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ON THE SPECTRUM OF A NONLINEAR OPERATOR ASSOCIATED WITH CALCULATION OF THE NORM OF A LINEAR VECTOR-FUNCTIONAL

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Abstract. An explicit formula is presented for the norm if $1 \le p \le \infty$ and for the quasi-norm if $0 < p < 1$ of a linear vector-functional $L : H \to l_p$ on a Hilbert space H and the set of all extremal elements is described. All eigenvalues and eigenvectors of a nonlinear homogeneous operator entering the corresponding Euler's equation, are written out explicitly.

1 Introduction

Let H be a complex Hilbert space with the inner product $(x, y), x, y \in H$, and $l : H \to \mathbb{C}$ be a continuous linear functional on H. By the well-known Riesz Theorem it has the form $Lx = (x, e), x \in H$, where the element e is uniquely defined by the functional l. Moreover, $\|l\|_{H\to\mathbb{C}} = \|e\|$ and each extremal element x, that is an element $x \in H, x \neq 0$, for which $|lx| = \|l\|_{H\to\mathbb{C}}\, \|x\|$ has the form $x = c\, e$ where $c \in \mathbb{C}, c \neq 0$. (See, for example, [3], Section 3.8 for detailed proof, corollaries and applications.)

We consider the case of a linear vector-functional

$$L = \{l_k\}_{k=1}^m ,$$

where $m \in \mathbb{N}$ or $m = \infty$ and $l_k : H \to \mathbb{C}$ are continuous linear functionals on H. By the Riesz Theorem there exist uniquely defined elements $e_k \in H$ such that

$$Lx = \{(x, e_k)\}_{k=1}^m , \quad x \in H . \tag{1.1}$$

Let $\{e_k\}_{k=1}^m$ be an orthogonal system of non-zero elemets in H, i. e.

$$(e_i, e_k) = 0 , \quad i, k = \overline{1, m},\, i \neq k ; \quad (e_k, e_k) > 0 , \quad k = \overline{1, m}$$

and let, for $z = \{z_k\}_{k=1}^m \subset \mathbb{C}$ and $0 < p \le \infty$,

$$\|z\|_p = \begin{cases} \left(\sum\limits_{k=1}^{m} |z_k|^p \right)^{\frac{1}{p}} & \text{if} \quad 0 < p < \infty, \\ \sup\limits_{k=\overline{1,m}} |z_k| & \text{if} \qquad p = \infty. \end{cases}$$

We consider the problem of calculating $\|L\|_p$, the norm if $1 \le p \le \infty$ and the quasi-norm if $0 < p < 1$, of the vector-functional L as an operator acting from H to l_p, that is

$$\|L\|_p \equiv \|L\|_{H\to l_p} = \sup_{x\in H, x\neq 0} \frac{\|Lx\|_p}{\|x\|} = \sup_{x\in H, x\neq 0} \frac{\left(\sum\limits_{k=1}^{m} |(x,e_k)|^p\right)^{\frac{1}{p}}}{\|x\|},$$

and of describing for the case $\|L\|_p < \infty$ the corresponding set E_p of all extremal elements, that is

$$E_p = \left\{x \in H : \ x \neq 0, \quad \frac{\|Lx\|_p}{\|x\|} = \|L\|_p\right\}.$$

We derive Euler's equation for this extremal problem and investigate the nonlinear homogeneous operator entering this equation. We find all its eigenvalues and all corresponding eigenvectors.

2 Main results

Lemma 2.1. *Let $0 < p \le \infty$ and $\|L\|_p < \infty$. If $0 < p < 2$ and $x \in E_p$, then $(x, e_k) \neq 0$ for all $k = \overline{1, m}$. If $2 \le p \le \infty$, then there exists $x \in E_p$ such that $(x, e_k) = 0$ for some $k = \overline{1, m}$.*

Lemma 2.2. (Euler's equation) *Let $0 < p < \infty$. If $\|L\|_p < \infty$ and $x \in E_p$, then*

$$\sum_{k=1}^{m} |(x,e_k)|^p < \infty, \ \sum_{k=1}^{m} |(x,e_k)|^{2(p-1)} \|e_k\|^2 < \infty$$

and there exists $\lambda = \lambda(x) \in \mathbb{C}$ such that

$$\|x\| \left(\sum_{k=1}^{m} |(x,e_k)|^p\right)^{\frac{1-p}{p}} \sum_{k=1}^{m} |(x,e_k)|^{p-2} (x,e_k)\, e_k = \lambda\, x\,. \tag{2.1}$$

(*By Lemma* 2.1 $(x, e_k) \neq 0, \ k = \overline{1, m}$, *for* $0 < p < 2$, *hence the quantities* $|(x, e_k)|^{2(p-1)}$, $|(x, e_k)|^{p-2}$ *respectively, are defined for all* $x \in E_p$. *If* $(x, e_k) = 0$ *for all* $k = \overline{1, m}$, *which is not excluded if* $p \ge 2$, *then it is assumed that the left-hand side of this equality is equal to* 0.)

We note the following particular cases of equality (2.1). For $p = 1$ equality (2.1) takes the form

$$\|x\| \sum_{k=1}^{m} \frac{(x,e_k)}{|(x,e_k)|}\, e_k = \lambda\, x$$

and for $p = 2$ equality (2.1) takes the form

$$\|x\| \left(\sum_{k=1}^{m} |(x,e_k)|^2\right)^{-\frac{1}{2}} \sum_{k=1}^{m} (x,e_k)\, e_k = \lambda\, x\,.$$

Equality (2.1) implies that

$$\lambda = \frac{\left(\sum\limits_{k=1}^{m} |(x,e_k)|^p\right)^{\frac{1}{p}}}{\|x\|} \geq 0\,.$$

Let, for $2 \leq p < \infty$,

$$D_p = \Big\{x \in H:\ \sum_{k=1}^{m} |(x,e_k)|^p < \infty,\ \sum_{k=1}^{m} |(x,e_k)|^{2(p-1)}\|e_k\|^2 < \infty\Big\}.$$

and, for $0 < p < 2$,

$$D_p = \Big\{x \in H:\ (x,e_k) \neq 0,\ k = \overline{1,m};\ \sum_{k=1}^{m} |(x,e_k)|^p\,,\ \sum_{k=1}^{m} |(x,e_k)|^{2(p-1)}\|e_k\|^2 < \infty\Big\}.$$

Consider the nonlinear homogeneous operator $A_p : D_p \to H$ defined by the left-hand side of equality (2.1):

$$A_p x = \|x\| \Big(\sum_{k=1}^{m} |(x,e_k)|^p\Big)^{\frac{1-p}{p}} \sum_{k=1}^{m} |(x,e_k)|^{p-2}(x,e_k)\, e_k\,. \tag{2.2}$$

Clearly, equation (2.1), together with the assumption $(x,e_k) \neq 0$, $k = \overline{1,m}$ for $0 < p < 2$ based on Lemma 1, is equivalent to the eigenvalue problem

$$A_p x = \lambda x\,, \quad x \in D_p\,. \tag{2.3}$$

Denote by Λ_p the set of all eigenvalues of the operator A_p. Note that $\Lambda_p \subset [0,\infty)$. The case $\lambda = 0$ is trivial. If $0 < p < 2$, then equation (2.1) and Lemma 1 imply that $0 \notin \Lambda_p$. Let $2 \leq p < \infty$ and let H_0 be the closed linear subspace spanned by the system $\{e_k\}_{k=1}^m$. If $H_0 = H$, then equality (2.1) with $\lambda = 0$ implies that $x = 0$, hence $0 \notin \Lambda_p$. If $H_0 \neq H$, then $0 \in \Lambda_p$ and each element $x \in H_0^{\perp}, x \neq 0$, is an eigenvector corresponding to the eigenvalue 0.

Denote by Λ_p^+ the set of all positive eigenvalues of the operator A_p.

Theorem 2.1. 1. *If $0 < p < 2$, then*

$$\Lambda_p^+ = \begin{cases} \varnothing & \text{if } \sum\limits_{k=1}^{m} \|e_k\|^{\frac{2p}{2-p}} = \infty\,, \\ \Big\{\Big(\sum\limits_{k=1}^{m} \|e_k\|^{\frac{2p}{2-p}}\Big)^{\frac{1}{p}-\frac{1}{2}}\Big\} & \text{if } \sum\limits_{k=1}^{m} \|e_k\|^{\frac{2p}{2-p}} < \infty\,. \end{cases}$$

For $\lambda = \Big(\sum\limits_{k=1}^{m} \|e_k\|^{\frac{2p}{2-p}}\Big)^{\frac{1}{p}-\frac{1}{2}} < \infty$ each corresponding eigenvector x has the form

$$x = \mu \sum_{k=1}^{m} a_k \|e_k\|^{\frac{2(p-1)}{2-p}} e_k\,,$$

where $\mu > 0, a_k \in \mathbb{C}, |a_k| = 1, k = \overline{1, m}$.

2. *If* $p = 2$, *then*

$$\Lambda_2^+ = \left\{ \|e_k\| \right\}_{k=1}^m$$

and for any $\lambda \in \Lambda_2^+$ *each corresponding eigenvector* x *has the form*

$$x = \sum_{k \in S_\lambda} c_k e_k \,,$$

where $c_k \in \mathbb{C}, 0 < \sum\limits_{k \in S_\lambda} |c_k|^2 \|e_k\|^2 < \infty$ *and*

$$S_\lambda = \{k = \overline{1, m} : \ \|e_k\| = \lambda\} \,.$$

3. *If* $2 < p < \infty$, *then*

$$\Lambda_p^+ = \left\{ \left(\sum_{k \in S} \|e_k\|^{\frac{2p}{2-p}} \right)^{\frac{1}{p} - \frac{1}{2}} \right\}_{\varnothing \neq S \subset \{1, \ldots, m\}} ,$$

where the set $\varnothing \neq S \subset \{1, \ldots, m\}$ *is such that* $\sum\limits_{k \in S} \|e_k\|^{\frac{2p}{2-p}} < \infty$.

For any $\lambda \in \Lambda_p^+$ *each corresponding eigenvector* x *has the form*

$$x = \mu \sum_{k \in S_\lambda} a_k \|e_k\|^{\frac{2(p-1)}{2-p}} e_k \,,$$

where $\mu > 0, a_k \in \mathbb{C}, |a_k| = 1, k \in S_\lambda$, *and the set* $S_\lambda \subset \{1, \ldots, m\}$ *is such that*

$$\left(\sum_{k \in S_\lambda} \|e_k\|^{\frac{2p}{2-p}} \right)^{\frac{1}{p} - \frac{1}{2}} = \lambda \,.$$

Corollary 2.1. *Let* $0 < p < \infty$. *If*

$$\sum_{k=1}^m \|e_k\|^{\frac{2p}{2-p}} = \infty \text{ for } 0 < p < 2 \quad \text{or} \quad \sup_{k = \overline{1,m}} \|e_k\| = \infty \text{ for } 2 \leq p < \infty \,,$$

then

$$\|L\|_p = \infty \,.$$

Otherwise

$$\|L\|_p = \sup \Lambda_p^+ < \infty \,.$$

Corollary 2.2. *Let* $\{e_k\}_{k=1}^m$ *be an orthonormal system in* H.

1. *If* $0 < p < 2$, *then*

$$\Lambda_p^+ = \begin{cases} \varnothing & \text{if} \quad m = \infty \,, \\ \left\{ m^{\frac{1}{p} - \frac{1}{2}} \right\} & \text{if} \quad m \in \mathbb{N} \,. \end{cases}$$

For $\lambda = m^{\frac{1}{p} - \frac{1}{2}}, m \in \mathbb{N}$, *each corresponding eigenvector* x *has the form*

$$x = \mu \sum_{k=1}^m a_k e_k \,,$$

where $\mu > 0, a_k \in \mathbb{C}, |a_k| = 1, k = \overline{1,m}$.

2. *If* $p = 2$, *then* $\Lambda_2^+ = \{1\}$ *and each eigenvector corresponding to the eigenvalue* 1 *has the form*

$$x = \sum_{k=1}^{m} c_k e_k \,,$$

where $c_k \in \mathbb{C}$ *and* $0 < \sum\limits_{k=1}^{m} |c_k|^2 < \infty$.

3. *If* $2 < p < \infty$, *then*

$$\Lambda_p^+ = \left\{ s^{\frac{1}{p}-\frac{1}{2}} \right\}_{s=1}^{m} .$$

For the eigenvalue $\lambda = s^{\frac{1}{p}-\frac{1}{2}}$ $(s = \overline{1,m})$ *each corresponding eigenvector has the form*

$$x = \mu \sum_{k=1}^{m} a_k e_k \,,$$

where $\mu > 0$, *exactly* s *coefficients* $a_k \in \mathbb{C}$ *are not equal to* 0, *and* $|a_k| = 1$ *for all nonzero coefficients.*

Theorem 2.2. 1. *If* $0 < p < 2$, *then*

$$\|L\|_p = \Big(\sum_{k=1}^{m} \|e_k\|^{\frac{2p}{2-p}} \Big)^{\frac{1}{p}-\frac{1}{2}} ,$$

and, if $2 \le p \le \infty$, *then*

$$\|L\|_p = \sup_{k=\overline{1,m}} \|e_k\| \,.$$

2. *Let* $\|L\|_p < \infty$. *If* $0 < p < 2$, *then*

$$E_p = \Big\{ \mu \sum_{k=1}^{m} a_k \|e_k\|^{\frac{2(p-1)}{2-p}} e_k : \mu > 0;\ a_k \in \mathbb{C}, |a_k| = 1, k = \overline{1,m} \Big\} .$$

If $2 \le p \le \infty$, *let*

$$K = \Big\{ k = \overline{1,m} : \|e_k\| = \sup_{s=\overline{1,m}} \|e_s\| \Big\} .$$

If $K = \varnothing$, *then* $E_p = \varnothing$. *If* $K \neq \varnothing$, *then*

$$E_2 = \Big\{ \sum_{k\in K} c_k e_k : c_k \in \mathbb{C}, 0 < \sum_{k\in K} |c_k|^2 \|e_k\|^2 < \infty \Big\}$$

and for $2 < p \le \infty$

$$E_p = \Big\{ c_k e_k : k \in K,\ c_k \in \mathbb{C}, c_k \neq 0 \Big\} .$$

Corollary 2.3. *Let* $\{e_k\}_{k=1}^m$ *be an orthonormal system in* H*. Then*

$$\|L\|_p = \begin{cases} \infty & \text{if} \quad 0<p<2, \quad m=\infty\,, \\ m^{\frac{1}{p}-\frac{1}{2}} & \text{if} \quad 0<p<2, \quad m\in\mathbb{N}\,, \\ 1 & \text{if} \quad 2\le p\le\infty, \quad m\in\mathbb{N} \text{ or } m=\infty\,. \end{cases}$$

Moreover,

$$E_p = \Big\{\mu\sum_{k=1}^{m} a_k e_k : \mu>0;\ a_k\in\mathbb{C}, |a_k|=1, k=\overline{1,m}\Big\}$$

for $0<p<2$ *and* $m\in\mathbb{N}$*,*

$$E_2 = \Big\{\sum_{k\in K} c_k e_k : c_k\in\mathbb{C}, 0<\sum_{k\in K}|c_k|^2<\infty\Big\}$$

and

$$E_p = \Big\{c_k e_k : k=\overline{1,m},\ c_k\in\mathbb{C}, c_k\neq 0\Big\}$$

for $2<p\le\infty$*.*

Remark 2. Theorem 2.2 is a corollary of Theorem 2.1 It is also possible to give a proof of Theorem 2.2 without applying Theorem 2.1, by using Hölder's and Jensen's inequalities and investigating the cases in which equalities are attained in these inequalities.

Remark 3. For the case $H = L_2(I^n)$, where $I^n = (0,1)^n$ is the unit cube in $\mathbb{R}^n$, $1<p<\infty$, $m\in\mathbb{N}$ and for the linear vector-functionals L defined by the functions

$$e_\alpha(t) = \frac{(-1)^{l-1}\alpha!Q_{\alpha,2}(t)}{\|Q_{\alpha,2}\|^2_{L_2(I^n)}}\,, \quad t\in\mathbb{R}^n\,,$$

where $\alpha=(\alpha_1,...,\alpha_n)$, α_j are non-negative integers, $|\alpha|=\alpha_1+\cdots+\alpha_n=l\in\mathbb{N}$, $\alpha!=\alpha_1!\cdots\alpha_n!$,

$$Q_{\alpha,2} = Q_{\alpha_1,2}\cdots Q_{\alpha_n,2}\,,$$

and $Q_{\sigma,2}$ is a polynomial of order $\sigma\in\mathbb{N}$ of least deviation from zero in $L_2(0,1)$, Theorems 2.1 and 2.2 were proved in [1], [2].

However, it appeared that, in fact, results in [1], [2] are particular cases of general statements of functional analysis for vector-functionals in Hilbert spaces formulated above.

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