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EMBEDDINGS OF WEIGHTED SOBOLEV CLASSES ON A JOHN DOMAIN

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Abstract. In this paper, embedding theorems for weighted Sobolev classes are obtained. Here weights are functions of distance to some h-set.

1 Introduction

Denote by $AC[t_0, t_1]$ the space of all absolutely continuous functions on an interval $[t_0, t_1]$. For $x \in \mathbb{R}^d$ and $a > 0$ we shall denote by $B_a(x)$ the closed Euclidean ball of radius a in $\mathbb{R}^d$ centered at the point x.

Definition 1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $a > 0$. We say that $\Omega \in \mathbf{FC}(a)$ if there exists a point $x_* \in \Omega$ such that, for any $x \in \Omega$, there exists a curve $\gamma_x : [0, T(x)] \to \Omega$ with the following properties:

1) $\gamma_x \in AC[0, T(x)]$, $\left|\frac{d\gamma_x(t)}{dt}\right| = 1$ a.e.,
2) $\gamma_x(0) = x$, $\gamma_x(T(x)) = x_*$,
3) $B_{at}(\gamma_x(t)) \subset \Omega$ for any $t \in [0, T(x)]$.

Definition 1.2. We say that Ω satisfies the John condition (and call Ω a John domain) if $\Omega \in \mathbf{FC}(a)$ for some $a > 0$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $g, v : \Omega \to (0, \infty)$ be measurable functions. For each measurable vector-valued function $\psi : \Omega \to \mathbb{R}^l$, $\psi = (\psi_k)_{1\le k\le l}$, and for each $p \in [1, \infty]$, we put $\|\psi\|_{L_p(\Omega)} = \left\| \max_{1\le k\le l} |\psi_k| \right\|_p$.

Let $\overline{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d := (\mathbb{N}\cup\{0\})^d$, $|\overline{\beta}| = \beta_1 + \ldots + \beta_d$. For $r \in \mathbb{N}$ and for any distribution f defined on Ω we write $\nabla^r f = \left(\partial^r f/\partial x^{\overline{\beta}}\right)_{|\overline{\beta}|=r}$ (here partial derivatives are understood in the sense of distributions), and denote by $l_{r,d}$ the number of components of the vector-valued distribution $\nabla^r f$.

We call the set

$$W^r_{p,g}(\Omega) = \left\{f : \Omega \to \mathbb{R} \middle|\, \exists \psi : \Omega \to \mathbb{R}^{l_{r,d}} : \|\psi\|_{L_p(\Omega)} \le 1, \nabla^r f = g \cdot \psi\right\}$$

a weighted Sobolev class and denote the corresponding function ψ by $\frac{\nabla^r f}{g}$.

A weighted Lebesgue space is defined by $L_{q,v}(\Omega) = \{f : \Omega \to \mathbb{R} | \ \|f\|_{L_{q,v}(\Omega)} < \infty\}$, where $\|f\|_{L_{q,v}(\Omega)} = \|fv\|_{L_q(\Omega)}$ (it is assumed that f is measurable on Ω).

We denote by $\mathbb{H}$ the set of all nondecreasing positive functions defined on $(0, 1]$.

Definition 1.3. (see [3]). Let $\Gamma \subset \mathbb{R}^d$ be a nonempty compact set and $h \in \mathbb{H}$. We say that Γ is an h-set if there are a constant $c_* \geq 1$ and a finite countably additive measure μ on $\mathbb{R}^d$ such that $\operatorname{supp} \mu = \Gamma$ and $c_*^{-1} h(t) \leq \mu(B_t(x)) \leq c_* h(t)$ for any $x \in \Gamma$ and $t \in (0, 1]$.

Given $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we set $\operatorname{dist}(x, E) = \inf\{|x - y| : \ y \in E\}$.

Let $g(x) = \varphi_g(\operatorname{dist}(x, \Gamma))$, $v(x) = \varphi_v(\operatorname{dist}(x, \Gamma))$, $\varphi_g, \ \varphi_v : (0, \infty) \to (0, \infty)$. Suppose that there exists $c_0 \geq c_*$ such that

$$\frac{h(t)}{h(s)} \leq c_0, \quad \frac{\varphi_g(t)}{\varphi_g(s)} \leq c_0, \quad \frac{\varphi_v(t)}{\varphi_v(s)} \leq c_0, \quad j \in \mathbb{N}, \ \ t, \ s \in [2^{-j-1}, \ 2^{-j+1}].$$

Let $\mathcal{G}$ be a graph that has neither multiple edges nor loops. We denote by $\mathbf{V}(\mathcal{G})$ the vertex set of $\mathcal{G}$, and by $\mathbf{E}(\mathcal{G})$, the edge set. We identify pairs of adjacent vertices with edges that connect them. Given a function $f : \mathbf{V}(\mathcal{G}) \to \mathbb{R}$, we set $\|f\|_{l_p(\mathcal{G})} = \left(\sum_{\xi \in \mathbf{V}(\mathcal{G})} |f(\xi)|^p \right)^{1/p}$. Denote by $l_p(\mathcal{G})$ the space of functions $f : \mathbf{V}(\mathcal{G}) \to \mathbb{R}$ with finite norm $\|f\|_{l_p(\mathcal{G})}$.

Let $\mathcal{T} = (\mathcal{T}, \xi_0)$ be a tree rooted at ξ_0. We introduce a partial order on $\mathbf{V}(\mathcal{T})$ as follows: we say that $\xi' > \xi$ if there exists a simple path $(\xi_0, \xi_1, \ldots, \xi_n, \xi')$ such that $\xi = \xi_k$ for some $k \in \overline{0, n}$; by the distance between ξ and ξ' we mean the quantity $\rho_{\mathcal{T}}(\xi, \xi') = \rho_{\mathcal{T}}(\xi', \xi) = n + 1 - k$. In addition, we set $\rho_{\mathcal{T}}(\xi, \xi) = 0$. For $j \in \mathbb{Z}_+$ and $\xi \in \mathbf{V}(\mathcal{T})$, let

$$\mathbf{V}_j(\xi) := \mathbf{V}_j^{\mathcal{T}}(\xi) := \{\xi' \geq \xi : \ \rho_{\mathcal{T}}(\xi, \xi') = j\}.$$

Given $\xi \in \mathbf{V}(\mathcal{T})$, we denote by $\mathcal{T}_\xi = (\mathcal{T}_\xi, \xi)$ the subtree in $\mathcal{T}$ with the vertex set $\{\xi' \in \mathbf{V}(\mathcal{T}) : \xi' \geq \xi\}$.

For every cube K and $s \in \mathbb{Z}_+$, we denote by $\Xi_s(K)$ the partition of K into 2^{sd} pairwise disjoint cubes of the same size, $\Xi(K) := \bigcup_{s \in \mathbb{Z}_+} \Xi_s(K)$.

Definition 1.4. Let $\Theta \subset \Xi\left(\left[-\frac{1}{2}, \ \frac{1}{2}\right]^d\right)$ be a family of pairwise disjoint cubes, let $\mathcal{G}$ be a graph, and let $F : \mathbf{V}(\mathcal{G}) \to \Theta$ be a one-to-one mapping. We say that F is consistent with the structure of the graph $\mathcal{G}$ if, for any adjacent vertices $\xi', \ \xi'' \in \mathbf{V}(\mathcal{G})$, the set $\Gamma_{\xi',\xi''} := F(\xi') \cap F(\xi'')$ has dimension $d - 1$.

Let $(\mathcal{T}, \xi_*)$ be a tree, and let $F : \mathbf{V}(\mathcal{T}) \to \Theta$ be a one-to-one mapping consistent with the structure of the tree $\mathcal{T}$. For any adjacent vertices $\xi', \ \xi''$, we denote by $\mathring{\Gamma}_{\xi',\xi''}$ the interior of $\Gamma_{\xi',\xi''}$ with respect to the induced topology on the affine span of $\Gamma_{\xi',\xi''}$. For each subtree $\mathcal{T}'$ of $\mathcal{T}$, we put

$$\Omega_{\mathcal{T}',F} = \left(\cup_{\xi \in \mathbf{V}(\mathcal{T}')} \operatorname{int} F(\xi)\right) \cup \left(\cup_{(\xi',\xi'') \in \mathbf{E}(\mathcal{T}')} \mathring{\Gamma}_{\xi',\xi''}\right).$$

Let $\Theta(\Omega)$ be a Whitney covering of Ω (see [6, p. 562]). The following result is proved in [7].

Theorem A. *Let $a > 0$, $\Omega \subset \left[-\frac12, \frac12\right]^d$, $\Omega \in \mathbf{FC}(a)$. Then there exist a tree $(\mathcal{T}, \xi_0)$ and a one-to-one mapping $F : \mathbf{V}(\mathcal{T}) \to \Theta(\Omega)$ consistent with the structure of $\mathcal{T}$ and a number $b_* = b_*(a, d) > 0$ such that $\Omega_{\mathcal{T}',F} \in \mathbf{FC}(b_*)$ for any subtree $\mathcal{T}'$ of $\mathcal{T}$.*

2 Main results

Let X, Y be sets, $f_1, f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \underset{y}{\lesssim} f_2(x, y)$ (or $f_2(x, y) \underset{y}{\gtrsim} f_1(x, y)$) if, for any $y \in Y$, there exists $c(y) > 0$ such that $f_1(x, y) \le c(y) f_2(x, y)$ for each $x \in X$; $f_1(x, y) \underset{y}{\asymp} f_2(x, y)$ if $f_1(x, y) \underset{y}{\lesssim} f_2(x, y)$ and $f_2(x, y) \underset{y}{\lesssim} f_1(x, y)$.

Let $\{(\mathcal{D}_{j,i}, \hat\xi_{j,i})\}_{j\ge j_*, i\in \tilde I_j}$ be a partition of the tree $\mathcal{T}$ into non-empty subtrees. We say that the tree $\mathcal{D}_{j',i'}$ succeeds the tree $\mathcal{D}_{j,i}$ if $\hat\xi_{j,i} < \hat\xi_{j',i'}$ and $\{\xi \in \mathcal{T} : \hat\xi_{j,i} \le \xi < \hat\xi_{j',i'}\} \subset \mathbf{V}(\mathcal{D}_{j,i})$. We say that this partition is regular if $j' = j+1$ for any $j, j' \ge j_*$, $i \in \tilde I_j$, $i' \in \tilde I_{j'}$ such that $\mathcal{D}_{j',i'}$ succeeds the tree $\mathcal{D}_{j,i}$.

In [8] the number $\overline{s} = \overline{s}(a, d) \in \mathbb{N}$ is defined and the regular partition $\{(\mathcal{D}_{j,i}, \hat\xi_{j,i})\}_{j\ge j_*, i\in \tilde I_j}$ of the tree $\mathcal{T}$ is constructed. This partition satisfies the following conditions: 1. $\operatorname{diam}\Omega_{\mathcal{D}_{j,i},F} \underset{a,d}{\asymp} 2^{-\overline{s}j}$ for any $j \ge j_*$, $i \in \tilde I_j$; 2. $\operatorname{dist}(x, \Gamma) \underset{a,d}{\asymp} 2^{-\overline{s}j}$ for any $x \in \Omega_{\mathcal{D}_{j,i},F}$.

Here we may assume that $\overline{s} \ge 4$. Given $j \ge j_*$, $t \in \tilde I_j$, $l \in \mathbb{Z}_+$, we define $\tilde I^l_{j,t}$ by

$$\tilde I^l_{j,t} = \{i \in \tilde I_{j+l} : \hat\xi_{j+l,i} \ge \hat\xi_{j,t}\}.$$

Let $\mathcal{A}$ be the tree with vertex set $\{\eta_{j,i}\}_{j\ge j_*, i\in \tilde I_j}$ and edge set defined by $\mathbf{V}^{\mathcal{A}}_1(\eta_{j,i}) = \{\eta_{j+1,s}\}_{s\in \tilde I^1_{j,i}}$. In [9] it is proved that $\operatorname{card}\mathbf{V}^{\mathcal{A}}_l(\eta_{j,i}) \underset{a,d,c_0}{\lesssim} \frac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}(j+l)})}$, $j \ge j_*$, $l \in \mathbb{Z}_+$.

Denote by $\mathcal{P}_{r-1}(\mathbb{R}^d)$ the space of polynomials on $\mathbb{R}^d$ of degree not exceeding $r-1$. For a measurable set $E \subset \mathbb{R}^d$ we put $\mathcal{P}_{r-1}(E) = \{f|_E : f \in \mathcal{P}_{r-1}(\mathbb{R}^d)\}$.

Given $j \ge j_*$, $i \in \tilde I_j$, we set

$$u(\eta_{j,i}) = u_j = \varphi_g(2^{-\overline{s}j}) \cdot 2^{-\left(r-\frac{d}{p}\right)\overline{s}j}, \quad w(\eta_{j,i}) = w_j = \varphi_v(2^{-\overline{s}j}) \cdot 2^{-\frac{d\overline{s}j}{q}}, \tag{2.1}$$

$\Omega[\eta_{j,i}] := \Omega_{\mathcal{D}_{j,i},F}$. Given a subtree $\mathcal{D} \subset \mathcal{A}$, we write $\Omega[\mathcal{D}] = \cup_{\xi\in\mathbf{V}(\mathcal{D})}\Omega[\xi]$.

Define the summation operator $S_{u,w,\mathcal{D}}$ by

$$S_{u,w,\mathcal{D}} f(\xi) = w(\xi)\sum_{\xi'\le\xi} u(\xi') f(\xi'), \quad \xi \in \mathbf{V}(\mathcal{D}), \quad f : \mathbf{V}(\mathcal{D}) \to \mathbb{R}.$$

Given $1 \le p, q \le \infty$, by $\mathfrak{S}^{p,q}_{\mathcal{D},u,w}$ we denote the operator norm of $S_{u,w,\mathcal{D}} : l_p(\mathcal{D}) \to l_q(\mathcal{D})$.

Throughout the paper, we assume that

$$1 < p \le \infty, \quad 1 \le q < \infty, \quad \delta := r + \frac{d}{q} - \frac{d}{p} > 0.$$

Let $\mathfrak{Z} = (a, d, r, p, q, h, c_0, g, v)$.

Theorem 2.1. *Let u, w be defined by (2.1) and let $\mathfrak{S}^{p,q}_{\mathcal{A},u,w} < \infty$. Then $W^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and for any vertex $\xi_* \in \mathbf{V}(\mathcal{A})$ there is a linear continuous operator $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any subtree $\mathcal{D} \subset \mathcal{A}$ rooted at ξ_* and for any function $f \in W^r_{p,g}(\Omega)$*

$$\|f - Pf\|_{L_{q,v}(\Omega[\mathcal{D}])} \underset{\mathfrak{Z}}{\lesssim} \mathfrak{S}^{p,q}_{\mathcal{D},u,w} \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega[\mathcal{D}])}.$$

The proof is the same as in [8, 9]. In order to estimate $\mathfrak{S}^{p,q}_{\mathcal{D},u,w}$, we apply results obtained in [1, 2, 5, 10] and the family of partitions of Γ constructed in [4].

Theorem 2.2. *Let $1 < p < q < \infty$ and let u, w be defined by (2.1). Suppose that there are $l_0 \in \mathbb{N}$ and $\lambda \in (0, 1)$ such that $\frac{\|w\|_{l_q(\mathcal{A}_{\xi'})}}{w(\xi)} \le \lambda$, $\xi \in \mathbf{V}(\mathcal{A})$, $\xi' \in \mathbf{V}^{\mathcal{A}}_{l_0}(\xi)$. Let*

$$\sup_{\xi \in \mathbf{V}(\mathcal{A})} u(\xi) \|w\|_{l_q(\mathcal{A}_\xi)} < \infty.$$

Then $W^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and for any vertex $\xi_* \in \mathbf{V}(\mathcal{A})$ there exists a linear continuous operator $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any subtree $\mathcal{D} \subset \mathcal{A}$ rooted at ξ_* and for any function $f \in W^r_{p,g}(\Omega)$

$$\|f - Pf\|_{L_{q,v}(\Omega[\mathcal{D}])} \underset{\mathfrak{Z}}{\lesssim} \sup_{\xi \in \mathbf{V}(\mathcal{D})} u(\xi) \|w\|_{l_q(\mathcal{D}_\xi)} \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega[\mathcal{D}])}.$$

Theorem 2.3. *Let $p \ge q$, $\xi_* \in \mathbf{V}^{\mathcal{A}}_{j_0}(\xi_0)$, and let u, w be defined by (2.1). We set $\hat{w}_j = w_j \cdot \left(\frac{h(2^{-\overline{s}j_0})}{h(2^{-\overline{s}j})}\right)^{\frac{1}{q}}$, $\hat{u}_j = u_j \cdot \left(\frac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}j_0})}\right)^{\frac{1}{p}}$ for any $j \ge j_0$. Let*

$$M_{\hat{u},\hat{w}}(j_0) := \sup_{j_0 \le j < \infty} \left(\sum_{i=j}^{\infty} \hat{w}_i^q\right)^{\frac{1}{q}} \left(\sum_{i=j_0}^{j} \hat{u}_i^{p'}\right)^{\frac{1}{p'}} < \infty \quad \text{for} \quad p = q,$$

$$M_{\hat{u},\hat{w}}(j_0) := \left(\sum_{j=j_0}^{\infty} \left(\left(\sum_{i=j}^{\infty} \hat{w}_i^q\right)^{\frac{1}{p}} \left(\sum_{i=j_0}^{j} \hat{u}_i^{p'}\right)^{\frac{1}{p'}}\right)^{\frac{pq}{p-q}} \hat{w}_j^q\right)^{\frac{1}{q}-\frac{1}{p}} < \infty \quad \text{for} \quad 1 \le q < p \le \infty.$$

Then $W^r_{p,g}(\Omega[\mathcal{A}_{\xi_*}]) \subset L_{q,v}(\Omega[\mathcal{A}_{\xi_*}])$ and there is a linear continuous operator $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any subtree $\mathcal{D} \subset \mathcal{A}_{\xi_*}$ rooted at ξ_* and for any function $f \in W^r_{p,g}(\Omega)$

$$\|f - Pf\|_{L_{q,v}(\Omega[\mathcal{D}])} \underset{\mathfrak{Z}}{\lesssim} M_{\hat{u},\hat{w}}(j_0) \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega[\mathcal{D}])}.$$

In conclusion, we notice that in [9] the functions h, φ_g, φ_v were defined as follows:

$$h(t) = t^\theta |\log t|^\gamma \tau(|\log t|),$$

$$\varphi_g(t) = t^{-\beta_g} |\log t|^{-\alpha_g} \rho_g(|\log t|), \quad \varphi_v(t) = t^{-\beta_v} |\log t|^{-\alpha_v} \rho_v(|\log t|),$$

where τ, ρ_g, $\rho_v : (0, \infty) \to (0, \infty)$ are absolutely continuous functions such that

$$\lim_{y\to\infty} \frac{y\tau'(y)}{\tau(y)} = \lim_{y\to\infty} \frac{y\rho_g'(y)}{\rho_g(y)} = \lim_{y\to\infty} \frac{y\rho_v'(y)}{\rho_v(y)} = 0,$$

$0 \le \theta < d$, $\beta_v < \frac{d-\theta}{q}$, $\beta_g + \beta_v = \delta - \theta \left(\frac{1}{q} - \frac{1}{p}\right)_+$. In [9] the inequality $\alpha_g + \alpha_v > (1-\gamma)\left(\frac{1}{q} - \frac{1}{p}\right)_+$ was shown to be sufficient for the embedding $W^r_{p,g}(\Omega)$ into $L_{q,v}(\Omega)$. For $\beta_v < \frac{d-\theta}{q}$ estimating the norm of summation operator is fairly simple. However, this simple method cannot be applied for $\beta_v = \frac{d-\theta}{q}$.

Let $\theta > 0$, $\beta_v = \frac{d-\theta}{q}$, $\alpha_v > \frac{1-\gamma}{q}$. Then Theorems 2.2, 2.3 yield that the inequalities

$$\alpha_g + \alpha_v > \frac{1}{q} \ \ (\text{for} \ \ p < q) \ \ \text{and} \ \ \alpha_g + \alpha_v > 1 + (1-\gamma)\left(\frac{1}{q} - \frac{1}{p}\right) \ \ (\text{for} \ \ p \ge q)$$

are sufficient conditions for the embedding $W^r_{p,g}(\Omega)$ in $L_{q,v}(\Omega)$ (this can be proved by quite straightforward calculation).

An informative historical account of embeddings of weighted Sobolev spaces may be found in [8].

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References

[1] K.F. Andersen, H.P. Heinig, *Weighted norm inequalities for certain integral operators.* SIAM J. Math. Anal. 14 (1983), no. 4, 834–844.

[2] G. Bennett, *Some elementary inequalities. III,* Quart. J. Math. Oxford Ser. (2), 42(1991) no. 166, 149–174.

[3] M. Bricchi, *Existence and properties of h-sets,* Georgian Mathematical Journal. 9 (2002), no. 1, 13–32.

[4] M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. 60/61 (1990), no. 2, 601–628.

[5] H.P. Heinig, *Weighted norm inequalities for certain integral operators, II*, Proc. AMS. 95 (1985), no. 3, 387–395.

[6] G. Leoni, *A first Course in Sobolev Spaces*, Graduate studies in Mathematics, vol. 105. AMS, Providence, Rhode Island, 2009.

[7] A.A. Vasil'eva, *Widths of weighted Sobolev classes on a John domain*, Proc. Steklov Inst. Math. 280 (2013), 91–119.

[8] A.A. Vasil'eva, *Embedding theorem for weighted Sobolev classes on a John domain with weights that are functions of the distance to some h-set*, Russ. J. Math. Phys. 20 (2013), no. 3, 360–373.

[9] A.A. Vasil'eva, *Embedding theorem for weighted Sobolev classes on a John domain with weights that are functions of the distance to some h-set*, Russ. J. Math. Phys. 21 (2014), no. 1, 112–122.

[10] A.A. Vasil'eva, *Estimates for norms of two-weighted summation operators on a tree under some conditions on weights*, arXiv:1311.0375.

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