a lattice $\mathbb{M}(X)$-isometry such that the only injective sublattice of ${}^{\epsilon}X$ containing $\iota(X)\rangle$ is ${}^{\epsilon}X$ itself, cf. [13] and [38].

Problem 4. *Find a suitable concept of injective envelope of a Banach lattice and clarify whether or not every Banach lattice admits an injective envelope.*

An *orthogonally additive convex modular* [27, § 3.3] on a vector lattice X is an operator $\Theta : X \to \Lambda$ satisfying (for all $x, y \in X$ and $a \in [0, \mathbb{1}]$):

(1) $\Theta(x) = 0 \Longleftrightarrow x = 0$;

(2) $|x| \le |y| \Longrightarrow \Theta(x) \le \Theta(y)$;

(3) $\Theta(ax + (\mathbb{1} - a)y) \le a\Theta(x) + (ax + (\mathbb{1} - a)\Theta(y)$;

(4) $|x| \wedge |y| = 0 \Longrightarrow \Theta(x + y) = \Theta(x) + \Theta(y)$.

Say that an orthogonally additive convex modular $\Theta : X \to \Lambda$ *admits factorization* through injective Banach lattice L, whenever $\Theta = \Phi \circ \theta$ for a strictly positive Maharam operator $\Phi : L \to \Lambda$ with the Levi property and an orthogonally additive (nonlinear) embedding $\theta : X \to L$.

Problem 5. *Find conditions under which an orthogonally additive convex modular admits factorization through injective Banach lattice.*

An *Orlicz* $\mathbb{B}$*-lattice* is a $\mathbb{B}$-cyclic Banach lattice X (cf. [27, § 3.3] if there is an orthogonally additive convex modular $\Theta : X \to \Lambda = \Lambda(\mathbb{B})$ with $\|x\| = \inf\{\alpha > 0 : \Theta(x/\alpha) \le \mathbb{1}\}$ $(x \in X)$.

Problem 6. *Prove a representation theorem for Orlicz* $\mathbb{B}$*-lattices making use of the above representation of injective Banach lattices.*

Say that a downward directed set $A \subset X$ is $\mathbb{B}$-*convergent to zero* if for every $0 < \varepsilon \in \mathbb{R}$ there exists a partition of unity $(\pi_a)_{a \in A}$ in $\mathbb{B}$ such that $\|\pi_a a\| \le \varepsilon$ for all $a \in A$. The norm in X is said to be order $\mathbb{B}$-*continuous* (or X is called order $\mathbb{B}$-*continuous*) if every downward directed set $A \subset X$ with $\inf A = 0$ is $\mathbb{B}$-convergent to zero. If X is a $\mathbb{B}$-cyclic Banach lattice and $\mathscr{X}$ is its Boolean-valued representation then X is order $\mathbb{B}$-continuous if and only if $\mathscr{X}$ is order continuous within $\mathrm{V}^{(\mathbb{B})}$ [38, Theorem 4.5].

Problem 7. *Develop a theory of order* $\mathbb{B}$*-continuous* $\mathbb{B}$*-cyclic Banach lattices.*

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