ISSN 2077–9879

Eurasian Mathematical Journal

2014, Volume 5, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 5, Number 3 (2014), 9 – 45

DECOMPOSITIONS FOR LOCAL MORREY SPACES

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Communicated by V.I. Burenkov

Key words: local Morrey spaces, atomic decomposition, molecular decomposition.

AMS Mathematics Subject Classification: 41A17, 42B25, 42B35.

Abstract. We develop and apply a decomposition theory for generic local Morrey spaces. Our results are smooth and nonsmooth decompositions, which follows from the fact that local Morrey spaces are isomorphic to local Hardy-Morrey spaces and local Triebel-Lizorkin-Morrey spaces in the generic case. As an application of our results, we consider a bilinear estimate for the fractional integral operators.

1 Introduction

In 1938, C. Morrey considered an estimate for the solutions of partial differential equations. His result grew up to an embedding result, which is nowadays called Morrey's lemma [18]. Later, in 1969, Peetre proposed to consider that lemma from the viewpoint of functonal analysis. In fact, from Morrey's lemma we are led to consider some important normed spaces, which are called Morrey spaces [27].

In 1975, D. Adams established that Morrey spaces can describe the boundedness property of Riesz potentials. Recently, the behaviour of Riesz potentials has been studied by the use of the local Morrey spaces. The definition of local Morrey spaces is as follows. Here and in the sequel we write $B(r) = \{|y| < r\}$ for $r > 0$. Let $1 < p < \infty$ and $0 \le \lambda < n$. For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ one defines the norm $\|f\|_{LM_{p,\lambda}}$ by:

$$\|f\|_{LM_{p,\lambda}} \equiv \sup_{r>0} \left(\frac{1}{r^\lambda} \int_{B(r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

One defines the space $LM_{p,\lambda}(\mathbb{R}^n)$ as the set of all measurable functions f for which the norm $\|f\|_{LM_{p,\lambda}}$ is finite. We also denote by $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^n)$ the set of all cubes whose axes are parallel to the coordinate axes. The indicator function of a set E is denoted by χ_E.

In this paper, we shall establish and apply the following three theorems. Two of them are related to non-smooth synthesis and decomposition.

Theorem 1.1. *Let $1 < p < q < \infty$ and $\lambda, \rho \in [0, n)$ satisfy*

$$\frac{n-\lambda}{p} > \frac{n-\rho}{q}. \tag{1.1}$$

Assume that $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty} \subset LM_{q,\rho}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ *and*

$$\|a_j\|_{LM_{q,\rho}} \leq \|\chi_{Q_j}\|_{LM_{q,\rho}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{LM_{p,\lambda}} < \infty. \tag{1.2}$$

Then the series $\sum_{j=1}^{\infty} \lambda_j a_j$ *converges in the Schwartz space* $\mathcal{S}'(\mathbb{R}^n)$ *of tempered distributions and in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and satisfies the estimate*

$$\left\|\sum_{j=1}^{\infty} \lambda_j a_j\right\|_{LM_{p,\lambda}} \leq C \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{LM_{p,\lambda}}, \tag{1.3}$$

where $C > 0$ *depends only on* n, p, q, λ *and* ρ.

Theorem 1.2. *Let* $L \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $1 < p < \infty$ *and* $\lambda \geq 0$. *Let* $f \in LM_{p,\lambda}(\mathbb{R}^n)$. *Then there exists a triplet* $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$, $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ *and* $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n)$ *such that* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$|a_j| \leq \chi_{Q_j}, \quad \int_{\mathbb{R}^n} x^{\alpha} a_j(x)\, dx = 0, \tag{1.4}$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *with* $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq L$, *where* $x^{\alpha} = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, *and for all* $v > 0$

$$\left\|\left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v}\right)^{1/v}\right\|_{LM_{p,\lambda}} \leq C_v \|f\|_{LM_{p,\lambda}}. \tag{1.5}$$

Here the constant $C_v > 0$ *is independent of* f.

To formulate the third decomposition result, we recall first the definition of atoms and molecules.

Definition 1.1 (Dyadic cubes). *For* $\nu \in \mathbb{Z}$ *and* $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, *define* $Q_{\nu,m} = \prod_{j=1}^{n} [2^{-\nu} m_j, 2^{-\nu} m_j + 2^{-\nu})$. *The cube* $Q_{\nu,m}$ *with* $\nu \in \mathbb{Z}$ *and* $m \in \mathbb{Z}^n$ *is called a dyadic cube of generation* m. *The collection of all such dyadic cubes is denoted by* $\mathcal{D}$. *The collection* $\mathcal{D}_{\nu}$ *is defined as the set of all dyadic cubes of the form* $Q_{\nu,m}$ *with* $m \in \mathbb{Z}^n$.

Definition 1.2 (Atom). *A* C^K*-function* a *is called an* (s,p)*-atom,* $s \in \mathbb{R}, 0 < p < \infty$, *if the following support, cancellation and smoothness conditions are satisfied for some cube* $Q \in \mathcal{Q}$:

1. $\mathrm{supp}(a) \subset 3Q$,
2. $\int_{\mathbb{R}^n} x^{\alpha} a(x)\, dx = 0$ for $|\alpha| \leq L$,
3. $\|\partial^{\alpha} a\|_{L_{\infty}} \leq |Q|^{s - \frac{n}{p} - |\alpha|}$ *if* $|\alpha| \leq K$.

Now we define a molecule. To this end we use the following notation:

$$\langle x\rangle \equiv \sqrt{1+|x|^2}$$

for $x \in \mathbb{R}^n$.

Definition 1.3 (Molecule). *A C^K-function m is called an (s,p)-molecule, $s \in \mathbb{R}$, $0 < p \le \infty$, if the following oscillation and decay conditions hold for some point $x_0 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}$, where M is a sufficiently large constant:*

1. $\displaystyle\int_{\mathbb{R}^n} x^\alpha m(x)\,dx = 0$ for $|\alpha| \le L$,
2. $|\partial^\alpha m(x)| \le 2^{-\nu(s-n/p)+\nu|\alpha|}\langle 2^\nu(x-x_0)\rangle^{-M-|\alpha|}$ if $|\alpha| \le K$.

Here and below we call m a molecule centered at $Q_{\nu,m}$, if $x_0 = 2^{-\nu}m$.

We call the set $\chi_Q^{(p)} = |Q|^{-\frac{1}{p}}\chi_Q$ a p-normalized indicator.

With these definitions in mind, let us formulate the third theorem. In (1.6), it is understood that

$$\sum_{Q\in\mathcal{D}} s_Q a_Q = \sum_{\nu=-\infty}^{\infty}\left(\sum_{m\in\mathbb{Z}^n} s_{Q_{\nu,m}} a_{Q_{\nu,m}}\right)$$

and the convergence takes place in $\mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.3 (Smooth atomic decomposition). *Suppose that $1 < p < \infty$, $\lambda \in [0,n)$, $K \in \mathbb{N}$ and $L \in \mathbb{N}$.*

(A) *Any function $f \in LM_{p,\lambda}(\mathbb{R}^n)$ admits the following decomposition: for each dyadic cube Q there exists a $(0,p)$-molecule m_Q centered at Q and a complex number s_Q such that*

$$f = \sum_{Q\in\mathcal{D}} s_Q m_Q \tag{1.6}$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$ and that the coefficients $\{s_Q\}$ depend linearly on f. Moreover, they can be chosen in such a way that

$$\|f\|_{LM_{p,\lambda}} \simeq \left\|\left(\sum_Q |s_Q\chi_Q^{(p)}|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}}, \tag{1.7}$$

that is,

$$C^{-1}\|f\|_{LM_{p,\lambda}} \le \left\|\left(\sum_Q |s_Q\chi_Q^{(p)}|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}} \le C\|f\|_{LM_{p,\lambda}},$$

where the constant C depends only on p and λ.

(B) *Conversely, if the right-hand side of (1.7) is finite, then*

$$\sum_{Q\in\mathcal{D}} s_Q a_Q, \quad \sum_{Q\in\mathcal{D}} s_Q m_Q \in LM_{p,\lambda}(\mathbb{R}^n),$$

where the a_Q are $(0,p)$-atoms and the m_Q are $(0,p)$-molecules.

Theorem 1.5. [9, Theorem 1.2] *Suppose that the real parameters p, λ, L satisfy*

$$1 < p < \infty, \quad 0 \le \lambda < n, \quad L \in \mathbb{N}_0.$$

Let $f \in GM_{p,\lambda}(\mathbb{R}^n)$. Then there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, for all multi-indices α with $|\alpha| \le L$ condition (1.4) is satisfied

$$\left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{GM_{p,\lambda}} \le C_v \|f\|_{GM_{p,\lambda}}$$

and that for all $v > 0$. Here the constant $C_v > 0$ is independent of f.

We describe how we organize the remaining part of this paper. Theorems 1.1 and 1.2 are proved in Section 7. We prove Theorem 1.3 in Section 8. Section 2 is devoted to preliminary facts. Section 3 is a virtual starting point, where we consider the maximal inequality, and extend the result to the vector-valued version. In Section 4, we consider the boundedness property of fundamental operators. In Section 5, we investigate the structure of the local Morrey spaces from the viewpoint of functional analysis. We are mainly interested in predual spaces; our method heavily depends on [52]. We shall specify the predual space of the local Morrey space $LM_{p,\lambda}(\mathbb{R}^n)$ for $1 < p < \infty$ and $\lambda \ge 0$, and use this observation in Section 6. Section 6 is a characterization of the local Morrey space $LM_{p,\lambda}(\mathbb{R}^n)$ for $1 < p < \infty$ and $\lambda \ge 0$. We also consider an embedding result in Section 7. Lemma 7.1 is a key estimate. Theorem 1.1 and Lemma 7.1, proofs of which can be found in Section 7, require an approach different from [23, 24]. Smooth decompositions are dealt in Section 8.

2 Preliminaries

2.1 Structure of local Morrey spaces

Since we will have to compare spaces $LM_{p,\lambda}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ in Section 6, we start with the following observation. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of test functions; see Definition 2.1 below.

Lemma 2.1. *Let $1 < p < \infty$ and $\lambda \ge 0$. Then for all $\kappa \in \mathcal{S}(\mathbb{R}^n)$ and $f \in LM_{p,\lambda}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |\kappa(x) f(x)| \, dx \le C \|f\|_{LM_{p,\lambda}} \sup_{x \in \mathbb{R}^n} (1+|x|)^{2n+\lambda+1} |\kappa(x)|, \tag{2.1}$$

where C does not depend on κ and f.

Proof. We decompose the left-hand side as follows:

$$\begin{aligned}\int_{\mathbb{R}^n}|\kappa(x)f(x)|dx &\le \int_{B(1)}|\kappa(x)f(x)|dx+\sum_{j=1}^{\infty}\int_{B(j+1)\setminus B(j)}|\kappa(x)f(x)|dx\\ &\le \|\kappa\|_{L^\infty(B(1))}\|f\|_{L^1(B(1))}\\ &\quad+\sum_{j=1}^{\infty}\frac{1}{j^{2n+\lambda+1}}\int\limits_{B(j+1)\setminus B(j)}|x|^{2n+\lambda+1}|\kappa(x)||f(x)|dx\\ &\le C\|f\|_{LM_{p,\lambda}}\left(\sup_{x\in\mathbb{R}^n}(1+|x|)^{2n+\lambda+1}|\kappa(x)|\right),\end{aligned}$$

where C depends only on n, p and λ. □

2.2 A convolution estimate

We shall need the following lemma for convolutions in Section 6.

Proposition 2.1. *Suppose that ρ is a positive decreasing function on $[0,\infty)$ and that*

$$\tau(x)=\rho(|x|)\text{ for all }x\in\mathbb{R}^n. \tag{2.2}$$

Then for all $t>0$ and $x\in\mathbb{R}^n$

$$\left[t^{-n}\tau(t^{-1}\cdot)*|f|\right](x)\le\|\tau\|_{L^1}Mf(x) \tag{2.3}$$

for all $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

2.3 Grand maximal estimate

Our idea is to convert the norm of $LM_{p,\lambda}(\mathbb{R}^n)$ to one of Hardy type. To this end, we recall the definition of the grand maximal function $\mathcal{M}f$ as well as the topology of $\mathcal{S}(\mathbb{R}^n)$.

Definition 2.1. 1. *The topology on $\mathcal{S}(\mathbb{R}^n)$ is defined by the norms $\{\rho_N\}_{N\in\mathbb{N}}$ where*

$$\rho_N(\varphi)\equiv\sum_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}(1+|x|)^N|\partial^\alpha\varphi(x)|\quad(\varphi\in\mathcal{S}(\mathbb{R}^n)).$$

Define $\mathcal{F}_N\equiv\{\varphi\in\mathcal{S}(\mathbb{R}^n):\rho_N(\varphi)\le1\}$ for $N\in\mathbb{N}_0$.

2. *The space $\mathcal{S}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}(\mathbb{R}^n)$.*
3. *Let $f\in\mathcal{S}'(\mathbb{R}^n)$. The grand maximal operator $\mathcal{M}$ is defined by*

$$\mathcal{M}f(x)=\mathcal{M}_Nf(x)\equiv\sup\{|t^{-n}\varphi(t^{-1}\cdot)*f(x)|:t>0,\varphi\in\mathcal{F}_N\}$$

for all $x\in\mathbb{R}^n$.

We recall the following lemma, which will be key to this paper. We refer to [39] for the proof. By $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, we denote the set of all compactly supported infinitely continously differentiable functions in $\mathbb{R}^n$. The set of all polynomials of degree less than or equal to d is denoted by $\mathcal{P}_d(\mathbb{R}^n)$.

Lemma 2.2. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $d \in \mathbb{N}_0$ and $j \in \mathbb{Z}$. Then there exist an index set K_j, collections of cubes $\{Q_{j,k}\}_{k \in K_j}$ and functions $\{\eta_{j,k}\}_{k \in K_j} \subset C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, which are all indexed by K_j for every j, and a decomposition*

$$f = g_j + b_j, \quad b_j = \sum_{k \in K_j} b_{j,k},$$

such that the following properties hold.

1) $g_j, b_j, b_{j,k} \in \mathcal{S}'(\mathbb{R}^n)$.
2) *Define $\mathcal{O}_j \equiv \{y \in \mathbb{R}^n : \mathcal{M}f(y) > 2^j\}$ and consider its Whitney decomposition. Then the cubes $\{200\, Q_{j,k}\}_{k \in K_j}$ have the bounded intersection property, and*

$$\mathcal{O}_j = \bigcup_{k \in K_j} Q_{j,k} = \bigcup_{k \in K_j} 200\, Q_{j,k}. \tag{2.4}$$

3) *Consider the partition of unity $\{\eta_{j,k}\}_{k \in K_j}$ with respect to $\{Q_{j,k}\}_{k \in K_j}$. Then each function $\eta_{j,k}$ is supported in $Q_{j,k}$ and*

$$\sum_{k \in K_j} \eta_{j,k} = \chi_{\{y \in \mathbb{R}^n \,:\, \mathcal{M}f(y) > 2^j\}}, \quad 0 \le \eta_{j,k} \le 1.$$

4) *The distribution g_j satisfies the inequality:*

$$\mathcal{M}g_j(x) \le C_1 \left(\mathcal{M}f(x)\chi_{\mathcal{O}_j{}^c}(x) + 2^j \sum_{k \in K_j} \frac{{\ell_{j,k}}^{n+d+1}}{(\ell_{j,k} + |x - x_{j,k}|)^{n+d+1}} \right) \tag{2.5}$$

for all $x \in \mathbb{R}^n$.

5) *Each distribution $b_{j,k}$ is given by $b_{j,k} = (f - c_{j,k})\eta_{j,k}$ with a certain polynomial $c_{j,k} \in \mathcal{P}_d(\mathbb{R}^n)$ satisfying*

$$\langle f - c_{j,k}, \eta \cdot P \rangle = 0 \text{for all } q \in \mathcal{P}_d(\mathbb{R}^n),$$

and

$$\mathcal{M}b_{j,k}(x) \le C_2 \left(\mathcal{M}f(x)\chi_{Q_{j,k}}(x) + 2^j \cdot \frac{{\ell_{j,k}}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}} \chi_{\mathbb{R}^n \setminus Q_{j,k}}(x) \right) \tag{2.6}$$

for all $x \in \mathbb{R}^n$.

In the above, $x_{j,k}$ and $\ell_{j,k}$ denote the center and the edge-length of $Q_{j,k}$, respectively, and C_1 and C_2 depend only on n.

Furthermore, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then g is an $L^\infty(\mathbb{R}^n)$ function whose norm is less than or equal to 2^{-j}.

2.4 Plancherel-Polya-Nikol'skii inequality

Recall that, for a measurable function f and $r > 0$, the maximal operator $M^{(r)}$ is defined by

$$M^{(r)}f(x) = \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^r \, dy \right)^{\frac{1}{r}}.$$

Denote by $(\mathcal{S}'(\mathbb{R}^n))^{\overline{B(x_0,1)}}$ the Schwartz distributions f whose Fourier transform $\mathcal{F}f$ is supported in $\overline{B(x_0,1)}$.

The following theorem will be used in Section 8; we refer to [43] for the proof.

Theorem 2.1. *Let $f \in (\mathcal{S}'(\mathbb{R}^n))^{B(x_0,1)}$. Then*

$$\sup_{y \in \mathbb{R}^n} \frac{|\nabla f(x-y)|}{1+|y|^{\frac{n}{r}}} \le C \sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}}, \tag{2.7}$$

$$\sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}} \le C M^{(r)} f(x) \tag{2.8}$$

for all $x \in \mathbb{R}^n$.

2.5 Moment condition

We need the following estimate, whose proof can be found in [3, p. 466].

Lemma 2.3. *Let $\nu, \mu \in \mathbb{Z}$ with $\nu \ge \mu$, $M > 0$ and $L \in \mathbb{N}_0$, and $N > M + L + n$. Suppose that a $C^L(\mathbb{R}^n)$-function φ and x_φ are such that*

$$|\nabla^L \varphi(x)| \le \frac{2^{\mu(n+L)}}{(1+2^\mu|x-x_\varphi|)^M}$$

for all $x \in \mathbb{R}^n$. Assume, in addition, that ψ is a measurable function such that

$$\int_{\mathbb{R}^n} x^\beta \psi(x) \, dx = 0, \text{ if } |\beta| \le L-1$$

and that, for some $x_\psi \in \mathbb{R}^n$,

$$|\psi(x)| \le \frac{2^{\nu n}}{(1+2^\nu|x-x_\psi|)^N}$$

for all $x \in \mathbb{R}^n$. Then

$$\left| \int_{\mathbb{R}^n} \varphi(x)\psi(x) \, dx \right| \le C \frac{2^{\mu n - (\nu-\mu)L}}{(1+2^\mu|x_\varphi - x_\psi|)^M}.$$

2.6 Rademacher functions

The next lemma is useful when we consider the $\ell^2(\mathbb{Z})$-norm.

Lemma 2.4. *Let $0 < p < \infty$. Define $\tilde{r}_k(t) \equiv (-1)^{[2^k t]}$ for $k \in \mathbb{N}$, $t \in [0,1]$. Rearrange $\{\tilde{r}_k\}_{k=1}^\infty$ to have $\{r_j\}_{j=-\infty}^\infty$. Then for any $\ell^2(\mathbb{Z})$-complex sequences $\{a_j\}_{j=-\infty}^\infty$, we have*

$$\left\| \sum_{j=-\infty}^{\infty} a_j r_j \right\|_{L^p[0,1]} \simeq \left(\sum_{j=-\infty}^{\infty} |a_j|^2 \right)^{\frac{1}{2}}.$$

See [48] for the proof.

3 Maximal inequalities

In this section, we consider the Hardy-Littlewood maximal operator M. Recall that M is defined for measurable functions f by the formula

$$Mf(x) = \sup_{B \in \mathcal{B}_x} \frac{1}{|B|} \int_B |f(y)|\, dy \quad (x \in \mathbb{R}^n),$$

where $\mathcal{B}_x$ denotes the set of all balls containing the point x.

The aim of this section is to extend the well-known inequalities

$$\int_{\mathbb{R}^n} Mf(x)^p\, dx \le c_{p,n} \int_{\mathbb{R}^n} |f(x)|^p\, dx \tag{3.1}$$

and

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} Mf_j(x)^q \right)^{\frac{p}{q}} dx \le c_{p,q,n} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{p}{q}} dx, \tag{3.2}$$

where $c_{p,n}$ and $c_{p,q,n}$ are independent of f and $f_j, j \in \mathbb{N}$, respectively. Here the parameters p and q satisfy $1 < p, q < \infty$. When $1 < p < q = \infty$, we have a counterpart to (3.2);

$$\int_{\mathbb{R}^n} \left(M \left[\sup_{j \in \mathbb{N}} |f_j| \right](x) \right)^p dx \le c_{p,n} \int_{\mathbb{R}^n} \left(\sup_{j \in \mathbb{N}} |f_j(x)| \right)^p dx. \tag{3.3}$$

Note that (3.3) is a direct consequence of (3.1) and the pointwise estimate

$$M \left[\sup_{j \in \mathbb{N}} |f_j| \right](x) \le \sup_{j \in \mathbb{N}} |f_j(x)|. \tag{3.4}$$

Our main result in this section is as follows:

Theorem 3.1. *Let $1 < p < \infty$, $1 < q < \infty$ and $0 \le \lambda < n$. Then we have*

$$\|Mf\|_{LM_{p,\lambda}} \le C \|f\|_{LM_{p,\lambda}} \tag{3.5}$$

and

$$\left\| \left(\sum_{j=1}^{\infty} (Mf_j)^q \right)^{\frac{1}{q}} \right\|_{LM_{p,\lambda}} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{LM_{p,\lambda}}. \tag{3.6}$$

Here, the constant C in (3.5) depends only on p, and n and the one in (3.6) depends only on p, q and n. In particular,

$$\left\| M\left[\sup_{j\in\mathbb{N}}|f_j|\right]\right\|_{LM_{p,\lambda}} \le C\left\|\sup_{j\in\mathbb{N}}|f_j|\right\|_{LM_{p,\lambda}}. \tag{3.7}$$

Proof. Analogous to (3.3), we can deduce (3.7) by using (3.4) and (3.5). By setting $f_1 = f, f_2 = f_3 = \cdots = 0$ in (3.6), we can obtain (3.5). Hence, we concentrate on proving (3.6).

Estimate (3.6) leads to the proof of the inequality;

$$\left(\frac{1}{r^\lambda}\int_{B(r)}\left(\sum_{j=1}^\infty Mf_j(y)^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}} \le C\left\|\left(\sum_{j=1}^\infty |f_j|^q\right)^{\frac{1}{q}}\right\|_{LM_{p,\lambda}}, \tag{3.8}$$

where the constant $C > 0$ depends on p, q, λ and n, but not on r and f_j.

We define $f_{j,1} \equiv f_j\chi_{B(5r)}$ and $f_{j,2} \equiv f_j - f_{j,1}$. Then we can decompose estimate (3.8) into two parts:

$$\left(\frac{1}{r^\lambda}\int_{B(r)}\left(\sum_{j=1}^\infty Mf_{j,1}(y)^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}} \le C\left\|\left(\sum_{j=1}^\infty |f_j|^q\right)^{\frac{1}{q}}\right\|_{LM_{p,\lambda}}, \tag{3.9}$$

$$\left(\frac{1}{r^\lambda}\int_{B(r)}\left(\sum_{j=1}^\infty Mf_{j,2}(y)^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}} \le C\left\|\left(\sum_{j=1}^\infty |f_j|^q\right)^{\frac{1}{q}}\right\|_{LM_{p,\lambda}}. \tag{3.10}$$

Estimate (3.9) follows from (3.2). Indeed, by (3.2), we obtain

$$\begin{aligned}\left(\frac{1}{r^\lambda}\int_{B(r)}\left(\sum_{j=1}^\infty Mf_{j,1}(y)^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}} &\le \left(\frac{1}{r^\lambda}\int_{\mathbb{R}^n}\left(\sum_{j=1}^\infty Mf_{j,1}(y)^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}}\\ &\le (c_{p,q,n})^{\frac{1}{p}}\left(\frac{1}{r^\lambda}\int_{\mathbb{R}^n}\left(\sum_{j=1}^\infty |f_{j,1}(y)|^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}}.\end{aligned}$$

If we use the definition of $f_{j,1}$, then we obtain

$$\begin{aligned}\left(\frac{1}{r^\lambda}\int_{\mathbb{R}^n}\left(\sum_{j=1}^\infty |f_{j,1}(y)|^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}} &= \left(\frac{1}{r^\lambda}\int_{B(5r)}\left(\sum_{j=1}^\infty |f_{j,1}(y)|^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}}\\ &= 5^{\frac{n}{p}}\left(\frac{1}{(5r)^\lambda}\int_{B(5r)}\left(\sum_{j=1}^\infty |f_{j,1}(y)|^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}}.\end{aligned}$$

In terms of the local Morrey norm, we conclude;

$$\left(\frac{1}{r^\lambda}\int_{B(r)}\left(\sum_{j=1}^\infty Mf_{j,1}(y)^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}} \le (5^n c_{p,q,n})^{\frac{1}{p}}\left\|\left(\sum_{j=1}^\infty |f_j|^q\right)^{\frac{1}{q}}\right\|_{LM_{p,\lambda}}.$$

As for (3.10), we need the following pointwise estimate:

$$Mf_{j,2}(x) = \sup_{B\in\mathcal{B}_x}\frac{1}{|B|}\int_{B\setminus B(5r)}|f_j(y)|\,dy \le 3^n \sup_{R>2r}\frac{1}{|B(R)|}\int_{B(R)}|f_j(y)|\,dy.$$

Keeping this in mind, we choose $R_j^* \in (2r,\infty)$ so that

$$Mf_{j,2}(x) \le \frac{4^n}{|B(R_j^*)|}\int_{B(R_j^*)}|f_j(y)|\,dy.$$

Let us set

$$R_j \equiv 2^{1+[\log_2 R_j^*/r]}r.$$

Then we have

$$Mf_{j,2}(x) \le \frac{8^n}{|B(R_j)|}\int_{B(R_j)}|f_j(y)|\,dy.$$

Thus, we obtain

$$Mf_{j,2}(x) \le 8^n\sum_{k=1}^\infty\frac{1}{|B(2^k r)|}\int_{B(2^k r)}|f_j(y)|\,dy.$$

Thus, it follows that

$$\begin{aligned}&\left(\frac{1}{r^\lambda}\int_{B(r)}\left(\sum_{j=1}^\infty Mf_{j,2}(x)^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}}\\&\le Cr^{(n-\lambda)/p}\left(\sum_{j=1}^\infty\left(\sum_{k=1}^\infty\frac{1}{|B(2^k r)|}\int_{B(2^k r)}|f_j(y)|\,dy\right)^q\right)^{\frac{1}{q}}.\end{aligned}$$

By the Minkowski inequality,

$$\begin{aligned}&\left(\frac{1}{r^\lambda}\int_{B(r)}\left(\sum_{j=1}^\infty Mf_{j,2}(x)^q\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}}\\&\le Cr^{(n-\lambda)/p}\sum_{k=1}^\infty\frac{1}{|B(2^k r)|}\int_{B(2^k r)}\left(\sum_{j=1}^\infty|f_j(y)|^q\right)^{\frac{1}{q}}dy.\end{aligned}$$

Since $\lambda < n$, we obtain (3.10). □

4 Singular integral inequalities

We consider the boundedness of the singular integral operator needed in Section 8.

Definition 4.1. *An $L^2(\mathbb{R}^n)$-bounded linear operator T is said to be a (generalized) Calderón-Zygmund operator if it satisfies the following conditions.*

(1) *There is a measurable function $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for all $L^\infty(\mathbb{R}^n)$-functions with compact supports,*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy \text{ for all } \ x \notin \mathrm{supp}(f). \tag{4.1}$$

(2) *The kernel function K satisfies the following estimates: for some $C > 0$*

$$|K(x,y)| \le C \frac{1}{|x-y|^n}\,, \tag{4.2}$$

if $x \neq y$, and

$$|K(x,z) - K(y,z)| + |K(z,x) - K(z,y)| \le C \frac{|x-y|}{|x-z|^{n+1}}\,, \tag{4.3}$$

if $0 < 2|x-y| < |z-x|$.

In this paper, we use the following typical example of a (generalized) Calderón-Zygmund operator. (See the proof in [17, p. 649–650])

Lemma 4.1. *Let $\tau \in \mathcal{S}(\mathbb{R}^n)$ be a function supported away from the origin. Set $\tau_j \equiv \tau(2^{-j}\cdot)$ for $j \in \mathbb{Z}$. Let $\varepsilon = \{\varepsilon_j\}_{j\in\mathbb{Z}}$ be a sequence taking its values in the set $\{-1,0,1\}$.*

1. *Define*

$$K_\varepsilon \equiv \sum_{j=-\infty}^{\infty} \varepsilon_j \mathcal{F}^{-1}[\tau_j].$$

(a) *For all $x \in \mathbb{R}^n$,*

$$|K_\varepsilon(x)| \le \sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j](x)| \le C|x|^{-n}. \tag{4.4}$$

(b) *For all $x \in \mathbb{R}^n$ and $k = 1, 2, \ldots, n$,*

$$|\partial_k K_\varepsilon(x)| \le \sum_{j=-\infty}^{\infty} |\partial_k \mathcal{F}^{-1}[\tau_j](x)| = \sum_{j=-\infty}^{\infty} \left| \frac{\partial \mathcal{F}^{-1}[\tau_j]}{\partial x_k}(x) \right| \le C|x|^{-n-1}. \tag{4.5}$$

In (4.4) and (4.5), the constant C does not depend on ε.

2. *Define*

$$T_\varepsilon f \equiv \sum_{j=-\infty}^{\infty} \varepsilon_j \mathcal{F}^{-1}[\tau_j] * f$$

for $f \in L^2(\mathbb{R}^n)$.

(a) *The series defining* $T_\varepsilon f$ *converges in the topology of* $L^2(\mathbb{R}^n)$.

(b) *Let* $f \in L^2(\mathbb{R}^n)$ *have a compact support. If* $x \notin \mathrm{supp}(f)$, *then*

$$T_\varepsilon f(x) = \int_{\mathbb{R}^n} K_\varepsilon(x-y) f(y)\, dy.$$

5 Predual space $LH_{q,\rho}(\mathbb{R}^n)$

In this section we consider the local predual Morrey space $LH_{q,\rho}(\mathbb{R}^n)$ based on the idea of Zorko [52].

Definition 5.1. *Let* $1 < q < \infty$ *and* $0 \le \rho < n$.

1. *A* $L^q(\mathbb{R}^n)$ *function* b *is said to be a centered* (q,ρ)*-block if it has support in* $B(r)$ *for some* $r > 0$ *and* $\|b\|_{L^q} \le r^{-\rho/q'}$.

2. *The local predual Morrey space* $LH_{q,\rho}(\mathbb{R}^n)$ *is the set of all elements* f *in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *such that*

$$f = \sum_{j=1}^{\infty} \lambda_j b_j,$$

where $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ *and each* b_j *is a* (q,ρ)*-block.*

Let $1 < a < q$. By the Hölder inequality, we have

$$\left(\int_{B(r)} |H(x)|^a\, dx\right)^{\frac{1}{a}} \le \left(\int_{B(r)} |H(x)|^q\, dx\right)^{\frac{1}{q}} |B(r)|^{\frac{1}{a}-\frac{1}{q}}.$$

Thus, if a is defined by

$$\frac{n}{a} - \frac{n}{q} = \frac{\rho}{q'},$$

then $\|b\|_{L^a} \le {v_n}^{\frac{1}{a}-\frac{1}{q}}$ for any (q,ρ)-block, where v_n denotes the volume of the unit ball in $\mathbb{R}^n$. Thus, $LH_{q,\rho}(\mathbb{R}^n)$ is embedded into $L^a(\mathbb{R}^n)$.

In this section, we aim to prove the following result:

Theorem 5.1. *Let* $1 < p < \infty$ *and* $0 \le \lambda < n$.

1. *Let* $f \in LM_{p,\lambda}(\mathbb{R}^n)$. *Then, for any* $g \in LH_{p',\lambda}(\mathbb{R}^n)$, *we have* $f \cdot g \in L^1(\mathbb{R}^n)$ *and the mapping*

$$g \in LH_{p',\lambda}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x) g(x)\, dx \in \mathbb{C}$$

defines a continuous linear functional L_f *on* $LH_{p',\lambda}(\mathbb{R}^n)$. *The operator norm of* L_f *equals* $\|f\|_{LM_{p,\lambda}}$.

2. *Conversely, any continuous linear functional* L *on* $LH_{p',\lambda}(\mathbb{R}^n)$ *can be realized as* $L = L_f$ *with a certain* $f \in LM_{p,\lambda}(\mathbb{R}^n)$. *In addition, if* f_1 *and* $f_2 \in LM_{p,\lambda}(\mathbb{R}^n)$ *define the same functional, then* $f_1 = f_2$ *almost everywhere.*

Recall that the following result is well known as the duality $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$, and observe that Theorem 5.1 covers the duality $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$ as a special case when $\lambda = 0$.

Proposition 5.1. *Let $1 < p < \infty$.*

1. *Let $f \in L^p(\mathbb{R}^n)$. Then for any $g \in L^{p'}(\mathbb{R}^n)$, $f \cdot g \in L^1(\mathbb{R}^n)$ and the mapping*

$$g \in L^{p'}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x)g(x)\,dx \in \mathbb{C}$$

defines a continuous linear functional L_f on $L^{p'}(\mathbb{R}^n)$. The operator norm of L_f equals $\|f\|_{L^p}$.

2. *Conversely, any continuous linear functional L on $L^{p'}(\mathbb{R}^n)$ can be realized as $L = L_f$ with a certain $f \in L^p(\mathbb{R}^n)$. In addition, if f_1 and $f_2 \in L^p(\mathbb{R}^n)$ define the same functional, then $f_1 = f_2$ almost everywhere.*

Proof of Theorem 5.1. The first statement is a corollary of the following inequality:

$$\int_{\mathbb{R}^n} |f(x)b(x)|\,dx \le \|f\|_{LM_{p,\lambda}},$$

whenever b is a (p', λ)-block.

To prove the second statement, let L be a continuous linear functional on $LH_{p,\lambda}(\mathbb{R}^n)$. For any $R \in (0, \infty)$, the functional

$$g \in L^{p'}(\mathbb{R}^n) \mapsto L(g\chi_{B(R)}) \in \mathbb{C}$$

is a bounded linear mapping with norm less than or equal to $R^{-\lambda/p}$, because $\dfrac{R^{-\lambda/p}}{\|g\|_{L^{p'}}} g\chi_{B(R)}$ is a (p', λ)-block. Thus,

$$|L(g\chi_{B(R)})| \le R^{\lambda/p}\|g\|_{L^{p'}}$$

for all $R > 0$. Hence, for each $R > 0$, according to Proposition 5.1(1), we obtain a measurable function f_R such that

$$L(g\chi_{B(R)}) = \int_{\mathbb{R}^n} f_R(x)g(x)\,dx, \quad \|f_R\|_{L^p} \le R^{\lambda/p}.$$

Observe that

$$L(g\chi_{B(R)}) = L(g\chi_{B(R)}\chi_{B([R+1])}) = \int_{\mathbb{R}^n} f_{[R+1]}(x)g(x)\chi_{B(R)}(x)\,dx$$

and the uniqueness of f_j (see Proposition 5.1(2)) implies that

$$f_R(x) = \chi_{B(R)}(x)f_{[R+1]}(x).$$

Thus, there exists a measurable function f such that $f_R(x) = \chi_{B(R)}(x)f(x)$ for all $R > 0$. Moreover, $f \in LM_{p,\lambda}(\mathbb{R}^n)$. □

6 Characterization of local Hardy Morrey spaces in terms of the grand maximal operator and the heat kernel

The next proposition characterizes the space $LM_{p,\lambda}(\mathbb{R}^n)$ in terms of the heat kernel. Let $t > 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ and define

$$e^{t\Delta}f(x) \equiv \left\langle f, \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x-\cdot|^2}{4t}\right)\right\rangle \quad (x \in \mathbb{R}^n).$$

We say that $f \in HLM_{p,\lambda}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\sup_{t>0}|e^{t\Delta}f| \in LM_{p,\lambda}(\mathbb{R}^n)$. We define

$$\|f\|_{HLM_{p,\lambda}} \equiv \left\|\sup_{t>0}|e^{t\Delta}f|\right\|_{LM_{p,\lambda}}.$$

Let $1 < p < \infty$ and $0 \le \lambda < n$. Then one defines the local Hardy Morrey space $HLM_{p,\lambda}(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the norm

$$\|f\|_{HLM_{p,\lambda}} \equiv \left\|\sup_{t>0}|e^{t\Delta}f|\right\|_{LM_{p,\lambda}} < \infty.$$

Let us show that $LM_{p,\lambda}(\mathbb{R}^n)$ and $HLM_{p,\lambda}(\mathbb{R}^n)$ are isomorphic by proving the following proposition.

Proposition 6.1. *Let $1 < p < \infty$ and $0 \le \lambda < n$.*

1. *If $f \in LM_{p,\lambda}(\mathbb{R}^n)$, then $f \in HLM_{p,\lambda}(\mathbb{R}^n)$ and*

$$\|f\|_{LM_{p,\lambda}} \le \|f\|_{HLM_{p,\lambda}} \le C\|f\|_{LM_{p,\lambda}}, \tag{6.1}$$

where $C > 0$ is independent of f.

2. *If $f \in HLM_{p,\lambda}(\mathbb{R}^n)$, then f is represented by a measurable function g which belongs to $LM_{p,\lambda}(\mathbb{R}^n)$.*

Proof. 1. We can easily verify that $LM_{p,\lambda}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ by using Lemma 2.1. Also, we have

$$\sup_{t>0}|e^{t\Delta}f| \le Mf$$

by virtue of Proposition 2.1. Due to Theorem 3.1 on the $LM_{p,\lambda}(\mathbb{R}^n)$-boundedness of the Hardy-Littlewood maximal operator $f \in HLM_{p,\lambda}(\mathbb{R}^n)$ and that the right-hand-side inequality in (6.1) follows.

2. Recall that the dual of $LH_{p',\lambda}(\mathbb{R}^n)$ is isomorphic to $LM_{p,\lambda}(\mathbb{R}^n)$ as we have established in Theorem 5.1. Let $L : f \in LM_{p,\lambda}(\mathbb{R}^n) \mapsto L_f \in (LH_{p',\lambda}(\mathbb{R}^n))^*$ be an isomorphism in Theorem 5.1. We shall make use of the general result due to Banach and Alaoglu: If X is a Banach space, then the unit ball of X^* is weakly-* (sequentially) compact. By the assumption, $\{e^{t\Delta}f\}_{t>0}$ forms a bounded set in $LM_{p,\lambda}(\mathbb{R}^n)$. Consider a sequence $\{t_j\}_{j=1}^\infty$ in $[0,1]$ which decreases to 0. Then $\{L_{e^{t_j\Delta}f}\}_{j=1}^\infty$ forms a bounded set in $(LH_{p',\lambda}(\mathbb{R}^n))^*$. Thus, by the Banach-Alaoglu theorem, there exists a positive

sequence, which we denote again by $\{t_j\}_{j=1}^{\infty}$, such that $L_{e^{t_j\Delta}f}$ converges to $G = L_g \in (LH_{p',\lambda}(\mathbb{R}^n))^*$ for some $g \in LM_{p,\lambda}(\mathbb{R}^n)$ in the weak-* sense. Observe that

$$\|f\|_{LM_{p,\lambda}} = \|L_f\|_{(LH_{p',\lambda})^*} \le \liminf_{j\to\infty} \|L_{e^{t_j\Delta}f}\|_{(LH_{p',\lambda})^*} = \liminf_{j\to\infty} \|e^{t_j\Delta}f\|_{LM_{p,\lambda}}. \tag{6.2}$$

Moreover, since $f \in \mathcal{S}'(\mathbb{R}^n)$, $e^{t_j\Delta}f$ converges to $f \in \mathcal{S}'(\mathbb{R}^n)$. Thus, we conclude that $\mathcal{S}'(\mathbb{R}^n) \ni f = g \in LM_{p,\lambda}(\mathbb{R}^n)$.

The left inequality in (6.1) follows since the space $LM_{p,\lambda}(\mathbb{R}^n)$ is isomorphic to the dual of $LH_{p',\lambda}(\mathbb{R}^n)$. Thus, from (6.2),

$$\|f\|_{LM_{p,\lambda}} \le \left\|\sup_{t>0} |e^{t\Delta}f|\right\|_{LM_{p,\lambda}} = \|f\|_{HLM_{p,\lambda}}.$$

□

Proposition 6.2. *Let $1 < p < \infty$ and $0 \le \lambda < n$.*

1. *If $f \in LM_{p,\lambda}(\mathbb{R}^n)$, then $\mathcal{M}f \in LM_{p,\lambda}(\mathbb{R}^n)$ and*

$$C^{-1}\|f\|_{LM_{p,\lambda}} \le \|\mathcal{M}f\|_{LM_{p,\lambda}} \le C\|f\|_{LM_{p,\lambda}}, \tag{6.3}$$

where $C > 0$ is independent of f.

2. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. If $\mathcal{M}f \in LM_{p,\lambda}(\mathbb{R}^n)$, then f is represented by a measurable function g which belongs to $LM_{p,\lambda}(\mathbb{R}^n)$.*

Proof. The implication (1) $\Longrightarrow$ (2) follows from the pointwise inequality $\mathcal{M}f(x) \le CMf(x)$. The converse implication (2) $\Longrightarrow$ (1) follows from the inequality $|e^{t\Delta}f(x)| \le C\mathcal{M}f(x)$. Indeed, from this pointwise estiamte, we conclude that $\sup_{t>0}|e^{t\Delta}f(\cdot)| \in LM_{p,\lambda}(\mathbb{R}^n)$. Thus, by applying Proposition 6.1 we have $f \in LM_{p,\lambda}(\mathbb{R}^n)$. □

7 Non-smooth decomposition

7.1 Norm estimate

We shall now prove Theorem 1.1. Let us write

$$f = \sum_{j=1}^{\infty} \lambda_j a_j.$$

Proof. To prove (1.3), we resort to the duality:

$$\|f\|_{LM_{p,\lambda}} = \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \,:\, \|g\|_{LH_{p',\lambda}} = 1\right\}.$$

For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \ge N$. Let us assume in addition, without loss of generality, that all a_j are non-negative. Fix a positive (p',λ)-block g with the associated ball B. We may suppose that $g \ge 0$ a.e., since f is non-negative.

Assume first that each Q_j contains B as a proper subset. If we group the j's such that Q_j are identical, we can assume that Q_j is centered at the origin and satisfies $|Q_j| = 2^{jn}|B|$ for each $j \in \mathbb{N}$. Then we have

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = \sum_{j=1}^{\infty} \lambda_j \int_B a_j(x)g(x)\,dx \le C\sum_{j=1}^{\infty} \lambda_j \|a_j\|_{L^p(B)}\|g\|_{L^{p'}}.$$

By the size condition for a_j and g, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,dx &\le C\sum_{j=1}^{\infty} \lambda_j |B|^{\frac{\lambda}{np}+\frac{1}{p}-\frac{1}{q}}\|a_j\|_{L^q(B)} \\ &\le C\sum_{j=1}^{\infty} \lambda_j |B|^{\frac{\lambda}{np}+\frac{1}{p}-\frac{1}{q}-\frac{\rho}{nq}}\|a_j\|_{LM_{q,\rho}} \\ &\le C\sum_{j=1}^{\infty} \lambda_j |B|^{\frac{\lambda}{np}+\frac{1}{p}-\frac{1}{q}-\frac{\rho}{nq}}\|\chi_{Q_j}\|_{LM_{q,\rho}}.\end{aligned}$$

Since Q_j is centered at the origin, we obtain

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx \le C\left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{LM_{p,\lambda}} \sum_{j=1}^{\infty}\left(\frac{|B|}{|B_j|}\right)^{\frac{\lambda}{np}+\frac{1}{p}-\frac{1}{q}-\frac{\rho}{nq}} = C\left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{LM_{p,\lambda}}$$

by virtue of (1.1).

Conversely, assume that B contains each Q_j. Then we have

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = \sum_{j=1}^{\infty} \lambda_j \int_{Q_j} a_j(x)g(x)\,dx \le C\sum_{j=1}^{\infty} \lambda_j \|a_j\|_{L^q}\|g\|_{L^{q'}(Q_j)}.$$

Denote by $B_j = B(r_j)$ the ball which is centered at the origin and which contains Q_j. Then we have

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,dx &\le C\sum_{j=1}^{\infty} \lambda_j \|a_j\|_{L^q}\|g\|_{L^{q'}(Q_j)} \\ &\le C\sum_{j=1}^{\infty} \lambda_j {r_j}^{-\rho/q}\|a_j\|_{LM_{q,\lambda}}\|g\|_{L^{q'}(Q_j)} \\ &\le C\sum_{j=1}^{\infty} \lambda_j {r_j}^{-\rho/q}\|\chi_{Q_j}\|_{LM_{q,\lambda}}\|g\|_{L^{q'}(Q_j)} \\ &\le C\sum_{j=1}^{\infty} \lambda_j |Q_j|^{1/q}\|g\|_{L^{q'}(Q_j)}.\end{aligned}$$

By using the Hardy-Littlewood maximal operator, we obtain

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx \le C\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(x)\chi_B(x) M[|g|^{q'}](x)^{\frac{1}{q'}}\,dx.$$

Note that

$$\|(M[|g|^{q'}])^{1/q'}\|_{H_{p',\lambda}} \le C\|g\|_{H_{p',\lambda}} \le C,$$

assuming $p < q$. If we apply Theorem 5.1, then we have the desired result. □

7.2 Nonsmooth decomposition of functions

The following lemma is the key to the decomposition of local Morrey spaces as is mentioned in Section 1; the structure of local Morrey spaces comes into play here.

Lemma 7.1. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. With the same notation as in Lemma 2.2, we have*

$$|\langle b_j, \varphi\rangle| \le C_\varphi \left\{ \sum_{l=0}^{\infty} \left(\frac{1}{2^{ln}} \left\| \mathcal{M}f \cdot \chi_{\mathcal{O}_j} \right\|_{L^1(B(2^l))} \right)^{\theta} \right\}^{1/\theta} \tag{7.1}$$

and

$$|\langle g_j, \varphi\rangle| \le C_\varphi \left\{ \sum_{l=0}^{\infty} \left(\frac{1}{2^{ln}} \left\| \mathcal{M}f \cdot \chi_{\mathcal{O}_j} \right\|_{L^1(B(2^l))} \right)^{\theta} \right\}^{1/\theta}, \tag{7.2}$$

where $\theta = \frac{n+d+1}{n}$ and the constants C_φ in (7.1) *and* (7.2) *depends on φ but not on j or k.*

Proof. For sufficiently large constant $M = M_\varphi$, we have $\psi_x \equiv M^{-1}\varphi(x - \cdot) \in \mathcal{F}_N$ for all $x \in B(1)$, so that

$$|\langle b_j, \varphi\rangle| = |b_j * \psi_x(z)|_{z=x} \le M \inf_{x \in B(1)} \mathcal{M}b_j(x).$$

Thus, we have

$$|\langle b_j, \varphi\rangle| \le C \inf_{x \in B(1)} \mathcal{M}b_j(x) \le C \inf_{x \in B(1)} \sum_{k \in K_j} \mathcal{M}b_{j,k}(x).$$

Observe also that

$$CM\chi_{B(x_B,r)}(x) \ge \frac{r^n}{r^n + |x - x_B|^n} \ge \frac{r^n}{|x - x_B|^n}\chi_{\mathbb{R}^n \setminus B(x_B,r)}(x) \quad (x \in \mathbb{R}^n).$$

It then follows from (2.6) that

$$\begin{aligned}\sum_{k \in K_j} \mathcal{M}b_{j,k}(x) &\le C \sum_{k \in K_j} \left(\mathcal{M}f(x)\chi_{Q_{j,k}}(x) + 2^j \cdot \frac{{\ell_{j,k}}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}} \chi_{\mathbb{R}^n \setminus Q_{j,k}}(x) \right) \\ &\le C \left(\mathcal{M}f(x)\chi_{\mathcal{O}_j}(x) + 2^j \sum_{k \in K_j} M\chi_{Q_{j,k}}(x)^{\frac{n+d+1}{n}} \right).\end{aligned}$$

Thus, from this pointwise estimate and (3.10), we deduce that

$$\begin{aligned}\|\mathcal{M}b_j\|_{L^1(B(1))} &\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}+2^j\sum_{k\in K_j}(M\chi_{Q_{j,k}})^{\frac{n+d+1}{n}}\right\|_{L^1(B(1))}\\ &\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^1(B(1))}+C\left\|2^j\sum_{k\in K_j}(M\chi_{Q_{j,k}})^{\frac{n+d+1}{n}}\right\|_{L^1(B(1))}\\ &\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^1(B(1))}+C\left\{\sum_{l=0}^{\infty}\left(\frac{1}{2^{ln}}\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^1(B(2^l))}\right)^{\theta}\right\}^{1/\theta}.\end{aligned}$$

In the same way, we can prove (7.2): indeed, we obtain

$$\begin{aligned}\|\mathcal{M}g_j\|_{L^1(B(1))} &\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j{}^c}\right\|_{L^1(B(1))}+C\left\|\sum_{k\in K_j}\frac{2^j\cdot{\ell_{j,k}}^{n+d+1}}{(\ell_{j,k}+|\cdot-x_{j,k}|)^{n+d+1}}\right\|_{L^1(B(1))}\\ &\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j{}^c}\right\|_{L^1(B(1))}+C\left\|\sum_{k\in K_j}2^j(M\chi_{Q_{j,k}})^{\frac{n+d+1}{n}}\right\|_{L^1(B(1))}\\ &\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j{}^c}\right\|_{L^1(B(1))}+C\left\{\sum_{l=0}^{\infty}\left(\frac{1}{2^{ln}}\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^1(B(2^l))}\right)^{\theta}\right\}^{1/\theta}.\end{aligned}$$

Thus, (7.2) is proved. □

Lemma 7.2. *In the notation of Lemma 2.2, in the topology of $\mathcal{S}'(\mathbb{R}^n)$, we have $g_j \to 0$ as $j \to -\infty$ and $b_j \to 0$ as $j \to \infty$. In particular,*

$$f=\sum_{j=-\infty}^{\infty}(g_{j+1}-g_j)$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$.

Proof. Observe that

$$\frac{1}{2^{ln}}\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\|_{L^1(B(2^l))}\le\frac{C}{2^{ln}}\|\mathcal{M}f\|_{L^1(B(2^l))}\le\frac{C}{2^{l(n-\lambda)/p}}\|f\|_{HLM_{p,\lambda}}.$$

Consequently, we may use the Lebesgue convergence theorem to conclude that $b_j \to 0$ as $j \to \infty$. Hence, it follows that $f = \lim_{j\to\infty} g_j$ in $\mathcal{S}'(\mathbb{R}^n)$.

Likewise, by using (7.2), we obtain $g_j \to 0$ as $j \to -\infty$ by the Lebesgue convergence theorem. Consequently, it follows that $f = \lim_{j\to\infty} g_j = \lim_{j,k\to\infty}\sum_{l=-k}^{j}(g_{l+1}-g_l)$ in $\mathcal{S}'(\mathbb{R}^n)$. □

We shall now prove Theorem 1.2.

Proof. For each $j \in \mathbb{Z}$, consider the level set

$$\mathcal{O}_j \equiv \{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2^j\}. \tag{7.3}$$

Then it follows immediately from the definition that

$$\mathcal{O}_{j+1} \subset \mathcal{O}_j. \tag{7.4}$$

If we apply Lemma 2.2, then f can be decomposed as

$$f = g_j + b_j, \quad b_j = \sum_k b_{j,k}, \quad b_{j,k} = (f - c_{j,k})\eta_{j,k}$$

where each $b_{j,k}$ is supported in a cube $Q_{j,k}$ as described in Lemma 2.2.

We know that

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j), \tag{7.5}$$

with the series converging in the sense of distributions from Lemma 7.2. Here, going through the same argument as the one in [39, p. 108–109], we have

$$f = \sum_{j,k} A_{j,k}, \quad g_{j+1} - g_j = \sum_k A_{j,k} \quad (j \in \mathbb{Z}) \tag{7.6}$$

in the sense of distributions, where each $A_{j,k}$, supported in $Q_{j,k}$, satisfies the pointwise estimate $|A_{j,k}(x)| \le C_0 2^j$ for some universal constant C_0 and the moment condition $\displaystyle\int_{\mathbb{R}^n} A_{j,k}(x)q(x)\,dx = 0$ for every $q \in \mathcal{P}_d(\mathbb{R}^n)$. With these observations in mind, let us set

$$a_{j,k} \equiv \frac{A_{j,k}}{C_0 2^j}, \quad \kappa_{j,k} \equiv C_0 2^j.$$

Then we shall obtain that each $a_{j,k}$ satisfies

$$|a_{j,k}| \le \chi_{Q_{j,k}}, \int_{\mathbb{R}^n} x^\alpha a_{j,k}(x)\,dx = 0 \quad (|\alpha| \le L)$$

and that $f = \displaystyle\sum_{j,k} \kappa_{j,k} a_{j,k}$ in the topology of $HLM_{p,\lambda}(\mathbb{R}^n)$, once we prove the estimate for the coefficients. Rearrange $\{a_{j,k}\}$ to obtain $\{a_j\}$. Do the same thing to $\{\lambda_{j,k}\}$.

To establish (1.5), we need to estimate

$$\alpha \equiv \left\| \left(\sum_{j=-\infty}^{\infty} |\lambda_j \chi_{Q_j}|^v \right)^{1/v} \right\|_{LM_{p,\lambda}}.$$

Since

$$\{(\kappa_{j,k}; Q_{j,k})\}_{j,k} = \{(\lambda_j; Q_j)\}_j,$$

we have

$$\alpha = \left\| \left(\sum_{j=-\infty}^{\infty} \sum_{k \in K_j} |\kappa_{j,k} \chi_{Q_{j,k}}|^v \right)^{1/v} \right\|_{LM_{p,\lambda}}.$$

By using the definition of κ_j, we then have

$$\alpha = C_0 \left\| \left(\sum_{j=-\infty}^{\infty} \sum_{k \in K_j} |2^j \chi_{Q_{j,k}}|^v \right)^{1/v} \right\|_{LM_{p,\lambda}} = C_0 \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jv} \sum_{k \in K_j} \chi_{Q_{j,k}} \right)^{1/v} \right\|_{LM_{p,\lambda}}.$$

Observe that (2.4), together with the bounded overlapping property, yields

$$\chi_{\mathcal{O}_j}(x) \le \sum_{k \in K_j} \chi_{Q_{j,k}}(x) \le \sum_{k \in K_j} \chi_{200\,Q_{j,k}}(x) \le C \chi_{\mathcal{O}_j}(x) \quad (x \in \mathbb{R}^n).$$

Thus, we have

$$\alpha \le C \left\| \left(\sum_{j=-\infty}^{\infty} \left(2^j \chi_{\mathcal{O}_j}\right)^v \right)^{1/v} \right\|_{LM_{p,\lambda}}.$$

Recalling that $\mathcal{O}_j \supset \mathcal{O}_{j+1}$ for each $j \in \mathbb{Z}$, we have

$$\sum_{j=-\infty}^{\infty} \left(2^j \chi_{\mathcal{O}_j}(x)\right)^v \simeq \left(\sum_{j=-\infty}^{\infty} 2^j \chi_{\mathcal{O}_j}(x) \right)^v \simeq \left(\sum_{j=-\infty}^{\infty} 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}}(x) \right)^v \quad (x \in \mathbb{R}^n).$$

Thus, we obtain

$$\alpha \le C \left\| \sum_{j=-\infty}^{\infty} 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} \right\|_{LM_{p,\lambda}}.$$

It follows by the definition of $\mathcal{O}_j$ that $2^j < \mathcal{M}f(x)$ for all $x \in \mathcal{O}_j$. Hence, we have

$$\alpha \le C \left\| \sum_{j=-\infty}^{\infty} \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} \mathcal{M}f \right\|_{LM_{p,\lambda}} \le C \|\mathcal{M}f\|_{LM_{p,\lambda}},$$

which is the desired result. □

7.3 Application – Olsen's inequality for local Morrey spaces

In this section we consider the following Olsen inequality for the fractional integral operator I_α, where I_α $(0 < \alpha < n)$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy \quad (x \in \mathbb{R}^n).$$

Theorem 7.1. *Suppose that we are given parameters* $p, \lambda, q, \rho, \alpha$ *satisfying conditions*

$$1 < p < q < \infty, \quad \alpha = \frac{n-\rho}{q}, \quad 0 < \alpha < \frac{n}{p}$$

and condition (1.1). *Then*

$$\|g \cdot I_\alpha f\|_{LM_{p,\lambda}} \le C\|g\|_{GM_{q,\rho}}\|f\|_{LM_{p,\lambda}},$$

where $C > 0$ *is independent of* f *and* g.

In particular, when $\alpha = 1$,

$$\|g \cdot f\|_{LM_{p,\lambda}} \le C\|g\|_{GM_{q,\rho}}\|\nabla f\|_{LM_{p,\lambda}}$$

for all appropriate functions f *and* g.

To prove this we need the following statement.

Lemma 7.3. [9, Lemma 4.2] *Let* $L \in \mathbb{N}_0$. *Suppose that* A *is an* $L^\infty(\mathbb{R}^n)$*-function supported on a cube* Q. *Assume, in addition, that* $\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0$ *for all multi-indices* β *with* $|\beta| \le L$. *Then*

$$|I_\alpha A(x)| \le C_{\alpha,L}\|A\|_{L^\infty}\ \ell(Q)^\alpha \sum_{k=1}^{\infty} \frac{1}{2^{k(n+L+1-\alpha)}}\chi_{2^kQ}(x) \quad (x \in \mathbb{R}^n). \tag{7.7}$$

Proof of Theorem 7.1. We decompose f according to Theorem 1.2 with sufficiently large L:

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ satisfy (1.4) and (1.5). Then by Lemma 7.3, we obtain

$$|g(x)I_\alpha f(x)| \le \sum_{j,k\in\mathbb{N}} \frac{\lambda_j}{2^{k(n+L+1-\alpha)}}\left(\ell(Q_j)^\alpha|g(x)|\chi_{2^kQ_j}(x)\right).$$

Therefore, we conclude

$$\|g \cdot I_\alpha f\|_{LM_{p,\lambda}} \le C\|g\|_{GM_{q,\rho}}\left\|\sum_{j,k\in\mathbb{N}} \frac{\lambda_j}{2^{k(n+L+1)}} \cdot \frac{\ell(2^kQ_j)^\alpha}{\|g\|_{GM_{q,\rho}}}|g|\chi_{2^kQ_j}\right\|_{LM_{p,\lambda}}.$$

For each $(j,k) \in \mathbb{N}\times\mathbb{N}$, write

$$\kappa_{jk} \equiv \frac{\lambda_j}{2^{k(n+L+1)}}, \quad b_{jk} \equiv \frac{\ell(2^kQ_j)^\alpha}{\|g\|_{GM_{q,\rho}}}|g|\chi_{2^kQ_j}.$$

Let us check that $\|b_{jk}\|_{LM_{q,\rho}} \le C\|\chi_{2^kQ_j}\|_{LM_{q,\rho}}$. If $2^{k+1}Q_j \ni 0$, then this is easy to check. Otherwise

$$\begin{aligned}
\|b_{jk}\|_{LM_{q,\rho}} &\le C\frac{\ell(2^kQ_j)^\alpha}{\|g\|_{GM_{q,\rho}}} \sup_{r>\ell(2^kQ_j)} \left(\frac{1}{r^\rho}\int_{B(r)\cap 2^kQ_j} |g(y)|^q\,dy\right)^{\frac{1}{q}} \\
&\le C\frac{\ell(2^kQ_j)^\alpha}{\|g\|_{GM_{q,\rho}}} \left(\frac{1}{(|c(Q_j)|+2^k\ell(Q_j))^\rho}\int_{2^kQ_j} |g(y)|^q\,dy\right)^{\frac{1}{q}} \\
&\le C\frac{\ell(2^kQ_j)^{\rho/q}}{(|c(Q_j)|+2^k\ell(Q_j))^{\rho/q}} \cdot \ell(2^kQ_j)^\alpha \\
&= C\frac{\ell(2^kQ_j)^{n/q}}{(|c(Q_j)|+2^k\ell(Q_j))^{\rho/q}} \\
&\le C\|\chi_{2^kQ_j}\|_{LM_{q,\rho}}.
\end{aligned}$$

Observe also that $q_0 > r_0$ and $q > r$. Thus, by Theorem 1.2, it follows that

$$\begin{aligned}
\|g\cdot I_\alpha f\|_{LM_{p,\lambda}} &\le C\|g\|_{GM_{q,\rho}} \left\|\sum_{j,k\in\mathbb{N}} \kappa_{jk}\chi_{2^kQ_j}\right\|_{LM_{p,\lambda}} \\
&= C\|g\|_{GM_{q,\rho}} \left\|\sum_{j,k\in\mathbb{N}} \frac{\lambda_j\chi_{2^kQ_j}}{2^{k(n+L+1)}}\right\|_{LM_{p,\lambda}}.
\end{aligned}$$

A geometric observation shows that the pointwise estimate $\chi_{2^kQ_j} \le 2^{kn}M\chi_{Q_j}$ holds. Thus, if we choose θ slightly larger than 1, then we have

$$\begin{aligned}
\|g\cdot I_\alpha f\|_{LM_{p,\lambda}} &\le C\|g\|_{GM_{q,\rho}} \left\|\sum_{j,k\in\mathbb{N}} \frac{\lambda_j(M\chi_{Q_j})^\theta}{2^{k(n(1-\theta)+L+1)}}\right\|_{LM_{p,\lambda}} \\
&\le C\|g\|_{GM_{q,\rho}} \left\|\sum_{j,k\in\mathbb{N}} \frac{\lambda_j\chi_{Q_j}}{2^{k(n(1-\theta)+L+1)}}\right\|_{LM_{p,\lambda}} \\
&\le C\|g\|_{GM_{q,\rho}}\|f\|_{LM_{p,\lambda}}.
\end{aligned}$$

□

8 Smooth decompositions

8.1 A characterization of the Morrey space $LM_{p,\lambda}(\mathbb{R}^n)$ in terms of Littlewood-Paley characterization

The following is a key ingredient for the proof of Theorem 1.3.

Theorem 8.1. *Let $1 < p < \infty$ and $0 \le \lambda < n$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the inequalities $\chi_{B(2)} \le \psi \le \chi_{B(4)}$. Define $\varphi \equiv \psi - \psi(2\cdot)$ and $\varphi_j \equiv \varphi(2^{-j}\cdot)$ for $j \in \mathbb{Z}$.*

1. *For any $f \in LM_{p,\lambda}(\mathbb{R}^n)$*

$$f = \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]$$

in the weak- topology of $LM_{p,\lambda}(\mathbb{R}^n)$, and the estimate*

$$\left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}} \simeq \|f\|_{LM_{p,\lambda}}$$

holds.

2. *Assume that $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies*

$$\left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}} < \infty.$$

Then the limit

$$F \equiv \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]$$

exists in the weak- topology of $LM_{p,\lambda}(\mathbb{R}^n)$, and the estimate*

$$\|F\|_{LM_{p,\lambda}} \simeq \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}}$$

holds.

In the lemma below, we consider the limit

$$\lim_{L_1, L_2 \to \infty} \sum_{j=-L_1}^{L_2} \mathcal{F}^{-1}[\varphi_j \mathcal{F} g].$$

The limit as $L_2 \to \infty$ does not cause any trouble, since it is a general fact for $g \in \mathcal{S}'(\mathbb{R}^n)$ that

$$\lim_{L_2 \to \infty} \sum_{j=0}^{L_2} \mathcal{F}^{-1}[\varphi_j \mathcal{F} g] = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F} g].$$

Thus, if $g \in LM_{p,\lambda}(\mathbb{R}^n)$, we need to handle carefully the limit as $L_1 \to \infty$. The next lemma shows that this is possible.

Lemma 8.1. *Let $1 < p < \infty$ and $0 \le \lambda < n$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function satisfying the inequalities $\chi_{B(2)} \le \psi \le \chi_{B(4)}$. Define $\varphi \equiv \psi - \psi(2\cdot)$ and $\varphi_j \equiv \varphi(2^{-j}\cdot)$ for $j \in \mathbb{Z}$. If $g \in LH_{p',\lambda}(\mathbb{R}^n)$, then*

$$g = \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F} g] \tag{8.1}$$

in the topology of $LH_{p',\lambda}(\mathbb{R}^n)$.

Proof. Let

$$S_J g \equiv \sum_{j=-J}^{J} \mathcal{F}^{-1}[\varphi_j \mathcal{F} g].$$

Since $g \in LH_{p',\lambda}(\mathbb{R}^n)$, we can take a complex ℓ^1-sequence $\lambda = \{\lambda_k\}_{k=1}^{\infty}$ and a collection $\{a_k\}_{k=1}^{\infty}$ of blocks such that $g = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ in $L^1_{\rm loc}(\mathbb{R}^n)$ and that $\sum_{k=-\infty}^{\infty} |\lambda_k| \le 2\|g\|_{LH_{p',\lambda}}$. To prove that (8.1) takes place in the topology of $LH_{p',\lambda}(\mathbb{R}^n)$, we take arbitrary $\varepsilon > 0$, and sufficiently large $K \in \mathbb{N}$ so that

$$\sum_{|k| \ge K+1} |\lambda_k| \le \varepsilon. \tag{8.2}$$

Set $g_K \equiv \sum_{k=-K}^{K} \lambda_k a_k$. Since S_J can be considered as a Calderón-Zygmund singular integral operator in the sense of Definition 4.1 with the related constants independent of J, it follows that

$$\|S_J g - S_J g_K\|_{LH_{p',\lambda}} \le C\|g - g_K\|_{LH_{p',\lambda}}.$$

Therefore

$$\begin{aligned}
\|S_J g - g\|_{LH_{p',\lambda}} &\le \|S_J g_K - g_K\|_{LH_{p',\lambda}} + \|S_J g - S_J g_K\|_{LH_{p',\lambda}} + \|g - g_K\|_{LH_{p',\lambda}} \\
&\le \|S_J g_K - g_K\|_{LH_{p',\lambda}} + (C+1)\varepsilon \\
&\le \sum_{k=-K}^{K} |\lambda_k| \cdot \|S_J a_k - a_k\|_{LH_{p',\lambda}} + (C+1)\varepsilon.
\end{aligned}$$

Let Q_j be a cube such that $\mathrm{supp}(a_k) \subset Q_k$. We choose $\varepsilon' \in (0,1]$ sufficiently close to 0, say $0 < \varepsilon' < n - n/p'$. Denote by $c(Q_k)$ the center of the cube Q_k. Let $x \notin 3Q_k$ and suppose that $|x - c(Q_k)| \le 2^{J+1}$. Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we then have

$$\begin{aligned}
&|S_J a_k(x) - a_k(x)| \le C \sum_{j=J+1}^{\infty} |\mathcal{F}^{-1}\varphi_j * a_k(x)| + C \sum_{j=-\infty}^{-J-1} |\mathcal{F}^{-1}\varphi_j * a_k(x)| \\
&\le C \int_{Q_k} \sum_{j=J+1}^{\infty} \frac{2^{jn}|a_k(y)|}{(1+2^j|x-y|)^{n+1}}\,dy + C \int_{Q_k} \sum_{j=-\infty}^{-J-1} \frac{2^{jn}|a_k(y)|}{(1+2^j|x-y|)^{n-\varepsilon'}}\,dy \\
&\le C \left(\sum_{j=J+1}^{\infty} \frac{2^{jn}}{(2^j|x-c(Q_k)|)^{n+1}} + \sum_{j=-\infty}^{-J-1} \frac{2^{jn}}{(1+2^j|x-c(Q_k)|)^{n-\varepsilon'}} \right) \|a_k\|_{L^1} \\
&\le C \left(\frac{1}{2^J|x-c(Q_k)|^{n+1}} + \frac{1}{2^{J\varepsilon'}|x-c(Q_k)|^{n-\varepsilon'}} \right) \|a_k\|_{L^1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|S_J a_k(x) - a_k(x)| \\
&\le |\chi_{3Q}(x)(S_J a_k(x) - a_k(x))| \\
&\quad + C\left(\frac{1}{2^J(\ell(Q_k)+|x-c(Q_k)|)^{n+1}} + \frac{1}{2^{J\varepsilon'}(\ell(Q_k)+|x-c(Q_k)|)^{n-\varepsilon'}}\right)\|a_k\|_{L^1}.
\end{aligned}$$

For the first term in the right-hand side, we can invoke the Littlewood-Paley theorem for $L^q(\mathbb{R}^n)$. Consequently, with k fixed, we have

$$\lim_{J\to\infty} \|S_J a_k - a_k\|_{LH_{p',\lambda}} = 0$$

and hence

$$\limsup_{J\to\infty} \|S_J g - g\|_{LH_{p',\lambda}} \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have (8.1). □

Lemma 8.2. *Let $1 < p < \infty$ and $0 \le \lambda < n$. Let $\tau \in \mathcal{S}(\mathbb{R}^n)$ be a radial function supported away from the origin, and for $j \in \mathbb{Z}$ denote $\tau_j \equiv \tau(2^{-j}\cdot)$.*

1. *For all $f \in LM_{p,\lambda}(\mathbb{R}^n)$,*

$$\left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j \mathcal{F} f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}} \le C\|f\|_{LM_{p,\lambda}}. \tag{8.3}$$

2. *For all $g \in LH_{p',\lambda}(\mathbb{R}^n)$ such that* $\operatorname{supp}(g)$ *is a compact set in $\mathbb{R}^n \setminus \{0\}$,*

$$\left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j \mathcal{F} g]|^2\right)^{\frac{1}{2}}\right\|_{LH_{p',\lambda}} \le C\|g\|_{LH_{p',\lambda}}. \tag{8.4}$$

In (8.3), (8.4) *$C > 0$ is independent of f, g respectively.*

Proof. Let $\{r_k\}_{k=-\infty}^{\infty}$ be as in Lemma 2.4.

We first prove (8.3). Note that

$$\left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j \mathcal{F} f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}} \le C\left\|\int_0^1 \left|\sum_{j=-\infty}^{\infty} r_j(t)\mathcal{F}^{-1}[\tau_j \mathcal{F} f]\right| dt\right\|_{LM_{p,\lambda}}$$

by virtue of Lemma 2.4. Thus, we have

$$\begin{aligned}
\left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j \mathcal{F} f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}} &\le C\int_0^1 \left\|\sum_{j=-\infty}^{\infty} r_j(t)\mathcal{F}^{-1}[\tau_j \mathcal{F} f]\right\|_{LM_{p,\lambda}} dt \\
&\le C\int_0^1 \|T_{\varepsilon(t)} f\|_{LM_{p,\lambda}}\, dt \\
&\le C\|f\|_{LM_{p,\lambda}}.
\end{aligned}$$

We now prove (8.4). Let $J \in \mathbb{N}$ be sufficiently large. Then

$$\begin{aligned}\left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j \mathcal{F} g]|^2 \right)^{\frac{1}{2}} \right\|_{LH_{p',\lambda}} &= \left\| \left(\sum_{j=-J}^{J} |\mathcal{F}^{-1}[\tau_j \mathcal{F} g]|^2 \right)^{\frac{1}{2}} \right\|_{LH_{p',\lambda}} \\ &\le C \left\| \int_0^1 \left| \sum_{j=-J}^{J} r_j(t) \mathcal{F}^{-1}[\tau_j \mathcal{F} g] \right| dt \right\|_{LH_{p',\lambda}}\end{aligned}$$

by Lemma 2.4. By the triangle inequality, we have

$$\begin{aligned}\left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j \mathcal{F} g]|^2 \right)^{\frac{1}{2}} \right\|_{LH_{p',\lambda}} &\le C \int_0^1 \left\| \sum_{j=-J}^{J} r_j(t) \mathcal{F}^{-1}[\tau_j \mathcal{F} g] \right\|_{LH_{p',\lambda}} dt \\ &\le C \int_0^1 \| T_{\{\chi_{[-J,J]}(j)\varepsilon_j(t)\}_{j\in\mathbb{Z}}} g \|_{LH_{p',\lambda}} \, dt \\ &\le C \|g\|_{LM_{p,\lambda}}.\end{aligned}$$

Thus, (8.4) follows and the proof is complete. □

Before we prove the remaining assertions, we need the following lemma for local Triebel-Lizorkin-Morrey spaces.

Lemma 8.3. *Let $1 < p < \infty$ and $0 \le \lambda < n$. Let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ be radial functions such that $\chi_{B(1)} \le \psi \le \chi_{B(2)}$ and $\varphi = \psi - \psi(2\cdot)$.*

1. *Assume that $f \in LM_{p,\lambda}(\mathbb{R}^n)$ satisfies*

$$0 \notin \operatorname{supp}(\mathcal{F} f), \quad \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}} < \infty.$$

Then

$$\|f\|_{LM_{p,\lambda}} \le C \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}}. \tag{8.5}$$

2. *Assume that $g \in LH_{p',\lambda}(\mathbb{R}^n)$ satisfies*

$$0 \notin \operatorname{supp}(\mathcal{F} g), \quad \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} g]|^2 \right)^{\frac{1}{2}} \right\|_{LH_{p',\lambda}} < \infty.$$

Then

$$\|g\|_{LH_{p',\lambda}} \le C \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} g]|^2 \right)^{\frac{1}{2}} \right\|_{LH_{p',\lambda}}. \tag{8.6}$$

In (8.5), (8.6) $C > 0$ is independent of f, g respectively.

Proof. 1. Let $f \in LM_{p,\lambda}(\mathbb{R}^n)$. By the duality, for all $f \in LM_{p,\lambda}(\mathbb{R}^n)$, we can find $g \in C^\infty_{\rm comp}(\mathbb{R}^n)$ such that

$$\|f\|_{LM_{p,\lambda}} \le 2\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \tag{8.7}$$

and that

$$\|g\|_{LH_{p',\lambda}} \le 2. \tag{8.8}$$

Indeed, by the duality, we can find $g_0 \in LH_{p',\lambda}(\mathbb{R}^n)$, such that

$$\|f\|_{LM_{p,\lambda}} < 2\left|\int_{\mathbb{R}^n} f(x)g_0(x)\,dx\right| \tag{8.9}$$

and that

$$\|g_0\|_{LH_{p',\lambda}} \le 1. \tag{8.10}$$

By choosing sufficiently large $R > 0$, we have

$$\|f\|_{LM_{p,\lambda}} < 2\left|\int_{\mathbb{R}^n} f(x)\chi_{B(R)}(x)g_0(x)\,dx\right|. \tag{8.11}$$

A mollification of g allows us to assume that (8.7) and (8.8) hold. In view of $0 \notin {\rm supp}(\mathcal{F}f)$, by setting $\tau \equiv \psi(2^{-1}\cdot) - \psi(4\cdot)$ and $\tau_j = \tau(2^{-j}\cdot)$ for $j \in \mathbb{Z}$, we obtain

$$\tau_j(\xi)\varphi_j(\xi) \equiv \varphi_j(\xi) \quad (\xi \in \mathbb{R}^n,\ j \in \mathbb{Z})$$

and

$$f = \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F}[\mathcal{F}^{-1}\tau_j\mathcal{F}f]] \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Thus, by using the $\mathcal{S}(\mathbb{R}^n)$- $\mathcal{S}'(\mathbb{R}^n)$ duality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,dx &= \langle f, g\rangle \\ &= \lim_{J\to\infty}\sum_{j=-J}^{J}\langle \mathcal{F}^{-1}[\varphi_j\mathcal{F}f], \mathcal{F}^{-1}[\tau_j\mathcal{F}g]\rangle \\ &= \lim_{J\to\infty}\int_{\mathbb{R}^n}\sum_{j=-J}^{J}\mathcal{F}^{-1}[\varphi_j\mathcal{F}f](x)\mathcal{F}^{-1}[\tau_j\mathcal{F}g](x)\,dx.\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \\ &\le \lim_{J\to\infty}\int_{\mathbb{R}^n}\left(\sum_{j=-J}^{J}|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f](x)|^2\right)^{\frac12}\left(\sum_{j=-J}^{J}|\mathcal{F}^{-1}[\tau_j\mathcal{F}g](x)|^2\right)^{\frac12}dx.\end{aligned}$$

By Theorem 5.1,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \\
&\le \left\|\left(\sum_{j=-\infty}^{\infty}|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}}\left\|\left(\sum_{j=-\infty}^{\infty}|\mathcal{F}^{-1}[\tau_j\mathcal{F}g]|^2\right)^{\frac{1}{2}}\right\|_{LH_{p',\lambda}}.
\end{aligned}$$

By virtue of (8.4) and (8.8), we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| &\le C\left\|\left(\sum_{j=-\infty}^{\infty}|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}}\|g\|_{LH_{p',\lambda}} \\
&= C\left\|\left(\sum_{j=-\infty}^{\infty}|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}}.
\end{aligned}$$

If we combine this with (8.7), we obtain that

$$\|f\|_{LM_{p,\lambda}} \le C\left\|\left(\sum_{j=-\infty}^{\infty}|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}}.$$

2. Let $g \in LH_{p',\lambda}(\mathbb{R}^n)$. By the Hahn-Banach theorem, we can find $f_0 \in LM_{p,\lambda}(\mathbb{R}^n)$ such that

$$\|g\|_{LH_{p',\lambda}} = \left|\int_{\mathbb{R}^n} f_0(x)g(x)\,dx\right| \tag{8.12}$$

and that

$$\|f_0\|_{LM_{p,\lambda}} = 1. \tag{8.13}$$

Since functions in $LH_{p',\lambda}(\mathbb{R}^n)$ having compact supports form a dense subset, by mollification, we can find $f \in C^\infty_{\rm comp}(\mathbb{R}^n)$ such that

$$\|g\|_{LH_{p',\lambda}} \le 2\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \tag{8.14}$$

and that

$$\|f\|_{LM_{p,\lambda}} \le 2. \tag{8.15}$$

In view $0 \notin {\rm supp}(\mathcal{F}g)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)g(x)\,dx = \langle f, g\rangle &= \lim_{J\to\infty}\sum_{j=-J}^{J}\langle \mathcal{F}^{-1}[\tau_j\mathcal{F}f], \mathcal{F}^{-1}[\varphi_j\mathcal{F}g]\rangle \\
&= \lim_{J\to\infty}\int_{\mathbb{R}^n}\sum_{j=-J}^{J}\mathcal{F}^{-1}[\tau_j\mathcal{F}f](x)\mathcal{F}^{-1}[\varphi_j\mathcal{F}g](x)\,dx.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right|\\
&\le \int_{\mathbb{R}^n} \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j \mathcal{F} f](x)|^2\right)^{\frac{1}{2}} \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} g](x)|^2\right)^{\frac{1}{2}} dx
\end{aligned}$$

By Theorem 5.1, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right|\\
&\le \left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\tau_j \mathcal{F} f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}} \left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} g]|^2\right)^{\frac{1}{2}}\right\|_{LH_{p',\lambda}}.
\end{aligned}$$

By virtue of (8.3) and (8.15), we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| &\le C\|f\|_{LM_{p,\lambda}} \left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} g]|^2\right)^{\frac{1}{2}}\right\|_{LH_{p',\lambda}}\\
&= C\left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} g]|^2\right)^{\frac{1}{2}}\right\|_{LH_{p',\lambda}}.
\end{aligned}$$

If we combine this with (8.14), then we have

$$\|g\|_{LH_{p',\lambda}} \le C\left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} g]|^2\right)^{\frac{1}{2}}\right\|_{LH_{p',\lambda}}.$$

□

Lemma 8.4. *Let $1 < q \le p < \infty$. Let $\varphi \in \mathcal{S}$ be a radial function and define $\varphi_j \equiv \varphi(2^{-j}\cdot)$ for $j \in \mathbb{Z}$. Assume that for $f \in \mathcal{S}'(\mathbb{R}^n)$*

$$\left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}} < \infty.$$

Then the limit

$$F = \lim_{J\to\infty} \sum_{j=-J}^{J} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]$$

exists in the weak topology of $LM_{p,\lambda}(\mathbb{R}^n)$, and

$$\|F\|_{LM_{p,\lambda}} \le C\left\|\left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}}, \tag{8.16}$$

where $C > 0$ is independent of f.

Proof. 1. Let

$$f_J = \sum_{j=-J}^{J} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f] \in LM_{p,\lambda}(\mathbb{R}^n).$$

We can apply Lemma 8.3 (1); recall that we are assuming that $0 \notin \mathrm{supp}(\mathcal{F}f)$. Thus, by Lemma 8.3 (1), we have

$$\|f_J\|_{LM_{p,\lambda}} \le C \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f_J]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}}.$$

Note that $\mathcal{F}^{-1}[\varphi_j \mathcal{F} f_J] = 0$ as long as $|J - j| > 1$. Thus,

$$\|f_J\|_{LM_{p,\lambda}} \le C \left\| \left(\sum_{j=-J-1}^{J+1} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f_J]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}}.$$

Note that $|\mathcal{F}^{-1}[\varphi_j \mathcal{F} f_J]| \le CM[\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]]$. Hence, by virtue of Theorem 3.1, the Fefferman-Stein vector-valued inequality, we have

$$\begin{aligned}
\|f_J\|_{LM_{p,\lambda}} &\le C \left\| \left(\sum_{j=-J-1}^{J+1} M[\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]]^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}} \\
&\le C \left\| \left(\sum_{j=-J-1}^{J+1} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}} \\
&\le C \left\| \left(\sum_{j=-\infty}^{\infty} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]|^2 \right)^{\frac{1}{2}} \right\|_{LM_{p,\lambda}}.
\end{aligned}$$

Since the constant C does not depend upon J, we can apply the Banach-Alaoglu theorem, which asserts that the unit ball of the dual space of a Banach space X is weakly-* compact. By virtue of the Banach-Alaoglu theorem, we can take a subsequence $\{f_{J_m}\}_{m=1}^{\infty}$ which converges in the weak-* topology to an element F in $LM_{p,\lambda}(\mathbb{R}^n)$; that is,

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} f_{J_m}(x) g(x)\, dx = \int_{\mathbb{R}^n} F(x) g(x)\, dx.$$

2. Let us prove $f_J = \sum_{j=-J}^{J} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]$ converges to F in the weak-* topology of $\mathcal{M}^p_q(\mathbb{R}^n)$. To this end, we choose $g \in LH_{p',\lambda}(\mathbb{R}^n)$, so we have

$$\int_{\mathbb{R}^n} F(x) g(x)\, dx = \lim_{m\to\infty} \int_{\mathbb{R}^n} F_{J_m}(x) g(x)\, dx$$

since we know that $F_{J_m}, m \in \mathbb{N}$ converges to a limit F in the weak-* topology of $\mathcal{M}^p_q(\mathbb{R}^n)$. If we use Lemma 8.1, then we obtain

$$\int_{\mathbb{R}^n} F(x) g(x)\, dx = \lim_{J\to\infty} \sum_{j=-J}^{J} \int_{\mathbb{R}^n} F(x) \mathcal{F}^{-1}[\varphi_j \mathcal{F} g](x)\, dx.$$

Since φ is radial, we deduce that $\mathcal{F}^{-1}\varphi = \mathcal{F}\varphi$ and

$$\int_{\mathbb{R}^n} F(x)g(x)\,dx = \lim_{J\to\infty}\int_{\mathbb{R}^n}\sum_{j=-J}^{J}\mathcal{F}^{-1}[\varphi_j\mathcal{F}F](x)g(x)\,dx.$$

By the properties of the Fourier transform, we have

$$\sum_{j=-J}^{J}\mathcal{F}^{-1}[\varphi_j\mathcal{F}F](x) = \frac{1}{(2\pi)^{n/2}}\sum_{j=-J}^{J}\mathcal{F}^{-1}\varphi_j * F(x) = \frac{1}{(2\pi)^{n/2}}\sum_{j=-J}^{J}\langle F, \mathcal{F}^{-1}\varphi_j(x-\cdot)\rangle.$$

By the definition of F_{J_m}, it follows that

$$\begin{aligned}\sum_{j=-J}^{J}\mathcal{F}^{-1}[\varphi_j\mathcal{F}F](x) &= \frac{1}{(2\pi)^{n/2}}\lim_{m\to\infty}\sum_{j=-J}^{J}\langle F_{J_m}, \mathcal{F}^{-1}\varphi_j(x-\cdot)\rangle\\ &= \frac{1}{(2\pi)^{n/2}}\lim_{m\to\infty}\sum_{j=-J}^{J}\sum_{k=-J_m}^{J_m}\langle \mathcal{F}^{-1}[\varphi_k\mathcal{F}f], \mathcal{F}^{-1}\varphi_j(x-\cdot)\rangle\\ &= \frac{1}{(2\pi)^{n/2}}\sum_{j=-J}^{J}\langle f, \mathcal{F}^{-1}\varphi_j(x-\cdot)\rangle\\ &= \sum_{j=-J}^{J}\mathcal{F}^{-1}[\varphi_j\mathcal{F}f](x).\end{aligned}$$

Thus, it follows that

$$\int_{\mathbb{R}^n} F(x)g(x)\,dx = \lim_{J\to\infty}\int_{\mathbb{R}^n}\sum_{j=-J}^{J}\mathcal{F}^{-1}[\varphi_j\mathcal{F}f]g(x)\,dx.$$

Since $g \in LH_{p',\lambda}(\mathbb{R}^n)$ is chosen arbitrarily, it follows that f_J converges to F in the weak-* topology of $\mathcal{M}^p_q(\mathbb{R}^n)$.

3. (8.16) is a consequence of Lemma 8.3(1). Indeed, since $\{f_J\}_{J=1}^{\infty}$ converges in the weak-* topology to F, we have

$$\|F\|_{LM_{p,\lambda}} \le \liminf_{m\to\infty}\|f_{J_m}\|_{LM_{p,\lambda}} \le C\left\|\left(\sum_{j=-\infty}^{\infty}|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f]|^2\right)^{\frac{1}{2}}\right\|_{LM_{p,\lambda}},$$

which proves (8.16). □

The next lemma concerns the uniqueness of F in Lemma 8.4 when $f \in LM_{p,\lambda}(\mathbb{R}^n)$.

Lemma 8.5. *Let $1 < q \le p < \infty$. If $f \in LM_{p,\lambda}(\mathbb{R}^n)$, then*

$$f = \lim_{J\to\infty}\sum_{j=-J}^{J}\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \tag{8.17}$$

in the weak- topology of $LM_{p,\lambda}(\mathbb{R}^n)$.*

Proof. In view of Lemmas 8.2 and 8.4, we see that

$$h = \lim_{J \to \infty} \sum_{j=-J}^{J} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f] \in LM_{p,\lambda}(\mathbb{R}^n),$$

where the convergence takes place in the weak-* topology of $LM_{p,\lambda}(\mathbb{R}^n)$. Since $\mathcal{F}(f-h)$ is supported at the origin, $f-h$ must be a polynomial. Since f and h both belong to $LM_{p,\lambda}(\mathbb{R}^n)$, we see that $f=h$. Consequently, we have (8.17). □

8.2 Smooth decompositions of Morrey spaces

Proof of Theorem 1.3(*A*). Let $f \in LM_{p,\lambda}(\mathbb{R}^n)$. Then we have

$$f = \psi(D) f + \sum_{j=1}^{\infty} \varphi_j(D) f$$

in the weak-* topology of $LM_{p,\lambda}(\mathbb{R}^n)$ according to Theorem 8.1.

Below we do not take the term $\psi(D)f$ into account because this term can be considered separately and then incorporated afterwards.

Let ρ be a function such that

$$\chi_{B(8)\setminus B(1)} \le \rho \le \chi_{B(16)\setminus B(1/2)}.$$

Then

$$\begin{aligned} \varphi_j(D) f(x) &= \rho_j(D)\varphi_j(D) f(x) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathcal{F}^{-1}[\rho_j](x-y)\varphi_j(D) f(y)\,dy \\ &= \sum_{m \in \mathbb{Z}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{2^{-j}m+[0,2^{-j}]^n} \mathcal{F}^{-1}[\rho_j](x-y)\varphi_j(D) f(y)\,dy. \end{aligned}$$

Let us set

$$\mu_{jm}(x) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{2^{-j}m+[0,2^{-j}]^n} \mathcal{F}^{-1}[\rho_j](x-y)\varphi_j(D) f(y)\,dy.$$

Then

$$\partial^\alpha \mu_{jm}(x) \equiv C_{\alpha,n} 2^{-j|\alpha|} \int_{2^{-j}m+[0,2^{-j}]^n} \mathcal{F}^{-1}[(2^{-j}\xi)^\alpha \rho_j](x-y)\varphi_j(D) f(y)\,dy$$

By Theorem 2.1, we have

$$\begin{aligned} &|\partial^\alpha \mu_{jm}(x)| \\ &\le C \inf_{y \in Q_{jm}} M[\varphi_j(D) f](y) 2^{-j|\alpha|} \int_{2^{-j}m+[0,2^{-j}]^n} |\mathcal{F}^{-1}[(2^{-j}\xi)^\alpha \rho_j](x-y)|(1+2^j|x-y|)^n\,dy \\ &\le C \frac{2^{-j|\alpha|}}{(1+2^j|x-2^{-j}m|)^N} \inf_{y \in Q_{jm}} M[\varphi_j(D) f](y). \end{aligned}$$

Therefore, if we set

$$\lambda_{jm} \equiv 2^{-jn/p} \inf_{y \in Q_{jm}} M[\varphi_j(D)f](y), \quad m_{jm} \equiv \frac{1}{\lambda_{jm}} \mu_{jm},$$

then each m_{jm} satisfies all requirements of the theorem and $f = \sum_{j=0}^{\infty} \lambda_{jm} m_{jm}$. □

Proof of Theorem 1.3(*B*). We consider

$$\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}} \lambda_{\nu m} m_{\nu m}.$$

Below we do not take into account the term with $j = 0$ because it can be easily incorporated. By Lemma 2.3, we have

$$\begin{aligned}
&|\lambda_{\nu m} \mathcal{F}^{-1}[\varphi_j] * m_{\nu m}(x)| \\
&\le 2^{\nu n/p} |\lambda_{\nu m}| 2^{\min(j,\nu)n - |\nu - j|} (1 + 2^{-\min(j,\nu)} |x - 2^{-\nu} m|)^{-n-1/2} \\
&\le 2^{\nu n/p} |\lambda_{\nu m}| 2^{\min(j,\nu)n - (n+1/2)|\nu - j|} (1 + 2^{-\nu} |x - 2^{-\nu} m|)^{-n-1/2} \\
&\le 2^{\nu n/p - |\nu - j|/2} M[\lambda_{\nu m} \chi_{\nu m}^{(p)}](x).
\end{aligned}$$

If we use Theorem 3.1, then we obtain the desired result. □

Acknowledgments

Both authors were partially supported by the Grant-in-Aid for Young Scientists (B) No. 24740085 of the Japan Society for the Promotion of Science. They were also partially supported by the Grant-in-Aid for Scientific Research (C), No. 24540194. They wish to express their thanks to Stefan Horocholyn for his careful reading of the paper.

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Received: 10.05.2014