ISSN 2077–9879

Eurasian Mathematical Journal

2014, Volume 5, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 5, Number 3 (2014), 46 – 57

RATE OF APPROXIMATION BY MODIFIED GAMMA - TAYLOR OPERATORS

A. Izgi

Communicated by E.D. Nursultanov

Key words: approximation, Gamma operators, modulus of continuity in weighted spaces, linear positive operators, Taylor polynomials.

AMS Mathematics Subject Classification: 41A10, 41A25, 41A30, 41A36.

Abstract. In this paper we consider the following modification of the Gamma operators which were first introduced in [8] (see [17], [18] and [8] respectively)

$$A_n(f;\, x) = \int_0^\infty K_n(x,\, t) f(t) dt$$

where

$$K_n(x,\, t) = \frac{(2n+3)!}{n!(n+2)!} \frac{t^n x^{n+3}}{(x+t)^{2n+4}}\,, \qquad x, t \in (0,\, \infty),$$

and the following modified Gamma-Taylor operators

$$A_{n,r}(f;\, x) = \int_0^\infty K_n(x,\, t) \left(\sum_{i=0}^{r} \frac{f^{(i)}(t)}{i!} (x-t)^i \right) dt.$$

We establish some approximation properties of these operators. At the end of the paper we also present some graphs allowing to compare the rate of approximation of f by $A_n(f;\, x)$ and $A_{n,r}(f;\, x)$ for certain n, r and x.

1 Introduction

Recently Izgi and Buyukyazici [8] (see [17], [18] and [8] respectively) defined the following operator

$$A_n(f; x) = \int\limits_0^\infty K_n(x,t) f(t) dt. \tag{1.1}$$

for any f for which the integral is convergent. Here

$$K_n(x,t) = \frac{(2n+3)!}{n!(n+2)!} \frac{x^{n+3} t^n}{(x+t)^{2n+4}},\ \ x, t \in (0, \infty),\ \ n = 1, 2, 3, ...$$

The rate of convergence of these operators for functions with derivatives of bounded variation was studied in [13]. In [12] the rate of pointwise convergence of these operators for functions of bounded variation was studied. In [14] direct local and global approximation results for these operators were obtained. The author of this paper studied these operators for Voronovskaya type asymptotic approximation in [10], also studied these operators for L^p-integrable functions [11] and studied these operators for bivariate functions in weighted spaces with the following operators [9]

$$A_{n,m}(f(u,\,v);x,y)=\int\limits_0^\infty\int\limits_0^\infty K_n(x,u)K_m(y,v)f(u,\,v)dudv.$$

In this paper, we consider new operators by combining modified Gamma operators and Taylor polynomials of r times differentiable functions f in weighted spaces on the interval $(0,\,b_n]$ which expands to $(0,\,\infty)$ when $n\to\infty$. We study the convergence of these new operators. At the end of paper we present some graphs allowing to compare the rate of approximation of f by $A_n(f;\,x)$ and $A_{n,r}(f;\,x)$ for certain $n,\,r$ and x.

Let $C^r(0,\infty)$ denote the set of all functions $f:(0,\infty)\to\mathbb{R}$ wich r $(r=0,1,2,...)$ order derivatives are continuous, where $C^0(0,\infty)\equiv C(0,\infty)$.

For any $f\in C^r(0,\infty)$ and arbitrary $t\in(0,\infty)$ we consider Taylor polynomials of order $r-$

$$T_r(f;\,x)=\sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i. \tag{1.2}$$

If we combine (1.1) and (1.2) ,we obatin

$$A_{n,r}(f;x)=\int\limits_0^\infty K_n(x,t)\sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i dt. \tag{1.3}$$

It is clear that $A_{n,0}(f;x)=A_n(f;x)$ (see [16]).

Note that $A_n(f;x)$ is linear positive operator , $A_{n,r}(f;x)$ is linear but not positive operator for $r\geq1$.

For our aim we use the weighted Korovkin type theorems, proved by A.D. Gadzhiev [4], [5] and we use the same notation as in [4].

Let $\rho(x)=1+x^2$, $x\in(-\infty,\infty)$. B_ρ denotes the set of all functions f defined on the real axis satisfying the condition

$$|f(x)|\leq M_f\rho(x) \tag{1.4}$$

where M_f is a constant depending only on f. B_ρ is a normed space with the norm

$$\|f\|_\rho=\sup_{x\in(-\infty,\,\infty)}\frac{|f(x)|}{\rho(x)},\ \ f\in B_\rho.$$

C_ρ denotes the subspace of all continuous functions in B_ρ and C_ρ^k denotes the subspace of all functions $f\in C_\rho$ for which

$$\lim_{|x|\to\infty}\frac{|f(x)|}{\rho(x)}=K_f<\infty.$$

$B_{\rho,(0,\,a_n]}, C_{\rho,(0,\,a_n]}$ and $C^k_{\rho,(0,\,a_n]}$ are defined as C_ρ , B_ρ and C^k_ρ respectively, only with the domain $(0,\, a_n]$ instead of $\mathbb{R}$ and the norm

$$\|f\|_{\rho,\,(0,\,a_n]} = \sup_{x\in(0,\,a_n]} \frac{|f(x)|}{\rho(x)} .$$

In the sequel it will be assumed that $\lim\limits_{n\to\infty} a_n = \infty$.

2 Auxiliary results

We need the following equalities for $A_n(f;x)$.

Lemma 2.1. ([8]). *For any $n, p \in \mathbb{N}$, $p \le n+2$*

$$A_n(t^p;x) = \frac{(n+p)!(n+2-p)!}{n!(n+2)!}x^p. \tag{2.1}$$

In particular

$$A_n(1;x) = 1, \tag{2.2}$$

$$A_n(t;x) = x - \frac{x}{n+2}\,, \tag{2.3}$$

$$A_n(t^2;x) = x^2 \tag{2.4}$$

$$A_n(t^3;x) = x^3 + \frac{3}{n}x^3, \tag{2.5}$$

$$A_n(t^4;x) = x^4 + \frac{4(2n+3)}{n(n-1)}x^4,\ n>1. \tag{2.6}$$

Remark 1. Several classical positive linear operators, e.g. the Bernstein, Baskakov and Szăsz-Mirakyan operators preserve the functions $e_0(x) = 1$, $e_1(x) = x$ but do not preserve $e_2(x) = x^2$. It has been shown that the error of approximation of f from certain function spaces, by operators which preserve $e_0(x) = 1$ and $e_2(x) = x^2$, is smaller than the operators which do not preserve $e_2(x) = x^2$, see [2, 15, 19].

Let us define

$$T_{n,s}(x) = A_n((t-x)^s;x) \ \text{ and } a_{n,p} = \frac{(n+p)!(n+2-p)!}{n!(n+2)!}.$$

Lemma 2.2.

$$T_{n,s}(x) = \left(\sum_{k=0}^{s}(-1)^k \binom{s}{k} \frac{(n+s-k)!(n+2-s+k)!}{n!(n+2)!}\right)x^s.$$

Proof. Since

$$\begin{aligned} T_{n,s}(x) &= A_n((t-x)^s;x) = \int_0^\infty K_n(x,t)(t-x)^s dt \\ &= \sum_{k=0}^{s}(-1)^k \binom{s}{k} x^k A_n(t^{s-k};x), \end{aligned}$$

the proof follows by Lemma 2.1, p replaced by $(s-k)$. □

The following equalities are obtained from Lemma 2.2:

$$A_n((t-x)^2;x) = \frac{2}{n+2}x^2, \tag{2.7}$$

$$A_n((t-x)^4;x) = \frac{12(n+4)}{(n+2)n(n-1)}x^4, \ n>1 \tag{2.8}$$

$$A_n((t-x)^6;x) = \frac{120(n^2+23n+48)}{n(n-1)(n-2)(n-3)(n+2)}x^6, \ n>3. \tag{2.9}$$

By equalities (2.7)- (2.9) and similar equalities for $A_n((t-x)^{2s};x)$ we get

$$T_{n,2s}(x) \le \alpha_s \frac{x^{2s}}{n^s}, \tag{2.10}$$

where α_s is a constant depending only on s.

Lemma 2.3. *For all n and $0 \le p \le 2$,*

$$a_{n,p} \le 1.$$

For and $p \ge 3$ and $n > p^2 + p - 3$

$$a_{n,p} < e.$$

Proof. It is easy to see $a_{n,p} \le 1$ for $0 \le p \le 2$ and $n > p^2 + p - 3$

$$\begin{aligned} a_{n,p} &= \frac{(n+p)(n+p-1)...(n+4)(n+3)}{n(n-1)...(n-(p-4))(n-(p-3))} \\ &= \left(1+\frac{p}{n}\right)\left(1+\frac{p}{n-1}\right)...\left(1+\frac{p}{n-(p-4)}\right)\left(1+\frac{p}{n-(p-3)}\right) \\ &\le \left(1+\frac{p}{n}\right)^{p-2} < e. \end{aligned}$$

□

Lemma 2.4. *For sufficiently large n satisfying the conditions of Lemma 2.3 the following inequalities hold:*

(*i*)

$$A_n(|t-x|^m;x) \leq \sqrt{\alpha_m}\,\frac{x^m}{n^{m/2}},$$

(*ii*)

$$A_n(|t-x|^m\, t^l;x) \leq \sqrt{\alpha_m\, e}\,\frac{x^{m+l}}{n^{m/2}},$$

(*iii*)

$$A_n(|t-x|^m\,(t-x)^j;x) \leq \sqrt{\alpha_m\,\alpha_j}\,\frac{x^{m+j}}{n^{(m+j)/2}},$$

(*iv*)

$$A_n(|t-x|^m\, t^l(t-x)^j;x) \leq (\alpha_m^2\,\alpha_{2j}e)^{1/4}\,\frac{x^{m+l+j}}{n^{(m+j)/2}},$$

where $l, m, j \in \mathbb{N}$.

Proof. (i) and (iii) follow by the using Hölder$'$s inequality and (2.10). Also by Hölder$'$s inequality, (2.10) and Lemma 2.3

$$\begin{aligned} A_n(|t-x|^m\, t^l;x) &\leq \sqrt{A_n((t-x)^{2m};x)}\sqrt{A_n(t^{2l};x)} \\ &\leq \sqrt{\alpha_m a_{n,2l}}\frac{x^{m+l}}{n^{(m+l)/2}} \leq \sqrt{\alpha_m e}\frac{x^{m+l}}{n^{(m+l)/2}} \end{aligned}$$

and

$$\begin{aligned} A_n(|t-x|^m\, t^l(t-x)^j;x) &\leq \sqrt{A_n((t-x)^{2m};x)}\sqrt{A_n(t^{2l};x)(t-x)^{2j}} \\ &\leq \sqrt{\alpha_m}\frac{x^m}{n^{m/2}}\sqrt{\sqrt{A_n(t^{4l};x)}\sqrt{A_n((t-x)^{4j};x)}} \\ &\leq (\alpha_m\alpha_{2j}e)^{1/4}\frac{x^{m+l+j}}{n^{(m+j)/2}}. \end{aligned}$$

□

3 Approximation of $\mathbf{A}_n(f;x)$ in weighted spaces

Let (b_n) be a sequence with positive terms, $b_{n+1} > b_n$,

$$\lim_{n\to\infty} b_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{b_n^2}{n} = 0\ . \tag{3.1}$$

By using (2.2)-(2.4), we have

$$A_n(\rho(t);x) = \rho(x).$$

Therefore, $\|A_n(f;x)\|_{\rho,(0,b_n]} \leq \|f\|_{\rho,(0,b_n]}$. Hence $\{A_n\}$ is a sequence of linear positive operators taking $C_\rho(0,b_n]$ into $B_\rho(0,b_n]$.

Theorem 3.1. *Let $f \in C_\rho^k(0,\infty)$. Then*

$$\lim_{n\to\infty} \|A_n f - f\|_{\rho,(0,b_n]} = 0.$$

Proof. By using (2.2)-(2.4), we have

$$\lim_{n\to\infty} \|A_n(1;x) - 1\|_{\rho,(0,b_n]} = 0,$$

$$\lim_{n\to\infty} \|A_n(t;x) - x\|_{\rho,(0,b_n]} = \lim_{n\to\infty} \sup_{0<x\le b_n} \frac{1}{n+3}\frac{x}{1+x^2} = 0,$$

$$\lim_{n\to\infty} \left\|A_n(t^2;x) - x^2\right\|_{\rho,(0,b_n]} = 0.$$

According to [5, Theorem B], the proof is completed. □

4 Rate of approximation of $A_n(f;x)$ and $A_{n,r}(f;x)$ in weighted spaces

Now we want to find the rate of approximation of the sequence of linear positive operators $\{A_n\}$ for $f \in C_\rho^k(0,b_n]$ and rate of approximation of linear operators $\{A_{n,r}\}$ for $f \in C^r(0,\infty)$, f, $f^{(r)} \in C_\rho^k(0,b_n]$. It is well known that, the first modulus of continuity

$$\omega(f;\delta) = \sup\{|f(t) - f(x)| : t,x \in [a,b], |t-x| \le \delta\}$$

does not tend to zero, as $\delta \to 0$, on any infinite interval.

In [6] a weighted modulus of continuity $\Omega_n(f;\delta)$ was defined, which tends to zero as $\delta \to 0$ on an infinite interval. A similar definition can be found in [1].

For each $f \in C_\rho^k(0,b_n]$ it is given by

$$\Delta_n(f;\delta) = \sup\left\{\frac{|f(x+h) - f(x)|}{(1+x^2)(1+h^2)} \ : \ |h| \le \delta,\ x \in (0,b_n]\right\}. \tag{4.1}$$

In [6] the following properties of $\Delta_n(f;\delta)$ were shown:

(*i*) $\lim_{\delta\to 0} \Delta_n(f;\delta) = 0$ for every $f \in C_\rho^k(0,b_n]$,

(*ii*) For every $f \in C_\rho^k(0,b_n]$ and $t,x \in (0,b_n]$,

$$|f(t) - f(x)| \le 2(1+\delta_n^2)(1+x^2)\Delta_n(f;\delta_n)S_n(t,x),$$

where $S_n(t,x) = \left(1 + \frac{|t-x|}{\delta_n}\right)(1+(t-x)^2)$.

It is easy to see that:

$$S_n(t,x) \le \begin{cases} 2(1+\delta_n^2), & \text{if } |t-x| \le \delta_n, \\ 2(1+\delta_n^2)\dfrac{(t-x)^4}{\delta_n^4}, & \text{if } |t-x| \ge \delta_n. \end{cases} \tag{4.2}$$

Theorem 4.1. *Let $f \in C_\rho^k(0,b_n]$. Then for all sufficiently large n*

$$\|A_n f - f\|_{\rho,(0,b_n]} \leq 592\Delta_n\left(f;\sqrt{\frac{b_n^2}{n}}\right).$$

Proof. If we use (2.2), then

$$\begin{aligned}|A_n(f;x)-f(x)| &\leq A_n(|f(t)-f(x)|\,;x)\\ &\leq 2(1+\delta_n^2)(1+x^2)\Delta_n(f;\delta_n).A_n(S_n(t,x);x)\end{aligned}$$

By (4.2) we get that $S_n(t,x) \leq 2(1+\delta_n^2)[1+\frac{(t-x)^4}{\delta_n^4}]$ for all $x \in (0,b_n]$, $t \in (0,\infty)$. Thus, for $n \geq 2$, $x \in (0,b_n]$, using (2.8), we get

$$\begin{aligned}|A_n(f;x)-f(x)| &\leq 4(1+\delta_n^2)^2(1+x^2)\left[1+\frac{1}{\delta_n^4}\frac{12(n+4)}{(n+2)n(n-1)}x^4\right]\Delta_n(f;\delta_n)\\ &\leq 4(1+\delta_n^2)^2(1+x^2)\left[1+\frac{36}{\delta_n^4}\frac{b_n^4}{n^2}\right]\Delta_n(f;\delta_n)\end{aligned}$$

Take here $\delta_n = \sqrt{\frac{b_n^2}{n}}$. Since $\lim\limits_{n\to\infty}\frac{b_n^2}{n} = 0$, we have that $\delta_n \leq 1$ for sufficiently large n, and the statement of the theorem follows. □

Remark 2. Theorems this kind were studied for different operators (for instance, for the Szasz-Mirakjian and Baskakov operators (see [6], [7]) for the norm $\|\cdot\|_{\rho^3}$. But in our Theorem we use the norm $\|\cdot\|_\rho$. Thus, Theorem 4.1 gives a better order approximation compared with analogous theorems which were proved in [6], [7].

Before Theorem 4.2 we need the following formula which is called a modified Taylor formula. By Taylor's theorem [20, pages, 391-392] we have

$$\begin{aligned}f(x) &= \sum_{i=0}^{r-1}\frac{f^{(i)}(t)}{i!}(x-t)^i + \int_t^x \frac{f^{(r)}(s)}{(r-1)!}(x-s)^{r-1}ds\\ &= \sum_{i=0}^{r-1}\frac{f^{(i)}(t)}{i!}(x-t)^i + \frac{(x-t)^r}{(r-1)!}\int_0^1\left(\frac{x-s}{x-t}\right)^{r-1}f^{(r)}(s)\frac{ds}{x-t}\end{aligned}$$

Let $s = t+u(x-t)$, then

$$\begin{aligned}f(x) &= \sum_{i=0}^{r-1}\frac{f^{(i)}(t)}{i!}(x-t)^i + \frac{(x-t)^r}{(r-1)!}\int_0^1(1-u)^{r-1}f^{(r)}(t+u(x-t))du\\ &= \sum_{i=0}^{r-1}\frac{f^{(i)}(t)}{i!}(x-t)^i\\ &\quad+\frac{(x-t)^r}{(r-1)!}\int_0^1(1-u)^{r-1}\left(f^{(r)}(t+u(x-t))-f^{(r)}(t)\right)du. \qquad (4.3)\end{aligned}$$

Theorem 4.2. *Let* $f,\ f^{(r)} \in C_\rho^k(0,b_n]$. *Then*

$$\|A_{n,r}f - f\|_{\rho,(0,b_n]} \leq M_r \left(\frac{b_n^2}{n}\right)^{r/2} \Delta_n\left(f^{(r)}; \sqrt{\frac{b_n^2}{n}}\right)$$

for all sufficiently large n*, where* M_r *depends only on* r $(r = 0, 1, 2, ...)$.

Proof. Using modified Taylor's formula (4.3) and equalities (2.2) and (1.3), we obtain

$$|A_{n,r}(f;x) - f(x)| \leq \int_0^\infty K_n(x,t)\frac{|x-t|}{(r-1)!}\Phi(r,t)dt$$

where $\Phi(r,t) = \int_0^1 (1-u)^{r-1} \left|f^{(r)}(t+u(x-t)) - f^{(r)}(t)\right| du$

Applying property (ii) of $\Delta_n(f,\delta)$, we get

$$\left|f^{(r)}(t+u(x-t)) - f^{(r)}(t)\right| \leq 2(1+\delta_n^2)(1+t^2)S_n(x,u,t)\Omega_n(f^{(r)},\delta_n),$$

where $x \in (0,b_n],\ t \in (0,\infty),\ u \in [0,1]$ and

$$S_n(x,u,t) = \left(1 + \frac{u}{\delta_n}|t-x|\right)\left(1+u^2(t-x)^2\right).$$

It is easy to see that

$$S_n(x,u,t) \leq \begin{cases} (1+u)(1+u^2\delta_n^2), & if\ \ |t-x| \leq \delta_n \\ (1+u)(1+u^2\delta_n^2)\dfrac{(t-x)^4}{\delta_n^4}, & if\ \ |t-x| \geq \delta_n \end{cases}$$

So, for all $x \in (0,b_n],\ t \in (0,\infty)$, and $u \in [0,1]$

$$S_n(x,u,t) \leq (1+u)(1+u^2\delta_n^2)\left[1 + \frac{(t-x)^4}{\delta_n^4}\right].$$

Thus,

$$|A_{n,r}(f;x) - f(x)| \leq M(r,\delta_n)\Delta_n(f^{(r)},\delta_n) \times A_n((1+t^2)|t-x|^r\left[1+\frac{(t-x)^4}{\delta_n^4}\right];x)$$

where $M(r,\delta_n) = 2(1+\delta_n^2)(\frac{1}{r!} + \frac{1}{(r+1)!} + \frac{\delta_n^2}{(r+2)!} + \frac{\delta_n^3}{(r+3)!})$.

Using Lemma 2.4, we get

$$\begin{aligned} |A_{n,r}(f;x) - f(x)| &\leq M(r,\delta_n)\Delta_n(f^{(r)},\delta_n) \times \left\{\sqrt{\alpha_r}\frac{x^r}{n^{r/2}} + \sqrt{\alpha_r e}\frac{x^{r+2}}{n^{r/2}} \right. \\ &\quad \left. + \frac{\sqrt{\alpha_r\alpha_4}}{\delta_n^4}\frac{x^{r+4}}{n^{(r+4)/2}} + \frac{(\alpha_r^2\alpha_8 e)^{1/2}}{\delta_n^4}\frac{x^{r+6}}{n^{(r+4)/2}}\right\} \\ &\leq M(r,\delta_n)\Delta_n(f^{(r)},\delta_n)(1+x^2) \times \left\{\sqrt{\alpha_r}\frac{x^r}{n^{r/2}} + \sqrt{\alpha_r e}\frac{x^r}{n^{r/2}} \right. \\ &\quad \left. + \frac{\sqrt{\alpha_r\alpha_4}}{\delta_n^4}\frac{x^{r+4}}{n^{(r+4)/2}} + \frac{(\alpha_r^2\alpha_8 e)^{1/2}}{\delta_n^4}\frac{x^{r+4}}{n^{(r+4)/2}}\right\} \end{aligned}$$

Thus, we have

$$\sup_{0<x\leq b_n} \frac{|A_{n,r}(f;x)-f(x)|}{1+x^2} \leq M(r,\delta_n)\Delta_n(f^{(r)},\delta_n)\left(\frac{b_n^2}{n}\right)^{r/2}\left\{A_r+\frac{B_r}{\delta_n^4}\left(\frac{b_n^2}{n}\right)^2\right\},$$

where $A_r=\sqrt{\alpha_r}+\sqrt{\alpha_r e}$, $B_r=\sqrt{\alpha_r\alpha_4}+(\alpha_r^2\alpha_8 e)^{1/2}$. Choosing $\delta_n=\sqrt{\frac{b_n^2}{n}}$ and taking into account that $\frac{b_n^2}{n}\leq 1$ for sufficiently large n, since $\lim_{n\to\infty}\frac{b_n^2}{n}=0$, we obtain

$M(r,\delta_n)\leq 4\left(\sum_{i=0}^{3}\frac{1}{(r+i)!}\right):=\beta_r$ and

$$\|A_{n,r}f-f\|_{\rho,(0,b_n]}\leq \beta_r(A_r+B_r)\left(\frac{b_n^2}{n}\right)^{r/2}\Delta_n\left(f^{(r)},\sqrt{\frac{b_n^2}{n}}\right).$$

Hence, the statement of the theorem follows with $M_r=\beta_r(A_r+B_r)$. □

Example 1. We compare the graphs of $A_{n,r}(f;x)$ and $A_n(f;x)$ where

$f(x)=x^{5/2}(1+\sin(x/1.07))$ for $n=10, r=1,\ 2$ (Fig. 1) and $n=50, r=1,2$ (Fig. 2).

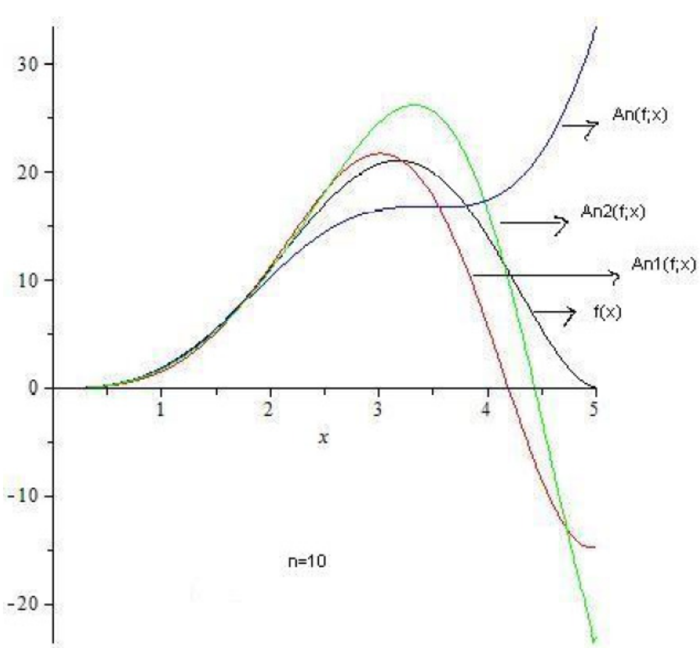

Fig.1

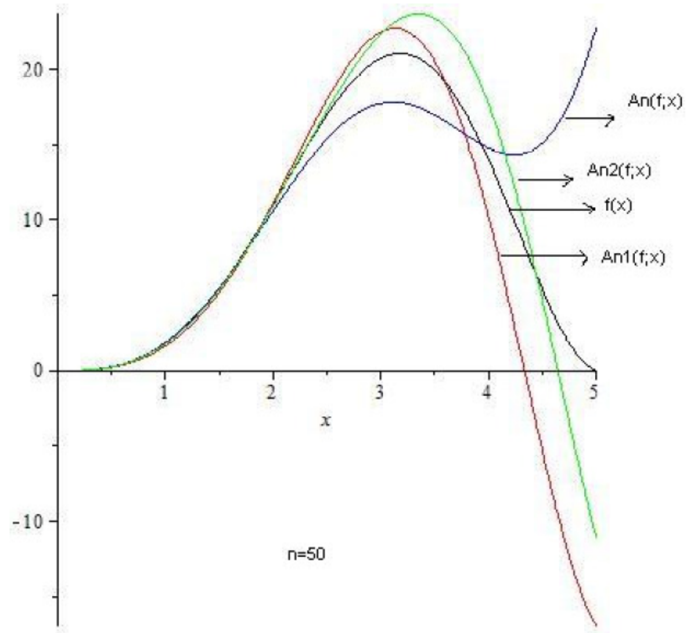

Fig. 2

References

[1] N.I. Akhieser, *Lectures on the theory of approximation,* OGIZ, Moscow-Leningrad, 1947(in Russian), *Theory of approximation,* (in English). Translated by Hymann, Frederick Ungar Publishing Co., New York, 1967, 208-226.

[2] O. Agratini, *On class of linear positive bivariate operators of King,* Studia Univ. "Babeş-Bolyai", Mathematica. L1(1) (2006), no. 4, 13-22.

[3] C. Brezinski, *Convergence acceleration during 20-th century,* J. Comput. Appl. Math. 122 (2000), 1-21.

[4] A.D. Gadzhiev, *Theorems of the type of P.P. Korovkin theorems,* Matem. Zametki. 20 (1976), no. 5, 781-786, (in Russian). Engl. translation in Math. Notes 20 (1976), no. 5-6, 996-998.

[5] A.D. Gadzhiev, I. Efendiev, E. Ibikli, *Generalized Bernstein-Chlodowsky polynomials,* Rocky Mt. J. Math. 28 (1988), no. 4, 1267-1277.

[6] N. Ispir, *On modified Baskakov operators on weighted spaces,* Turk. J. Math. 26 (2001), no. 3, 355-365.

[7] N. Ispir, C. Atakut, *Approximation by modified Szass-Mirakjan operators on weighted spaces,* Proc. Indian. Ac. Sci. (Math.Sci.) 112 (2002), no. 4, 571-578.

[8] A. Izgi, I. Buyukyazici, *Approximation on unbounded interval and order of approximation,* (in Turkish). Kastamonu Eğitim dergisi, Cilt II (2003), no. 2, 451-456.

[9] A. Izgi, *Order of approximation of functions in two variables by new type Gamma operators,* General Mathematics. 17 (2009), no. 1, 23-32.

[10] A. Izgi, *Voronovskaya type asymptotic approximation by modified Gamma operators,* Appl. Math. Comp. 217 (2011), 8061-8067.

[11] A. Izgi, *Approximation of L^p-integrable functions by Gamma type operators,* J. Anal. Comp. (submitted).

[12] H. Karslı, V. Gupta, A. Izgi, *Rate of pointwise convergence of a new kind of Gamma operators for functions of bounded variation,* Appl. Math. Letters. 22 (2009), no. 4, 505-510.

[13] H. Karslı, *Rate of convergence of new Gamma type operators for functions with derivatives of bounded variation,* Math. Comp. Modelling. 45 (2007), 617-624.

[14] H. Karslı M.A. Özarslan, *Direct local and global approximation results for operators of Gamma type,* Hacettepe J. Math. Stat. 39 (2010), no. 2, 241 253.

[15] J.P. King, *Positive linear operators which preserve x^2*, Acta Math. Hungar. 99 (2003) no. 3, 203-208.

[16] G.H. Kirov, *A generalization of the Bernstein polynomials,* Math. Balkanica (New Ser.) 6 (1992), no. 2, 147-153.

[17] A. Lupas, M.Muller, *Approximation Seigenschaften der Gamma operatören,* Math. Zeit. 98 (1967), 208-226 (in German).

[18] S.M. Mazhar, *Approximation by positive operators on infinite and finite intervals,* Mathematica Balkanica. 5 (1991), no.2, 99-104.

[19] L. Rempulska, K. Tomczzak, *Approximation by certain linear operators preserving x^2*, Turk. J. Math. 32 (2008), 1-11.

[20] M. Spivak, *Calculus,* Second Edition “Publish or Perish. Inc.”, 1980.

[21] B.S. Theodore, Ho Hsu, *A non-linear transformation for sequences and integrals,* Thesis master science, Graduate Faculty of the Texas Thechnological College, 1968. M. Sci. thesis

Aydın Izgi
Department of Mathematics
Harran University, Science and Arts Faculty
Osmanbey Kampüsü
63300-Ş.Urfa / Turkey
E-mail: aydinizgi@yahoo.com

Received: 31.08.2012