ISSN 2077–9879

Eurasian Mathematical Journal

2014, Volume 5, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Short communications

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 5, Number 2 (2014), 126 – 131

THE RAINWATER–SIMONS WEAK CONVERGENCE THEOREM FOR THE BROWN ASSOCIATED NORM

A.R. Alimov

Communicated by V.I. Burenkov

Key words: Rainwater–Simons convergence theorem; Brown associated norm; Menger connected set.

AMS Mathematics Subject Classification: 40A05.

Abstract. The Rainwater–Simons weak convergence theorem is extended to the convergence with respect to the associated norm (in the sense of Brown), the latter proved useful, inter alia, in testing the membership of a point to the Banach–Mazur hull of two points, which is the intersection of all closed balls containing these points.

1 Introduction

In recent years much attention has been paid to various extensions and applications of the Rainwater–Simons convergence theorem (see, for example, Hardtke [9], Nygaard [12], Kalenda [10]). In the present paper we study properties of the associated norm on normed linear spaces and extend the Rainwater–Simons theorem on weak convergence of sequences to the convergence with respect to the associated norm (in the sense of Brown). Associated norms are defined (see (1.1) below) on a broad class of normed linear spaces including all separable spaces.

In the 1980s, Aldric Brown [5], when working on the problem of connectedness of suns in finite-dimensional spaces, had introduced the so-called associated norm $|\cdot|$ (or, as we sometimes call it, the Brown associated norm). The importance of this norm in geometrical theory of approximation and in convexity follows by the fact that a point z lies in the Banach–Mazur hull $\mathrm{m}(x, y)$ of two points x, y (by definition, $\mathrm{m}(x, y)$ is the intersection of all closed balls containing x and y) if and only if z is $|\cdot|$-between x and y in the sense of metrical convexity (that is, $|x-y| = |x-z| + |z-y|$ with respect to the associated norm $|\cdot|$ to be defined later). Brown's studies have been continued by Franchetti and Roversi [8] and later by the author [1], [2], [3], who extended some of Brown's constructions and results to infinite-dimensional setting.

Consider the following class of spaces introduced by Franchetti and Roversi [8]:

$$(\textit{Ex-w}^*s) \qquad\qquad \operatorname{ext} S^* \ \text{ is } \ w^*\text{separable.}$$

In the definition of the class (*Ex-w*s*) we will always assume that

$$F = (f_i)_{i\in I} \subset \operatorname{ext} S^* \ \text{ is } \ w^*\text{-dense in } \operatorname{ext} S^*, \quad \operatorname{card} I \le \aleph_0, \quad F = -F.$$

The abbreviation (*Ex-w*s*) is taken from the German 'Die Extrempunktmenge der konjugierten Einheitskugel ist w^*-separabel'. According to a result of Lindenstrauss and Phelps, the set of extreme points of the unit ball of a reflexive infinite-dimensional Banach space is uncountable, but we shall see later that (*Ex-w*s*) contains all separable normed linear spaces.

Even though the question of massiveness of boundaries of normed linear spaces has been studied by many mathematicians, of whom we mention, *inter alia*, M. I. Kadets, V. P. Fonf, J. Lindenstrauss, R. R. Phelps, O. Nygaard, T. A. Abrahamsen, M. Põldvere (for a survey on thick and w^*-thick sets see, for example, [11]), the question of w^*-separability of $\operatorname{ext} S^*$ is not that well studied.

That any space in (*Ex-w*s*) has w^*-separable dual ball B^* is an easy consequence of the Krein–Milman theorem. Next, w^*-separability of B^* is equivalent [6] to the fact that X is isometrically isomorphic to a subspace of ℓ^∞. It is also worth noting that there are examples of $C(K)$-spaces (K is a non-separable Hausdorff compact set) or spaces of the form $X = \ell_1 \oplus \ell_2(\Gamma)$ ($|\Gamma| = \mathfrak{c}$) such that X^* is w^*-separable, but the dual unit ball B^* is not [4].

Further, it is well known that if X is a separable normed linear space, then the w^*-topology of B^* is metrizable. As a result, the dual ball B^* is w^*-separable ([7], Corollary 3.104). Hence, *any separable space lies the class* (*Ex-w*s*). The class (*Ex-w*s*) also contains the *non-separable* space ℓ^∞ (*qua* the space of continuous functions on the Stone–Čech compactification $\beta\mathbb{N}$ of the natural numbers— such compact set is separable but non-metrizable). Also note that $C(Q)$ on a non-separable Q and $c_0(\Gamma)$ on an uncountable Γ fail to lie in (*Ex-w*s*).

It would be interesting to characterize the class (*Ex-w*s*).

In summary,

> the class (*Ex-w*s*) contains all separable normed linear spaces (and in particular, $C(Q)$ on a metrizable compact set Q) and the non-separable space ℓ^∞.

Let $X \in$ (*Ex-w*s*), let $F = (f_i)_{i\in I}$ be the family of functionals in the definition of the class (*Ex-w*s*), let $(\alpha_i) \subset \mathbb{R}$, $\alpha_i > 0$, $i \in I$, and let $\sum \alpha_i < \infty$. Given $x \in X$, we set

$$|x| = \sum_{i\in I} \alpha_i |f_i(x)|. \tag{1.1}$$

It is easily seen that $|\cdot|$ defines a norm on X, which we shall call, following Brown [5], the *associated* norm on X. Clearly, $|x| \le \|x\| \sum_{i\in I} \alpha_i$.

The importance of the associated norm is seen by the following result, which is a direct and straightforward generalization of Corollary 3.2 of [5], which was proved by Brown for $\dim X < \infty$. In particular (Franchetti and Roversi [8], Alimov [3]), on a separable Banach space X the following conditions are equivalent, given $x, y \in X$:

a) $z \in \mathrm{m}(x,y)$;
b) $|f_i(x)-f_i(y)| = |f_i(x)-f_i(z)| + |f_i(z)-f_i(y)|$ for all $i \in I$, where $F=(f_i)_{i\in I}$ is the family in the definition of (*Ex-w*s*);
c) $|x-y| = |x-z| + |z-y|$ (that is, z is $|\cdot|$-between x and y).

2 The main result

The following theorem is the main result of the present paper, in which we extend the well-known Rainwater–Simons theorem (see, for example, § 3.11.8.5 in [7]) to the convergence in the associated norm $|\cdot|$ on spaces in the class (*Ex-w*s*) (in particular, on separable spaces). According to the Rainwater–Simons theorem, a bounded sequence (x_n) in a Banach space X weakly converges to an $x \in X$ if and only if the sequence $(f(x_n))$ converges to $f(x)$ for any functional f in an arbitrary fixed James boundary for X (for example, for all $f \in \operatorname{ext} S^*$). Even though the weak convergence is non-metrizable in general, we shall see that there is a norm on $X \in$ (*Ex-w*s*) with respect to which the convergence of *sequences* is equivalent to the weak convergence.

Here, we recall (see, for example, § 3.11.8 in [7]) that a subset A of the dual unit sphere S^* of X^* is called a (*James*) *boundary* for the space X if, for any $x \in X$, there exists an $f \in A$ such that $f(x) = \|x\|$. It is an easy consequence of Krein–Milman's theorem that the set $\operatorname{ext} S^*$ of extreme points of the dual unit ball is a boundary for X.

Theorem 2.1. *Let $X \in$ (Ex-w*s) be a Banach space, $F := (f_i)_{i\in I} \subset \operatorname{ext} S^*$ be the family of functionals in the definition of (Ex-w*s). Also let (x_n) be a $|\cdot|$-bounded sequence in X. Consider the following conditions*:

a) $x_n \xrightarrow{|\cdot|} x$;
b) $f_i(x_n) \to f_i(x)$ *for any* $i \in I$;
c) $x_n \xrightarrow{w} x$.

Then conditions a) *and* b) *are equivalent, either of which follows by* c). *If X^* is separable, then all three conditions are equivalent.*

Since $|x| \le \|x\| \sum_{i\in I} \alpha_i$, a $\|\cdot\|$-bounded sequence is necessarily $|\cdot|$-bounded.

Corollary 2.1. *If $\dim X = \infty$ and X^* is separable, then X is not $|\cdot|$-complete.*

Remark 1. That X^* is separable in b)⇒c) is essential. Indeed, let $X = \ell^1$. Consider finite sequences in ℓ^∞ consisting of 0 and ± 1. This set is countable and w^*-dense in $\operatorname{ext} S^*$. However, the convergence of elements in ℓ^1 on these sequences does not imply their weak convergence. This fact was noted by P. A. Borodin in disputing the results of the paper.

Corollary 2.2. *$(X, |\cdot|)$ is a Schur space.*

3 Proofs

To prove Corollary 2.2 we recall that, by definition, a space X is a *Schur space* if weakly convergent sequences in X are norm convergent; the space ℓ^1 is a classical example of a Schur space. Indeed, $|x| > \alpha_i|f_i(x)|$ for any i by the definition of the associated norm. Consequently, any $f_i \in F$ lies in $X^*_{|\cdot|}$. Now it follows by assertion b) of Theorem 2.1 that if (x_n) $w_{|\cdot|}$-converges, then (x_n) $|\cdot|$-converges. Thus, for a $|\cdot|$-bounded sequence (x_n)

$$x_n \xrightarrow{w_{|\cdot|}} x \iff x_n \xrightarrow{|\cdot|} x. \tag{3.1}$$

As a result, in $(X, |\cdot|)$ $w_{|\cdot|}$-compactness coincides with the strong $|\cdot|$-compactness. Thus, a reflexive $(X, |\cdot|)$ is finite-dimensional. This observation arose during conversations between the author and Prof. Olav Nygaard, to whom the author wishes to express his thanks.

The implication a)⇒b) in of Theorem 2.1 is quite clear: if $x_n \xrightarrow{|\cdot|} x$ (in the norm $|\cdot|$), then the sum $\sum_{i\in I} \alpha_i|f_i(x_n) - f_i(x)|$ is small for all sufficiently large n. Consequently, for each fixed j the difference $|f_j(x_n) - f_j(x)|$ is also small for such n.

Let us prove b)⇒a). For any n, we split the sum

$$\sum_{i\in I} \alpha_i|f_i(x_n) - f_i(x)|$$

into two sums: for $i \le N$ and for $i > N$ (N will be chosen later). By the hypothesis, the sequence (x_n) is uniformly $|\cdot|$-bounded, and hence, in the second sum, $|f_i(x_n)-f_i(x)| \le C$ (where C is independent of i, n). So, the second sum is bounded from above by the sum $\sum_{i>N} C\alpha_i < \infty$. Given $\varepsilon > 0$ we choose an N for which the second sum is smaller than ε. The first sum is finite, and there we choose large n.

That c)⇒b) is clear. Assume that X^* is separable and prove that b)⇒c). Note that the following assertions are equivalent for a Banach space X:

- X has a separable boundary;
- the boundary $\operatorname{ext} B^*$ is separable;
- space X^* is separable.

Here, the converse implication are straightforward (in a normed linear space, the (strong) separability of a set is inherited by its arbitrary subsets), and the first assertion implies the last one in view of the well-known Godefroy–Rodé theorem (see, for example, [7], Theorem 3.122), which states that $B^* = \overline{\operatorname{conv}}^{\|\cdot\|} A$ (here, A is a separable boundary for X). As a corollary, the ball B^* is separable and so is the space X^*.

Further, we shall need the concept of an (I)-generating set introduced by Fonf and Lindenstrauss. By definition, a set $C \subset B^*$ (I)-*generates* the dual ball B^* if

$$B^* = \overline{\operatorname{conv}}\Big(\bigcup_i \overline{\operatorname{conv}}^{w^*} C_i\Big) \tag{3.2}$$

for any representation $C = \bigcup C_i$ as a countable union of sets C_i. In this definition, 'I' comes from the Latin *intermedius* and is explained by the fact that

$$B^* = \overline{\operatorname{conv}} C \implies C \text{ (I)-generates } B^* \implies B^* = \overline{\operatorname{conv}}^{w^*} C.$$

Let $C_i := \{f_1, \dots, f_i\}$, $i \in I$ (here, $F := (f_i)_{i\in I}$ is the family of functionals in the definition of the class (*Ex-w*s*)). Clearly, $F = \bigcup C_i$. By Kadets–Fonf (Godefroy–Rodé)'s theorem,

$$B^* = \overline{\operatorname{conv}}^{\|\cdot\|} \operatorname{ext} B^*. \tag{3.3}$$

The space X^* is separable, and hence so is $\operatorname{ext} B^*$. Next, $\operatorname{ext} S^* \subset \overline{F}^{w^*}$ by the definition of the class (*Ex-w^*s*), and hence, $\operatorname{ext} S^* \subset \big(\bigcup_i \overline{\operatorname{conv}}^{\,w^*} C_i\big)$. Finally, F (I)-generates the ball B^* by (3.3) and (3.2).

Now it remains to employ one result due to Nygaard [12] and Kalenda [10]. According to this result, if C (I)-generates the dual ball B^*, then C is a Rainwater set; that is, a set with the property: if a bounded sequence $(x_n) \subset X$ converges pointwise on B, then (x_n) converges weakly. The proof of Theorem 2.1 is complete. □

The same arguments as in the proof of implications a)⇒b) and b)⇒a) in Theorem 2.1 show that

$$(x_n) \text{ is } |\cdot|\text{-Cauchy} \Leftrightarrow (f_i(x_n)) \text{ is Cauchy in } \mathbb{R} \text{ for any } i \in I. \tag{3.4}$$

Hence, if X is $|\cdot|$-complete, then by (3.4) X is $|\cdot|$-weakly sequentially complete. It is easily verified that a weakly sequentially complete space is Banach. Finally, a weakly sequentially complete Banach space is either reflexive or contains a subspace, isomorphic to ℓ^1. This proves Corollary 2.1.

Acknowledgments

The author is indebted to Jan-David Hardtke, Olav Nygaard and Antonio Avilés for helpful discussion of this problem.

This research was carried out with the financial support of the Russian Foundation for Basic Research (grant no. 13-01-00022).

References

[1] A.R. Alimov, *Connectedness of suns in the space c_0*, Izvestiya: Mathematics, 69 (2005), no. 4, 651–665.

[2] A.R. Alimov, *Monotone path-connectedness of Chebyshev sets in the space $C(Q)$*, Sb. Math., 197 (2006), no. 9, 1259–127.

[3] A.R. Alimov, *Monotone path-connectedness and solarity of Menger-connected sets in Banach spaces*, Izvestiya: Mathematics, 78 (2014), no. 4, 3–18.

[4] A. Avilés, G. Plebanek, J. Rodríguez, *A weak* separable $C(K)^*$ space whose unit ball is not weak* separable*, Trans. Amer. Math. Soc. 366 (2014), 4733-4753.

[5] A.L. Brown, *Suns in normed linear spaces which are finite dimensional*, Math. Ann., 279 (1987), 87–101.

[6] E.N. Dancer, B. Sims, *Weak star separability*, Bull. Austral. Math. Soc., 20 (1979), 253–257.

[7] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis* Springer, New York, 2011.

[8] C. Franchetti, S. Roversi, *Suns, M-connected sets and P-acyclic sets in Banach spaces*, Preprint no. 50139, Instituto di Matematica Applicata "G. Sansone", 1988, 1–29.

[9] J.-D. Hardtke, *Rainwater–Simons type convergence theorems for generalized convergence methods*, Acta Comm. Univ. Tartuensis Math., 14 (2010), 65–74.

[10] O.F.K. Kalenda, *(I)-envelopes of unit balls and James' characterization of reflexivity*, Studia Math., 182 (2007), 29–40.

[11] O. Nygaard, *Thick sets in Banach spaces and their properties*, Quaest. Math., 29 (2006), no. 1, 59–72.

[12] O. Nygaard, *A remark on Rainwater's theorem*, Annales Math. Inform., 32 (2005), 125–127.

Alexey Rostislavovich Alimov
Faculty of Mechanics and Mathematics
Moscow State University
119991 Moscow, Russia
E-mail: alexey.alimov@gmail.com

Received: 21.05.2014

EURASIAN MATHEMATICAL JOURNAL
2014 – Том 5, № 2 – Астана: ЕНУ. – 139 с.
Подписано в печать 29.06.2014 г. Тираж – 120 экз.

Адрес редакции: 010008, Астана, ул. Мирзояна, 2,
Евразийский национальный университет имени Л.Н. Гумилева,
главный корпус, каб. 312
Тел.: +7-7172-709500 добавочный 31313

Дизайн: К. Булан

Отпечатано в типографии ЕНУ имени Л.Н.Гумилева

Свидетельство о постановке на учет печатного издания
Министерства культуры и информации Республики Казахстан
№ 10330 – Ж от 25.09.2009 г.