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GENERALIZED POTENTIALS OF DOUBLE LAYER IN PLANE THEORY OF ELASTICITY

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Abstract. Connected with the function-theoretic approach, generalized potentials of double layer are introduced for the Lamé system of plane anisotropic elasticity theory. These potentials are constructed for the displacement vector — a solution of the Lamé system, and as well for the conjugate vector–functions describing the stress tensor. There are obtained integral representations of these solutions via potentials mentioned above. As a corollary the first and the second boundary-value problems in different classes (Hölder, Hardy, the class of functions continuous in a closed domain) are reduced to equivalent systems of the boundary Fredholm equations in corresponding spaces.

1 The Lamé system

Let us consider the Lamé system [15, 14]

$$a_{11}\frac{\partial^2 u}{\partial x^2} + (a_{12}+a_{21})\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} = 0 \tag{1.1}$$

with constant matrix coefficients

$$a_{11} = \begin{pmatrix} \alpha_1 & \alpha_6 \\ \alpha_6 & \alpha_3 \end{pmatrix}, \qquad a_{12} = \begin{pmatrix} \alpha_6 & \alpha_4 \\ \alpha_3 & \alpha_5 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} \alpha_6 & \alpha_3 \\ \alpha_4 & \alpha_5 \end{pmatrix}, \qquad a_{22} = \begin{pmatrix} \alpha_3 & \alpha_5 \\ \alpha_5 & \alpha_2 \end{pmatrix}.$$

The elements α_j of the matrix coefficients, called modules of elasticity, satisfy the requirement of positive definiteness of the following matrix

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_6 \\ \alpha_4 & \alpha_2 & \alpha_5 \\ \alpha_6 & \alpha_5 & \alpha_3 \end{pmatrix}.$$

Together with this matrix it is convenient to introduce the adjoint matrix $\beta = (\det\alpha)\alpha^{-1}$ written in the same form:

$$\beta = \begin{pmatrix} \beta_1 & \beta_4 & \beta_6 \\ \beta_4 & \beta_2 & \beta_5 \\ \beta_6 & \beta_5 & \beta_3 \end{pmatrix}, \qquad \begin{array}{ll} \beta_1 = \alpha_2\alpha_3 - \alpha_5^2 & \beta_2 = \alpha_1\alpha_3 - \alpha_6^2, \\ \beta_3 = \alpha_1\alpha_2 - \alpha_4^2 & \beta_4 = \alpha_5\alpha_6 - \alpha_3\alpha_4, \\ \beta_5 = \alpha_4\alpha_6 - \alpha_1\alpha_5, & \beta_6 = \alpha_4\alpha_5 - \alpha_2\alpha_6. \end{array} \tag{1.2}$$

Then the Sylvester criterion of positive definiteness for the matrix α can be represented by the inequalities $\det\alpha > 0$ and $\alpha_i > 0,\ \beta_i > 0,\ 1 \le i \le 3$.

The vector $u = (u_1, u_2)$ characterizes the displacement vector, it is connected with the columns $\sigma_{(1)} = (\sigma_1, \sigma_3), \sigma_{(2)} = (\sigma_3, \sigma_2)$ of the stress tensor

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}$$

by the relations

$$\sigma_{(i)} = a_{i1}\frac{\partial u}{\partial x} + a_{i2}\frac{\partial u}{\partial y}, \quad i = 1, 2, \tag{1.3}$$

which present Hook's law content.

In the absence of body forces the matrix σ satisfies the equilibrium equations

$$\frac{\partial \sigma_{(1)}}{\partial x} + \frac{\partial \sigma_{(2)}}{\partial y} = 0,$$

which jointly with relation (1.3) reduce to the Lamé system.

Let us introduce the matrix trinomial $p(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2$ of system (1.1). It represents the symmetric matrix

$$p = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}$$

with the corresponding quadratic trinomials p_j. Besides them further an important role is played by another pair of polynomials q_2 and q_3 with the elements of adjoint matrix (1.2) as coefficients. Explicitly

$$\begin{gathered} p_1(z) = \alpha_1 + 2\alpha_6 z + \alpha_3 z^2, \\ p_2(z) = \alpha_3 + 2\alpha_5 z + \alpha_2 z^2, \quad p_3(z) = \alpha_6 + (\alpha_3 + \alpha_4)z + \alpha_5 z^2 \\ q_2(z) = \beta_4 - \beta_6 z + \beta_1 z^2, \quad q_3(z) = z q_2(z) = \beta_4 z - \beta_6 z^2 + \beta_1 z^3. \end{gathered} \tag{1.4}$$

The Lamé system is well known to be an elliptic one, i.e. its fourth order characteristic polynomial has no real roots. Therefore in the upper half-plane we have two roots ν_1, ν_2 for which two cases occur: (i) $\nu_1 \ne \nu_2$ and (ii) $\nu_1 = \nu_2 = \nu$.

We should pay special attention to the case of the Lamé system when the polynomials p_2 and p_3 in (1.4) are linearly dependent: $\alpha_2 p_3 - \alpha_5 p_2 = 0$, or, equivalently, $\alpha_2\alpha_6 = \alpha_3\alpha_5$ and $\alpha_2(\alpha_3 + \alpha_4) = 2\alpha_5^2$. Hence also $2\alpha_2\alpha_6^2 = \alpha_3^2(\alpha_3 + \alpha_4)$, $2\alpha_5\alpha_6 = \alpha_3(\alpha_3 + \alpha_4)$, so after elementary computing we obtain

$$2\det\alpha = (\alpha_1\alpha_2 - \alpha_3^2)(\alpha_3 - \alpha_4) > 0.$$

Since the inequalities $\alpha_1\alpha_2-\alpha_3^2<0$ and $\alpha_3<\alpha_4$ contradict to the inequality $\alpha_1\alpha_2-\alpha_4^2>0$, the special case is described by the relations

$$\alpha_2\alpha_6=\alpha_3\alpha_5,\ \alpha_2(\alpha_3+\alpha_4)=2\alpha_5^2;\quad \alpha_1\alpha_2>\alpha_3^2,\ \alpha_3>\alpha_4. \tag{1.5}$$

If $\alpha_3+\alpha_4=0$ these relations turn into

$$\alpha_5=\alpha_6=0,\ \alpha_3=-\alpha_4;\quad \alpha_1\alpha_2>\alpha_3^2 \tag{1.5$_0$}$$

and correspondingly the Lamé system can be diagonalized, i.e. it decomposes into two equations

$$\alpha_1\frac{\partial^2u_1}{\partial x^2}+\alpha_3\frac{\partial^2u_1}{\partial y^2}=0,\quad \alpha_3\frac{\partial^2u_2}{\partial x^2}+\alpha_2\frac{\partial^2u_2}{\partial y^2}=0.$$

The roots of the characteristic equation in the special case can be calculated explicitly. Indeed, by virtue of the linear dependence $\alpha_2p_3-\alpha_5p_2=0$ we have the equality $\alpha_2^2(p_1p_2-p_3^2)=(\alpha_2^2p_1-\alpha_5^2p_2)p_2$. Thus

$$\alpha_2\nu_k=-\alpha_5+i\delta_k,\quad k=1,2, \tag{1.6}$$

where $\delta_k>0$ are defined by the equalities

$$\delta_1^2=\alpha_2\alpha_3-\alpha_5^2+\frac{\alpha_2^2(\alpha_1\alpha_2-\alpha_3^2)}{\alpha_2\alpha_3-\alpha_5^2},\quad \delta_2^2=\alpha_2\alpha_3-\alpha_5^2.$$

In case (1.5$_0$) these equalities transform into

$$\nu_1=\sqrt{\frac{\alpha_1}{\alpha_3}},\quad \nu_2=\sqrt{\frac{\alpha_3}{\alpha_2}}. \tag{1.6$_0$}$$

By virtue of (1.2) and (1.5) the quantities $\alpha_2\alpha_3-\alpha_5^2$ and $\alpha_1\alpha_2-\alpha_3^2$ are positive, so $\delta_1>\delta_2$. Therefore the roots ν_1 and ν_2 are different, i.e. the special case is a particular case of (i).

Let us introduce the matrices

$$\text{(i) } J=\begin{pmatrix}\nu_1 & 0\\ 0 & \nu_2\end{pmatrix},\quad \text{(ii) } J=\begin{pmatrix}\nu & 1\\ 0 & \nu\end{pmatrix}. \tag{1.7}$$

for two cases (i) and (ii) for the roots of the characteristic equation. As it was established in [20] for the second order elliptic system of form (1.1) and particularly for the Lamé system the following statement is valid.

Lemma 1.1. *There exists a matrix $b\in\mathbb{C}^{2\times2}$ with non-vanishing columns such that*

$$a_{11}b+(a_{12}+a_{21})bJ+a_{22}bJ^2=0. \tag{1.8}$$

For this matrix the mapping

$$\eta\to(\operatorname{Re}b\eta,\operatorname{Re}bJ\eta) \tag{1.9}$$

from $\mathbb{C}^2$ to $\mathbb{R}^2\times\mathbb{R}^2$ is invertible. Any other matrix b_1 with the same properties is connected with b by the relation $b_1=bd$ where d is some invertible matrix commuting with J.

It should be noted that in accordance with (1.7) all matrices d commuting with J have the following form

$$\text{(i) } d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad \text{(ii) } d = \begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix}. \tag{1.10}$$

Let us connect the matrix

$$c = -(a_{21}b + a_{22}bJ). \tag{1.11}$$

with the matrix b. It is obvious that under transfer from b to $b_1 = bd$ where $dJ = Jd$ the similar relation $c_1 = cd$ also holds for matrices of this type.

Further for investigation of the main boundary-value problems for the Lamé system matrices b and c play the key role. Therefore various modifications of their explicit expressions through elasticity modules and roots of characteristic equation were given in [20] – [23]. All these formulas are equivalent in the sense that they are obtained from each other by multiplying by a suitable matrix of form (1.10). The most simple and in some sense complete form for them was obtained in [23] with their expressions being given for each case (i) and (ii) separately.

Theorem 1.1. *Apart from the special case the matrices b and c are given by the equalities*

$$b = \begin{pmatrix} p_2(\nu_1) & p_2(\nu_2) \\ -p_3(\nu_1) & -p_3(\nu_2) \end{pmatrix}, \quad c = \begin{pmatrix} -q_3(\nu_1) & -q_3(\nu_2) \\ q_2(\nu_1) & q_2(\nu_2) \end{pmatrix}, \tag{1.12i}$$

or

$$b = \begin{pmatrix} p_2(\nu) & p_2'(\nu) \\ -p_3(\nu) & -p_3'(\nu) \end{pmatrix}, \quad c = \begin{pmatrix} -q_3(\nu) & -q_3'(\nu) \\ q_2(\nu) & q_2'(\nu) \end{pmatrix}, \tag{1.12ii}$$

where the polynomials p_j and q_j are defined in (1.4).

In the special case for these matrices we have the expressions

$$b = \begin{pmatrix} \alpha_2 & 0 \\ -\alpha_5 & 1 \end{pmatrix}, \quad c = \frac{\delta_2}{\alpha_2} \begin{pmatrix} \alpha_5\delta_2 - i\delta_1\delta_2 & -(\delta_2 + i\alpha_5) \\ \alpha_2\delta_2 & -i\alpha_2 \end{pmatrix}, \tag{1.13}$$

where δ_1, δ_2 are the numbers entering (1.6).

In all cases the matrices b and c are invertible.

The matrices b and c satisfy relations (1.8) and (1.11) is verified directly. The columns of the matrix b being non-vanishing in all cases the corresponding mapping (1.9) is also invertible by virtue of Lemma 1.1. The invertibility property of matrices (1.13) is obvious but direct verification of matrices invertibility is rather difficult .

Let us note several important properties of the matrix c.

Lemma 1.2. (a) *The matrix c can be written in the following form*

$$c = c_0 d, \tag{1.14}$$

where

$$(i) \quad c_0 = \begin{pmatrix} -\nu_1 & -\nu_2 \\ 1 & 1 \end{pmatrix}, \quad (ii) \quad c_0 = \begin{pmatrix} -\nu & -1 \\ 1 & 0 \end{pmatrix},$$

with the invertible matrix d of form (1.10).

(b) *Let the space $X_k \subseteq \mathbb{C}^2$, $k = 0, 1, 2$, (over the field $\mathbb{R}$) consists of vectors η, for which* $\operatorname{Re} cJ^s\eta = 0$, $0 \le s \le k$. *Then* $\dim X_k = 2 - k$.

Proof. (a) Comparing (1.12) and (1.13) one can see that excluding the special case the following matrices

$$(i)\quad d=\begin{pmatrix} p(\nu_1) & 0\\ 0 & p(\nu_2)\end{pmatrix},\qquad (ii)\quad d=\begin{pmatrix} q_2(\nu) & q_2'(\nu)\\ 0 & q_2(\nu)\end{pmatrix}.$$

satisfy equality (1.14). In special case (1.5), (1.6) in the capacity of d we have

$$d=\begin{pmatrix} \delta_2^2 & 0\\ 0 & -i\delta_2\end{pmatrix}.$$

(b) The matrix c being invertible, it is necessary to consider only the cases $k=1$ and $k=2$. It suffices to verify that the block matrices

$$C_1=\begin{pmatrix} c & \overline{c}\\ cJ & \overline{cJ}\end{pmatrix},\qquad C_2=\begin{pmatrix} c & \overline{c}\\ cJ & \overline{cJ}\\ cJ^2 & \overline{cJ^2}\end{pmatrix} \tag{1.15}$$

have ranks 3, 4 respectively. Let us consider each of the two cases for the matrix c in (1.12) and (1.13) separately. In view of Proposition (a) of the lemma the conversion from c to c_0 does not have an impact on the rank of the matrices C_k. So without loss of generality we can replace c by c_0 in the definition of these matrices. It is easy to see that the first row of the matrix c_0 is opposite to the second row of the matrix c_0J and the pair of the matrices c_0J and c_0J^2 has an analogous property.

So if one cuts the fourth row from C_1 and the fourth and sixth ones from C_2 then the ranks of the obtained matrices which are denoted by $\widetilde{C}_1$, $\widetilde{C}_2$ respectively do not change. Up to row permutation and multiplication of some of them by -1 the matrix $\widetilde{C}_2$ coincides with the matrix W of the form

$$(i)\ W=[h(\nu_1),h(\nu_2),h(\overline{\nu_1}),h(\overline{\nu_2})],$$

$$(ii)\ W=[h(\nu),h'(\nu),h(\overline{\nu}),h'(\overline{\nu})],$$

where $h(z)$ is the column vector with the elements $z^j,\ j=0,1,2,3$. As for the matrix $\widetilde{C}_1$ it is equivalent (in the same sense) to the matrix which can be obtained from W by cutting the last row. In the case (i) the matrix W is the classical Vandermonde matrix and in the case (ii) it is the generalized Vandermonde matrix. In both cases their determinants are non-zero [24]. Therefore the statement on the rank of matrices (1.15) and the statement of the lemma are established completely. □

An elastic medium is said to be orthotropic if

$$\alpha_5=\alpha_6=0. \tag{1.16}$$

For this medium the coordinate lines are symmetry axes. In the orthotropic case polynomials (1.4) can be represented by simplified expressions, namely

$$\begin{gathered} p_1(z)=\alpha_1+\alpha_3z^2,\ p_2(z)=\alpha_3+\alpha_2z^2,\ p_3(z)=(\alpha_3+\alpha_4)z,\\ q_2(z)=\alpha_3(-\alpha_4+\alpha_2z^2),\quad q_3(z)=\alpha_3(-\alpha_4z+\alpha_2z^3). \end{gathered} \tag{1.17}$$

It should be mentioned that in the orthotropic medium the special case is defined by relations (1.5_0).

The orthotropic medium is called isotropic if in addition to (1.16) the relations

$$\alpha_1 = \alpha_2 = 2\alpha_3 + \alpha_4. \tag{1.18}$$

are also carried out. Hence jointly with the inequality $\alpha_4^2 < \alpha_1\alpha_2$ it follows that $\alpha_1 > \alpha_3$. It is easy to see that in the case under consideration the characteristic equation has the multiple root $\nu = i$, so we can use formulas (1.12). In view of (1.17), (1.18) these formulas give the following equalities

$$b = \begin{pmatrix} \alpha_3 - \alpha_1 & 2\alpha_1 i \\ (\alpha_3 - \alpha_1)i & \alpha_3 - \alpha_1 \end{pmatrix}, \quad c = 2\alpha_3 \begin{pmatrix} (\alpha_1 - \alpha_3)i & 2\alpha_1 - \alpha_3 \\ \alpha_3 - \alpha_1 & \alpha_1 i \end{pmatrix}.$$

According to Lemma 1.1 in the capacity of b and c we can also take the matrices which are obtained by multiplication of these equalities by the matrix

$$d = (\alpha_3 - \alpha_1)^{-1} \begin{pmatrix} 1 & 2\alpha_1(\alpha_1 - \alpha_3)^{-1} i \\ 0 & 1 \end{pmatrix}.$$

By carrying out elementary calculations we arrive at the formulas

$$b = \begin{pmatrix} 1 & 0 \\ i & -æ \end{pmatrix}, \quad c = \alpha_3 \begin{pmatrix} -2i & æ - 1 \\ 2 & i(æ + 1) \end{pmatrix} \tag{1.19}$$

with the positive constant $æ = (\alpha_1 + \alpha_3)/(\alpha_1 - \alpha_3)$.

2 The first and the second boundary-value problems

Let us consider the Lamé system in a domain D bounded by a Lyapunov contour $\Gamma \in C^{1,\nu}$, $0 < \nu < 1$. The main boundary conditions for this system are known [17] to consist of assigning either the displacement vector

$$u^+ = f, \tag{2.1}$$

or the normal component $\sigma^+ n = \sigma_{(1)}^+ n_1 + \sigma_{(2)}^+ n_2$ of the stress tensor σ on the boundary contour, where $n = (n_1, n_2)$ is the unit exterior normal to Γ and the upper sign $+$ indicates the boundary values of functions. In accordance with (1.3) the last boundary condition can be written in the form

$$n_1 \left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} \right)^+ + n_2 \left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} \right)^+ = g. \tag{2.2}$$

Therefore (2.1) corresponds to the Dirichlet problem for the Lamé system and (2.2) corresponds to the Neumann problem. These problems are also called the first and the second boundary value problems.

The domain D can be both finite or infinite. In the last case the following condition is imposed on the gradient of a solution u:

$$\operatorname{grad} u(z) = O(|z|^{-2}) \quad \text{as} \quad z \to \infty, \tag{2.3}$$

in particular, there exists the limit $u(\infty) = \lim_{z\to\infty} u(z)$ at infinity. In the sequel conditions of such type will arise frequently so it is convenient to say that a function $w \in C(D)$ has order k at infinity where k is an integer if it behaves as $O(|z|^k)$ as $z \to \infty$. In the cases $k = 0$ and $k = -1$ we also say that w is bounded, vanishes at infinity respectively.

Let us highlight the class of first order polynomial vectors $u(x, y) = \xi_0 + x\xi_1 + y\xi_2$ for which $a_{i1}\xi_1 + a_{i2}\xi_2 = 0, \ \ i = 1, 2$. It is obvious that the system rank is equal to 3 and any solution $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ of the homogeneous system $a\xi = 0$ has the form $\xi_1 = (0, \lambda)$, $\xi_2 = (-\lambda, 0)$. The corresponding polynomials $u = (u_1, u_2)$ mentioned above have

$$u_1(x, y) = \lambda_1 - \lambda y, \ u_2(x, y) = \lambda_2 + \lambda x, \quad \lambda_j, \lambda \in \mathbb{R}. \tag{2.4}$$

as their components. Polynomials of such type are said to be trivial solutions of the Lamé system. For them the left-hand side of (2.2) turns into zero. Clearly in the infinite domain case in accordance with (2.3) trivial solutions are reduced to constant vector valued functions. For general strongly elliptic systems solvability problems for the Dirichlet and the Neumann problems in the Hölder and Sobolev spaces are well-studied [10]. In particular for the Lamé system the following classical result holds.

Theorem 2.1. *Let a domain D be bounded by a contour $\Gamma \in C^{1,\nu}$. Then the Dirichlet problem is uniquly solvable in the class $C^{1,\mu}(\overline{D})$, $0 < \mu < \nu$.*

For the Neumann problem the homogeneous equation has only the trivial solution in this class and the nonhomogeneous equation is solvable if and only if the orthogonality condition to all trivial solutions

$$\int_\Gamma g(t)p(t)|dt| = 0 \tag{2.5}$$

is carried out. Hereinafter $|dt|$ means the arc length element.

The second boundaryvalue problem (2.2) can be written in the form of the first one with respect to the so-called conjugate function v, which is defined by the following relations:

$$\frac{\partial v}{\partial x} = -\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right), \quad \frac{\partial v}{\partial y} = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}. \tag{2.6}$$

Rewriting (1.1) in the form

$$\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) = 0 \tag{2.7}$$

we see that the necessary condition for the existence of the function v is carried out. Therefore up to an additive constant $\xi \in \mathbb{R}^l$ it is uniquely defined in each simply connected domain $D_0 \subseteq D$. In the whole domain this function can be multi-valued. Further under multi-valued functions we will mean functions with one-valued partial derivatives.

The unit tangent vector $e = e_1 + ie_2$ in the direction keeping the domain D on the left is connected with n by the equality $e = in$, so by virtue of (2.6) the tangent derivative

$$v'_e = e_1\frac{\partial v}{\partial x} + e_2\frac{\partial v}{\partial y}$$

on Γ coincides precisely with the left-hand side of (2.2). Therefore the boundary condition of the Neumann problem can be written in the form

$$(v^+)'_e = g. \tag{2.8}$$

It should be noted that the left-hand side here can be considered as the derivative of the function v on Γ with respect to the arc length parameter which is counted in the direction keeping the domain D on the left. In the case of multiply-connected domain D the function v as mentioned above can, in general, be multi-valued.

In the same way, defined by the condition $f'_e = g$, the antiderivative f of the function g can be non-existent on whole contour. However subtracting from v a suitable multi-valued function, which is conjugate to some solution in a neighborhood of $\overline{D}$, we can always achieve that the function v is one-valued and there exists the antiderivative $f \in C^1(\Gamma)$ of the right-hand side. The possibility of such choice will be visible by Lemma 4.1 below. As a result the previous boundary condition can be written in the form

$$v^+ = f + \chi, \tag{2.9}$$

where the function χ is constant at connected components of the contour Γ and it is subject to definition together with u. If $f = 0$ the problem is said to be homogeneous. It should be noted that for trivial solution (2.4) the conjugate function is constant in the domain D and consequently the homogeneous boundary condition (2.9) is satisfied. By virtue of (2.4) conditions (2.5) can be written in the form

$$\int_\Gamma g(t)|dt| = 0, \quad \int_\Gamma [(-\operatorname{Im} t)g_1(t) + (\operatorname{Re} t)g_2(t)]|dt| = 0,$$

and in the infinite domain case the second equality should be omited. As for the tangent derivative $g = f'_e$ of a function $f \in C^1(\Gamma)$, the first equality holds automatically and the second one takes the form

$$\int_\Gamma fn|dt| = 0, \tag{2.10}$$

after integration by parts. Here $n = (n_1, n_2)$ is the unit exterior normal to Γ. It is necessary to take into account that the tangent vector e is connected with n by the relation $i(n_1 + in_2) = e_1 + ie_2$.

It should be noted that the orthogonality condition holds always for $f = \chi$. Indeed let the contour Γ_0 bounding finite domain D_0 be one of the connected components of Γ. Then if the constant vector $\chi = (\chi_1, \chi_2)$ is considered as a function in D_0 we have

$$\int_{\Gamma_0} (\chi_1 n_1 + \chi_2 n_2)|dt| = 0$$

on the basis of the Green formula.

There are two main directions in the investigations of the boundary value problems of anisotropic plane elasticity. The first of them consists of using analytic functions by analogy with the Kolosov–Muskhelishvili formulas [17] in the isotropic case. This direction is represented in the works by S.G. Lekhnitskii, G.N. Savin, S.G. Mikhlin and others (see for example [15, 17, 9]). The second one is based on application of the potential method. It was developed by V.D. Kupradze [14], M.O. Baskheleishvili [4] and others.

The study of boundary-value problems for the second order plane elliptic systems of form (1.1) with coefficients $a_{ij} \in \mathbb{R}^{l\times l}$ is simplified considerably by using the so called $J-$analytic functions, solutions of the first order elliptic system

$$\frac{\partial\phi}{\partial y} - J\frac{\partial\phi}{\partial x} = 0, \tag{2.11}$$

instead of analytic vector–functions. In this case all eigenvalues of the matrix $J \in \mathbb{C}^{l\times l}$ lie in the upper half-plane. The set of these functions consists of the general solution of equation (1.1) representable in the form [24, 25]

$$u = \operatorname{Re} b\phi, \tag{2.12}$$

where the matrices $b, J \in \mathbb{C}^{l\times l}$ are selected as in Lemma 1.1 (it remains valid in the general case). Differentiating this relation and taking into account (2.11) we obtain:

$$\frac{\partial u}{\partial x} = \operatorname{Re} b\phi', \quad \frac{\partial u}{\partial y} = \operatorname{Re} b\frac{\partial\phi}{\partial y} = \operatorname{Re} bJ\phi', \tag{2.13}$$

where hereinafter ϕ' means the partial derivative with respect to x. According to Lemma 2.1 the mapping $\eta \to (\operatorname{Re} b\eta, \operatorname{Re} bJ\eta)$ is invertible. Its inverse can be written in the form $\eta = b^1\xi_1 + b^2\xi_2,\ \xi_j \in \mathbb{R}^l$ with some complex matrices b^k. Relations (2.13) are equivalent to

$$\phi' = b^1\frac{\partial u}{\partial x} + b^2\frac{\partial u}{\partial y} \tag{2.14}$$

respectively. The function ϕ can be multi-valued. It follows directly from (2.14) that the equality $u = 0$ in (2.12) is possible if and only if $\phi = \eta \in \mathbb{C}^l,\ \operatorname{Re} b\eta = 0$. It is easy to verify that in the notation of (1.11) the function

$$v = \operatorname{Re} c\phi, \tag{2.15}$$

is conjugate to u, i.e. it satisfies relations (2.6). Indeed similarly to (2.13) we have:

$$\frac{\partial v}{\partial x} = \operatorname{Re} c\phi', \quad \frac{\partial v}{\partial y} = \operatorname{Re} cJ\phi'. \tag{2.16}$$

In the notation of (1.11) equation (1.8) can be written in the form $cJ = a_{11}b + a_{12}bJ$. Substituting it in (2.16) with (2.13) we obtain relations (2.6).

In problems of elasticity theory the conjugate function plays a supporting role and can be used for determining the stress tensor σ. According to (1.3), (2.6) for the columns of this matrix we have the following relations

$$\sigma_{(1)} = \frac{\partial v}{\partial y}, \quad \sigma_{(2)} = -\frac{\partial v}{\partial x},$$

or element-wise

$$\sigma_1 = \frac{\partial v_1}{\partial y},\ \sigma_2 = -\frac{\partial v_2}{\partial x},\ \sigma_3 = \frac{\partial v_2}{\partial y} = -\frac{\partial v_1}{\partial x}. \tag{2.17}$$

The conjugate function $v = (v_1, v_2)$ satisfies the equation

$$\frac{\partial v_2}{\partial y} = -\frac{\partial v_1}{\partial x},$$

which follows directly from definition (2.6) and from the form of the matrices a_{ij} in (1.1).

It should be noted that there are other function-theoretic approaches to investigating boundary value problems for the Lamé system (see e. g. [11, 5, 6]).

To illustrate this let us discuss the connection of representations (2.12), (2.15) for an isotropic domain with the classical Kolosov – Muskhelishvili formulas expressing the displacement vector u and the stress tensor σ in terms of analytic functions (see also [25]). According to Section 1 in the case under consideration we have the multiple root $\nu = i$, the matrix J is a Jordan sell and the matrices b and c are given by equalities (1.19). Therefore representation (2.12) takes the form

$$u_1 = \text{Re } \phi_1, \quad u_2 = \text{Re } (i\phi_1 - æ\phi_2). \tag{2.18}$$

element-wise. Substituting equation (1.19) in (2.16) for the matrix c we obtain representations of the stress tensor components

$$\begin{aligned}\sigma_1 &= \text{Re } [2\phi_1' + i(æ - 3)\phi_2'],\\ \sigma_2 &= -\text{ Re } [2\phi_1' + i(æ + 1)\phi_2'],\\ \sigma_3 &= \text{Re } [2i\phi_1' - (æ - 1)\phi_2']\end{aligned} \tag{2.19}$$

for elements of matrix σ in (2.17). It should be noted that the matrix c has the property $(cJ)_{2k} = -c_{1k}$. According to it in component-wise form (2.18) the equality defining σ_3 occurs twice.

System (2.11) can be component-wise written in the form of the equations

$$\frac{\partial \phi_1}{\partial y} - i\frac{\partial \phi_1}{\partial x} - \frac{\partial \phi_2}{\partial x} = 0, \quad \frac{\partial \phi_2}{\partial y} - i\frac{\partial \phi_2}{\partial x} = 0.$$

For an arbitrary pair of analytic functions $\psi_k(z)$, $k = 1, 2$, the functions $\phi_1(z) = \psi_1(z) + y\psi_2'(z)$, $\quad \phi_2(z) = \psi_2(z)$, satisfy these equations and besides ψ can be uniquely reconstructed by the equalities $\psi_1(z) = \phi_1(z) - y\phi_2'(z)$, $\quad \psi_2(z) = \phi_2(z)$. Substituting these equations into (2.17), (2.19) we arrive at the following representation

$$u_1 = \text{Re}[\psi_1 + y\psi_2'],\ u_2 = \text{Re}[i(\psi_1 + y\psi_2') - æ\psi_2]$$

of the displacement vector components and

$$\begin{aligned}\sigma_1 &= \alpha_3 \,\text{Re}[2(\psi_1' + y\psi_2'') + i(æ - 3)\psi_2'],\\ \sigma_2 &= -\alpha_3 \,\text{Re}[2(\psi_1' + y\psi_2'') + i(æ + 1)\psi_2'],\\ \sigma_3 &= \alpha_3 \,\text{Re}[2i(\psi_1' + y\psi_2'') - (æ - 1)\psi_2']\end{aligned}$$

of the stress tensor components from analytic functions pair ψ_1, ψ_2 of the same variable z.

By using the linear substitution

$$\chi_1(z) = -i\psi_2(z),\ \chi_2(z) = -2\psi_1(z) + i\ae\psi_2(z) + iz\psi_2'(z)$$

these representations can be written in the form of the equalities

$$2(u_1 - iu_2)(z) = \ae\overline{\chi_1(z)} - z\chi_1'(z) - \chi_2(z),$$

$$(\sigma_1 + \sigma_2)(z) = 4\alpha_3 \operatorname{Re}\chi_1'(z), \quad (\sigma_2 - \sigma_1 + 2i\sigma_3)(z) = 2\alpha_3[\bar{z}\chi_1''(z) + \chi_1'(z)],$$

which represent the classical Kolosov–Muskhelishvili formulas [17].

3 The Douglis analytic functions

System (2.11) has been investigated by A. Douglis [8] for the Hankel matrices J in the framework of hypercomplex numbers and this system generalizes the Cauchy–Riemann system. It is convenient to connect the matrix

$$z_J = x1 + yJ, \tag{3.1}$$

with the complex number $z = x+iy$. Here $x = x1$ is a scalar matrix and the eigenvalues of J being in the upper half-plane, the matrix z_J is invertible if $z \neq 0$.

Solutions ϕ of system (2.11) are said to be $J-$ analytic functions because they can be described as functions belonging to the class $C^1(D)$ which have the generalized derivative

$$\phi'(z) = \lim_{t\to z}(t-z)_J^{-1}[\phi(t) - \phi(z)],$$

coinciding with the partial derivative with respect to x in each point $z \in D$.

If the domain D is infinite then similarly to (2.3) the condition for ϕ' is added to this definition. In particular the function ϕ is bounded at infinity and has the limit $\phi(\infty) = \lim \phi(z)$ as $z \to \infty$.

Let the contour Γ be positively oriented with respect to D and let the function $\phi \in C^1(D)$ satisfy (2.11) in the domain D and has order -2 at infinity in the infinite domain case. Then integrating equality (2.11) and using Green's formula we obtain the equality

$$\int_\Gamma dt_J\phi^+(t) = 0, \tag{3.2}$$

which plays the role of the Cauchy theorem. Here the matrix differential is defined analogously to (2.17), acts on a vector ϕ^+ in the usual way and therefore it stands in front of this vector. Hence, as in the classical analytic functions case, it implies the Cauchy formula

$$\frac{1}{2\pi i}\int_\Gamma (t-z)_J^{-1} dt_J\phi^+(t) = \phi(z),\ z \in D, \tag{3.3}$$

where it is supposed that in the unbounded domain case the function ϕ has order -1 at infinity. In particular, it follows from the last formula that the function $\phi \in C^\infty(D)$.

Denoting by $\phi^{(k)}$ the successive partial derivatives with respect to x, in view of (2.11) we have the equations

$$\frac{\partial^k \phi}{\partial x^{k-s} \partial y^s} = J^s \phi^{(k)}, \quad 0 \le s \le k \tag{3.4}$$

for the rest of the partial derivatives. It also follows from the Cauchy formula that the function, which is $J-$analytic all over the domain and vanishes at infinity, is identically equal to zero. As for ordinary analytic functions, from formulas (3.2), (3.3) the next proposition on analytic extension easily follows.

Let a slit L be a smooth arc with ends at points of the boundary contour Γ, which lies inside D except for these ends. If a function ϕ is continuous in D and $J-$analytic in $D \setminus L$, then this function is $J-$analytic all over the domain D.

All main results of the classical analytic functions theory which are based on the Cauchy integral hold for $J-$analytic functions [25]. For convenience let us present the basic ideas of this theory without proof. In a neighborhood of an isolated singular point a for a $J-$analytic function we have the Laurent expansion

$$\phi(z) = \sum (z-a)_J^k c_k, \quad c_k \in \mathbb{C}^l,$$

in integer powers of the matrix $(z-a)_J$. If ϕ is bounded in a neighborhood of this point then it is removable and the expansion becomes the corresponding Taylor expansion with the coefficients $c_k = \phi^{(k)}/k!$. The corresponding partial sums of this series are $J-$analytic polynomials

$$p(z) = \sum_{k=0}^{n} z_J^k c_k, \quad c_k \in \mathbb{C}^l.$$

We denote the class of all such polynomials by P_J^n. Obviously, for $n = 0$ it coincides with $\mathbb{C}^l$.

In the case of an unbounded domain D the infinitely remote point ∞ can be considered as an isolated one. We would remind that $J-$analyticity of a function ϕ in this domain implies that for the derivative ϕ' the condition similar to (2.3) is satisfied. In this case the Laurent expansion in a neighborhood of ∞ becomes a series in positive powers of z_J. Similarly if ϕ has order k at infinity i.e. ϕ can be estimated as $\phi(z) = O(|z|^k)$, then its Laurent expansion in a neighborhood of ∞ is a series in powers of z_J^i, $i \le k$. In particular, the function $z_J^{-k}\phi(z)$ is the Douglis analytic one in a neighborhood of ∞.

If a function ψ is defined and is $J-$analytic in a simply connected domain D then the integral

$$\phi(z) = \int_{z_0}^{z} dt_J \psi(t) \tag{3.5}$$

does not depend on a path of integration and defines a $J-$analytic function with the derivative $\phi' = \psi$. Generally speaking, in the case of multiply-connected domain D the antiderivative ϕ of the function ψ is multi-valued and allows branching when going over connected components of the domain boundary. It is obvious that formula (3.5) leads to a single-valued function if an only if

$$\int_{\Gamma'} dt_J \psi(t) = 0 \tag{3.6}$$

for each simple contour $\Gamma' \subseteq D$. In the general case the integral here could be interpreted as the variation of the function ϕ over the contour Γ'.

Let the domain D be bounded and its boundary consist of finite number m of connected components. Let us consider in D simple contours Γ'_j, $1 \le j \le m-1$, which keep inside the corresponding $m-1$ of these components. Then in view of the Cauchy theorem it is sufficient to verify condition (3.6) only for these contours. A similar proposition is valid for an infinite domain under the condition that ψ has order -2 at infinity. In this case it is possible to integrate in (3.5) from $z_0 = \infty$ and boundary components connected with Γ'_j can be chosen arbitrarily.

Similarly to (3.3) we can introduce the generalized Cauchy integral

$$(I\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma (t-z)_J^{-1} dt_J \varphi(t),$$

defining the $J-$analytic function $\phi = I\varphi$ with order -1 at infinity outside of the oriented contour and the corresponding singular Cauchy integral

$$(S\varphi)(t_0) = \frac{1}{\pi i}\int_\Gamma (t-t_0)_J^{-1} dt_J \varphi(t), \quad t_0 \in \Gamma,$$

which is undestood in the sense of the principal value. The operator I defined by this integral is bounded in the Hölder spaces $C^\mu(\Gamma) \to C^\mu(\overline{D})$, where D is any connected component of the complement to Γ, and the Sokhotskii-Plejmel formulas [3]

$$2\phi^\pm = \pm\varphi + S\varphi \tag{3.7}$$

are valid for its boundary values $\phi^\pm$ (the signs are defined by the contour orientation).

Hence, taking into account the proposition on analytic continuation, it is easy to solve the problem on the Cauchy integral representation of a function which is $J-$analytic outside the contour Γ and has finite order at infinity. If this function belongs to $C^\mu(\overline{D})$ where D is an arbitrary connected component of the complement to Γ and $\varphi = \phi^+ - \phi^-$, then $\phi = I\varphi + p$ with some $J-$analytic polynomial p.

If the density φ of the Cauchy integral belongs to the Lebesgue space $L^p(\Gamma)$, $1 < p < \infty$ then function $\phi = I\varphi$ belongs to the Hardy space $H^p(D)$. This space of $J-$analytic functions can be introduced by the following way. Let the contour Γ belong to class $C^{1,\nu}$ and let a sequence of contours $\Gamma_n \subseteq D,\ n = 1, 2, \ldots$ converge to Γ in in the metric of this class. Then $H^p(D)$ consists of all functions $J-$analytic in D for which the norm

$$|\phi| = \sup_n |\phi|_{L^p(\Gamma_n)} \tag{3.8}$$

is finite. As is shown in [3] the Cauchy integral is bounded $L^p(\Gamma) \to H^p(D)$ as linear operator $\varphi \to \phi$ and the Sokhotskii–Plejmel formula is valid for the boundary values. On the other hand any function $\phi \in H^p$ can be represented by the Cauchy integral with the density $\varphi \in L^p(\Gamma)$. Indeed let the domain $D_n \subseteq D$ be bounded by a contour Γ_n. Then for any fixed point $z \in D$ and sufficiently large n we can write the Cauchy formula

$$\phi(z) = \frac{1}{2\pi i}\int_{\Gamma_n} (t-z)_J^{-1} dt_J \phi(t).$$

It follows from (3.7) and the weak compactness [19] of the unit ball in a reflexive Banach space L^p, $p > 1$, that for any matrix–function $k(t) \in C(\overline{D})$ there is a function $\varphi \in L^p(\Gamma)$ and a subsequence n_k such that

$$\lim_{k\to\infty} \int_{\Gamma_{n_k}} k(t)dt_J\phi(t) = \int_{\Gamma} k(t)dt_J\varphi(t).$$

Therefore with $k(t) = (t-z)_J^{-1}$ and $n = n_k$ in the previous equality it is possible to pass to the limit as $k \to \infty$ and to present ϕ by the Cauchy integral as a result. In particular it follows that a function $\phi \in H^p(D)$ if and only if there exist angular limits almost everywhere on Γ which belong to $L^p(\Gamma)$ and the Cauchy formula holds its form. For these reasons the space H^p can be defined as the closure of the class of $J-$analytic functions being continuous in a close domain $\overline{D}$ in the norm

$$|\phi| = |\phi^+|_{L^p(\Gamma)},$$

which is equivalent to (3.8).

Results of [28] on $J-$analytic functions in the Hölder classes can be easy extended to the Hardy class. Without loss of generality matrix J can be considered as a triangular one.

Lemma 3.1. *Let a domain D (bounded or unbounded) be bounded by a simple Lyapunov contour Γ and the matrix J be triangular. Let a $J-$ analytic function $\phi \in H^p(D)$ be such that $\mathrm{Re}\phi^+$ is constant on Γ. Then ϕ is constant in the domain D.*

Proof. First we prove the lemma for the scalar case $l = 1$ when $J = \nu \in \mathbb{C}$ and ϕ is a solution of the equation

$$\frac{\partial\phi}{\partial y} - \nu\frac{\partial\phi}{\partial x} = 0.$$

Under the affine transformation $z = x + iy \to z_\nu = x + \nu y$ this equation becomes the Cauchy–Riemann equation defining analytic functions. Obviously the Hardy class is invariant under these transformations, therefore without loss of generality the function ϕ can be considered to be analytic. In this case the statement of the lemma is well known[12]. So in the scalar case the statement of the lemma has been established. In the general case $l > 1$ let us consider system of equations (2.11).

The matrix J, assumed for definiteness to be upper-triangular, can be coordinatewise rewritten in the form

$$\frac{\partial\phi_j}{\partial y} - \sum_{k=j}^{l} J_{kj}\frac{\partial\phi_k}{\partial x} = 0, \quad 1 \le k \le l$$

for the vector $\phi = (\phi_1, \ldots, \phi_l)$. By virtue of the proved facts from the last equation of this system it follows that the function ϕ_l is constant. Hence the equation number $(l-1)$ becomes a scalar equation for ϕ_{l-1} discussed above and $\nu = J_{l-1,l-1}$. Therefore for the same reasons the function ϕ_{l-1} is constant. Repeating these arguments we conclude that all functions ϕ_k are constant. □

Let us turn to the problem on representability of $J-$analytic functions $\phi \in H^p$ by the Cauchy integrals with real density.

Theorem 3.1. *Let a Lyapunov contour* Γ *bound a domain* D*, have positive orientation relative to* D *and consist of components* $\Gamma_1, \dots, \Gamma_m$*,* $m \geq 1$*, with the contour* Γ_m *enveloping all others in the case of a finite domain. Let the matrix* J *be triangular.*

Then any $J-$*analytic function* $\phi \in H^p(D)$ *can be represented in the form*

$$\phi = I\varphi + \eta, \quad \eta \in \mathbb{C}^l, \tag{3.9}$$

where real $l-$*vector-valued function* φ *belongs to* $L^p(\Gamma)$ *and, in the case of a bounded domain* D*, the vector* η *is purely imaginary.*

Herewith in this representation $\phi = 0$ *if and only if* $\eta = 0$ *and the function* φ *is constant on the contours* Γ_j *(vanishing on* Γ_m *in the case of a bounded domain* D*).*

Proof. is carried out in the same way as in the case of functions in the Hölder classes [28]. First we suppose that the domain D is finite and bounded by a simple contour (i.e. $m = 1$). Let $\widetilde{D}$ be the complement of $\overline{D}$ and let $\widetilde{I}\varphi$ be the Cauchy operator in the domain $\widetilde{D}$. Then on the basis of (3.7)

$$(I\varphi)^+ - (\widetilde{I}\varphi)^- = \varphi. \tag{3.10}$$

We claim that

$$\text{Re } (I\varphi)^+ = 0 \;\Rightarrow\; \varphi = 0, \tag{3.11}$$

$$\text{Re } (\widetilde{I}\varphi)^- = \xi \in \mathbb{R}^l \;\Rightarrow\; \xi = 0,\ \varphi \in \mathbb{R}^l. \tag{3.12}$$

Indeed if Re $(I\varphi)^+ = 0$ then by Lemma 3.1 the function $I\varphi$ is constant, and taking into account (3.9) we obtain that the function $\text{Im}\,(\widetilde{I}\varphi)^-$ is constant too. Using Lemma 3.1 once more we deduce that the function $\widetilde{I}\varphi$ is also constant and the density $\varphi = \xi \in \mathbb{R}^l$ is constant too. But then $I\varphi = \xi$ and because Re $(I\varphi)^+ = 0$ implication (3.11) is valid. Reasoning for the integral $\widetilde{I}\varphi$ is the same. As above we make certain that the functions $\widetilde{I}\varphi$ and φ are constant. Hence, because the first of them vanishes at infinity, (3.12) follows.

Let us consider operators the $M\varphi = \text{Re } (I\varphi)^+$ and $\widetilde{M}\varphi = \text{Re } (\widetilde{I}\varphi)^-$ acting in $L^p(\Gamma)$. According to (3.7) we have:

$$M\varphi = \text{Re } (\varphi + S\varphi)/2, \quad \widetilde{M}\varphi = \text{Re } (-\varphi + S\varphi)/2.$$

The complex conjugation operation $\varphi \to \overline{\varphi}$ induces the corresponding involution operator $N \to \overline{N}$ by the rule $\overline{N}\varphi = \overline{N\overline{\varphi}}$. With this notation

$$M = 1 + (S + \overline{S})/2, \quad \widetilde{M} = -1 + (S + \overline{S})/2. \tag{3.13}$$

If the dependence of the operator S on the matrix J is denoted by $S = S_J$ then $\overline{S} = -S_{\overline{J}}$ (the minus sign appears because of the multiplier $1/\pi i$ in front of the singular integral). Let S_0 be the classical Cauchy singular operator, corresponding to the scalar matrix

$J = i$, and $e = e_1 + ie_2$ be the unit tangent vector tothe contour Γ selected to be consistent with its orientation. Then in compliance with (3.1) we have

$$(S\varphi)(t_0) - (S_0\varphi)(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{k(t_0,t;J)}{t-t_0}\varphi(t)|dt|, \tag{3.14}$$

where $k(t_0,t;J) = (t-t_0)(t-t_0)_J e_J(t) - e(t)$ and $|dt|$ is the element of arc length. As Γ is a Lyapunov contour the matrix-function

$$k(t_0,t;J) = O(|t-t_0|^\nu) \tag{3.15}$$

for some $\nu > 0$, it follows that the operator $S - S_0$ is a compact operator in the space L^p. But then the operator $S + \overline{S} = S_J - S_{\overline{J}}$ has this property too. Therefore by virtue of the Riesz theorem [19] the operators M and $\widetilde{M}$ in (3.13) are Fredholm operators with zero indices. In combination with (3.11), (3.12) we conclude that the operator M is invertible and

$$\ker \widetilde{M} = \{0\}, \quad \mathbb{R}^l \cap \operatorname{im} \widetilde{M} = \{0\}. \tag{3.16}$$

Let further $\phi \in H^p(D)$ and $f = \operatorname{Re}\phi^+$. Assuming that $\varphi = M^{-1}\varphi$ we obtain $(\phi - I\varphi)^+ = 0$ and by Lemma 3.1 the function $\phi = I\varphi + i\xi$ with some $\xi \in \mathbb{R}^l$. If in this equality $\phi = 0$ then $M\varphi = 0$ and so $\varphi = 0$. Therefore the statement of the theorem is established in the case under consideration.

Let further $\widetilde{\phi} \in H^p(\widetilde{D})$ and $\widetilde{\phi}_0(z) = \widetilde{\phi}(z) - \widetilde{\phi}(\infty)$. We remind that the operator $\widetilde{M}$ is a Fredholm one with zero index. So taking into account (3.16) we obtain that function $f = \operatorname{Re}\widetilde{\phi}_0$ can be represented in the form $\widetilde{M}\varphi + \xi$ with some $\varphi \in L^p$ and $\xi \in \mathbb{R}^l$. Then $\operatorname{Re}(\widetilde{\phi}_0 - \widetilde{I}\varphi) = \xi$ and by Lemma 3.1 the function $\widetilde{\phi}_0 - \widetilde{I}\varphi$ is constant. As it vanishes at ∞ since $\widetilde{\phi} = \widetilde{I}\varphi$, it implies expansion (3.9) for $\widetilde{\phi}$ with $\eta = \widetilde{\phi}(\infty)$. The fact, that $\widetilde{\phi} = 0$ in this expansion implies that $\eta = 0$ and $\varphi \in \mathbb{R}^l$, is proved in the same way. Therefore the statement of the theorem is also established for the case of an infinite domain bounded by a simple contour.

Let us consider the general case of a contour Γ. A domain D is for definiteness assumed to be bounded. Let a domain D_j be bounded by a contour Γ_j and unbounded (bounded) for $1 \le j \le m-1$ (for $j = m$), so the domain D' is the union of domains $D_1, \ldots, D_m$. In accordance with the Cauchy formula the function $\phi \in H^p(D)$ can be represented as the sum

$$\phi(z) = \phi_1(z) + \ldots + \phi_m(z), \quad z \in D, \tag{3.17}$$

where $\phi_j \in H^p(D_j)$ are defined by the Cauchy integral

$$\phi_j(z) = \frac{1}{2\pi i}\int_{\Gamma_j}(t-z)_J^{-1} dt_J \phi^+(t), \quad z \in D_j.$$

The statement of the theorem is already applicable to the functions ϕ_j so

$$\phi_j(z) = \frac{1}{2\pi i}\int_{\Gamma_j}(t-z)_J^{-1} dt_J \varphi_j(t), \quad 1 \le j \le m-1,$$

$$\phi_m(z) = \frac{1}{2\pi i}\int_{\Gamma_m}(t-z)_J^{-1}dt_J\varphi_m(t) + i\xi, \quad \xi \in \mathbb{R}^l.$$

Substituting these relations in (3.17) we conclude the required expansion (3.9). If in this expansion $\phi = 0$ and functions $\phi_j \in H^p(D_j)$ are defined by the Cauchy integral with the density $\varphi|_{\Gamma_j}$ then the equality $-\phi_m = \phi_1 + \ldots + \phi_{m-1}$ allows us to continue $-\phi_m$ to the Douglis analytic function all over the plane vanishing at ∞. Therefore $\phi_m = 0$. In the same way it can be shown that $\phi_j = 0$ for all j.

Applying the theorem to ϕ_j we make certain that $\varphi|_{\Gamma_j} \in \mathbb{R}^l$ as $1 \le j \le m-1$, and $\xi = 0$, $\varphi|_{\Gamma_m} = 0$, i.e. the theorem is established completely. □

As in the case of general analytic functions the Riemann–Hilbert problem

$$\operatorname{Re} G\phi\big|_{\Gamma} = f, \tag{3.18}$$

can be considered for the Douglis analytic functions. Here $l \times l-$matrix–function $G \in C(\Gamma)$ is invertible all over Γ. This problem is considered in the space $H^p(D)$, $p > 1$ with the right-hand side $f \in L^p(\Gamma)$. The Fredholm property and the index of the problem are understood with respect to the $\mathbb{R}-$ linear operator $\phi \to \operatorname{Re} G\phi$ of its boundary condition.

Theorem 3.2. *Let a Lyapunov contour Γ consist of m connected components and the determinant of the matrix–function $G \in C(\Gamma)$ be non-zero all over Γ.*

Then problem (3.18) is a Fredholm one and its index æ is given by the formula

$$æ = -\frac{1}{\pi}\arg\det G\big|_{\Gamma} + (2-m)l, \tag{3.19}$$

where the increment of continuous argument branch on Γ is selected in the direction keeping the domain D on the left.

If $\Gamma \in C^{1,\nu}$ and $G \in C^{\nu}(\Gamma)$ then any solution $\phi \in H^p(D)$ of this problem with the right-hand side $f \in C^{\mu}(\Gamma)$, $0 < \mu < \nu$, belongs to the class $C^{\mu}(\overline{D})$. Under additional assumption $G \in C^{1,\nu}(\Gamma)$ the same statement is also valid for the classes $C^{1,\mu}$.

Proof. Without loss of generality the matrix J can be assumed to be Jordan and, in particular, triangular. Indeed, let the matrix $b \in \mathbb{C}^{l\times l}$ reducing J to the Jordan form J_0, i.e. $J_0 = b^{-1}Jb$. Then the substitution $\phi = b\phi_0$ transforms $J-$analytic functions into J_0-analytic. Let us note that under this substitution problem (3.18) turns to the same problem for J_0-analytic function ϕ_0 with the matrix $G_0 = Gb$.

So we can use Theorem 3.1. It follows by this theorem that the integral operator I acting from the space $L^p(\Gamma)$ of real $l-$ vector-functions in $H^p(D)$ is a Fredholm one and its index $\operatorname{ind} I = l(m-2)$. On the other hand in accordance with (3.7) for composition $N = 2RI$ of the operator R of problem (3.17) and I we have equality $N\varphi = \operatorname{Re}(\varphi + S\varphi)$. In terms of the involution operator of conjugation introduced when proving Theorem 3.1 we can write

$$N = G(1+S)/2 + \overline{G}(1+\overline{S})/2 = G(1+S_0)/2 + \overline{G}(1-S_0)/2 + K, \tag{3.20}$$

with the integral operator $2K = G(S-S_0) + \overline{G}(\overline{S}+S_0)$. This operator is defined by (3.14) with the matrix–function

$$k(t_0,t) = [G(t_0)k(t_0,t;J) - \overline{G(t_0)}k(t_0,t;\overline{J})]/2,$$

for which property (3.15) holds. Therefore by virtue of classical theory of singular operators with Cauchy kernel [18, 13] the operator N is Fredholm in L^p and its index is defined by the first summand in the right-hand side of (3.19). Hence by the common properties of Fredholm operators [19] the first statement of the theorem is obtained directly.

As for the second part of the theorem related to smoothness, let $\Gamma \in C^{1,\nu}$ and $0 < \mu < \nu$. Then the Cauchy operator $I : C^{\mu}(\Gamma) \to C^{\mu}(\overline{D})$ is bounded. Taking into account the Cauchy integral derivation formula established in [2] (see Lemma 5.2 below) we obtain that an analogous statement is valid for the classes $C^{1,\mu}$. Therefore it suffices to establish the statements of the theorem on smoothness only for the equation $N\varphi = f$ defined by operator (3.20). Let $\Gamma \in C^{1,\nu}, G \in C^{\nu}(\Gamma)$ and $f \in C^{\mu}(\Gamma)$. Then the function $k(t_0, t; J)$ in (3.14), and with it $k(t_0, t)$ belong to the class $C^{\nu}(\Gamma \times \Gamma)$. The integral operator K is shown in [29] be bounded $C(\Gamma) \to C^{\nu}(\Gamma)$ and particularly compact in $C^{\mu}(\Gamma)$. Therefore by virtue of the general theory [18, 13] the solution $\varphi \in L^p$ of this equation belongs to the class C^{μ}. A similar statement with respect to classes $C^{1,\mu}$ requires more delicate reasoning connected with singular integral derivation and it has been established in [2]. □

Let us make a special emphasis on the Riemann–Hilbert problem with the constant matrix G. In this case index formula (3.19) turns into æ $= l(2 - m)$. More precise consideration of this problem allows to outline the following property of the Hardy spaces.

Theorem 3.3. *Let a domain D be bounded by a contour Γ of class $C^{1,\nu}$ and let a sequence $\Gamma_n \subseteq D,\ n = 1, 2, \dots$ converge to Γ in the metric of this class. Let also a matrix $G \in \mathbb{C}^{l\times l}$ be invertible and a function ϕ $J-$analytic in the domain D be such that the real $l-$vector–function* Re $G\phi$ *has finite norm (3.8). Then $\phi \in H^p(D)$.*

Proof. Without loss of generality the matrix J can be assumed to be triangular. This can be justified as in the case of Theorem 3.2. Since the statement of the theorem is connected with the behavior of the function ϕ near connected components of contour, the domain D can be assumed to be bounded with the contour Γ consisting of two components. In this case it is convenient to slightly modify the operator I of the Cauchy integral setting

$$(I\varphi)(z) = \frac{1}{2\pi i}\left[\int_{\Gamma}(t - z)_J^{-1} dt_J \varphi(t) + \int_{\Gamma}\varphi(t)|dt|\right], \quad z \in D, \tag{3.21}$$

It follows by Theorem 3.1 that the operator $I : L^p(\Gamma) \to H^p(D)$ is invertible.

In our notation the next equality corresponds to formula (3.7)

$$2(I\varphi)^+ = \varphi + S\varphi, \quad (S\varphi)(t_0) = \frac{1}{\pi i}\int_{\Gamma}[(t - t_0)_J^{-1} e_J(t) + 1]\varphi(t)|dt|. \tag{3.22}$$

and for the operator $N\varphi = \operatorname{Re} G(I\varphi)^+$ we have the expression similar to (3.20):

$$N = G(1 + S)/2 + \overline{G}(1 + \overline{S})/2. \tag{3.23}$$

Therefore problem (3.18) is equivalent to the equation $N\varphi = 2f$ whose solution defines the solution $\phi = I\varphi$ of the problem. According to Theorem 3.2 the problem and, correspondingly, the operator N are Fredholm ones and have zero index. Let $\phi_1, \dots, \phi_k \in H^p(D)$ form a basis in the space of solutions of homogeneous problem (3.18). Without loss of generality one can assume that some subdomain D_0 with its boundary is inside all contours Γ_n. The real $l-$ vector-functions $\operatorname{Re}\phi_j$ as elements $C(\overline{D_0})$ are linearly independent. Indeed if $\operatorname{Re}\phi \equiv 0$ in the domain D_0 for some $J-$analytic function ϕ then the fact that $\phi \equiv 0$ is proved similarly to Lemma 3.1. Let us choose a system of real $l-$vector-functions $\psi_1, \dots, \psi_k$ biorthogonal to functions $\operatorname{Re}\phi_j(z),\ z \in D_0$. In other words

$$\int_{D_0} (\mathrm{Re}\ \phi_i)\psi_j dxdy = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. then homogeneous problem (3.18) supplemented by conditions

$$\int_{D_0} \psi_j \operatorname{Re}\phi \, dxdy = 0, \quad 1 \le j \le k,$$

has only zero solution. Let us consider the operator $L:\ L^p(\Gamma) \to \mathbb{R}^k$ defined by the formula

$$(L\varphi)_j = \int_{D_0} \operatorname{Re}\psi_j \operatorname{Re}(I\varphi) dxdy, \quad 1 \le j \le k.$$

Taking into account (3.21) we can write it in a more explicit form of the inner product

$$(L\varphi)_j = \int_{\Gamma} g_j(t)\varphi(t)|dt|, \quad 1 \le j \le k, \tag{3.24}$$

with the functions

$$g_j(t) = \frac{1}{2\pi}\int_{D_0} \operatorname{Im}[e_J(t)(t-z)_J^{-1} + 1]\psi_j(z) dxdy.$$

In this notation the operator $(N, L):\ L^p(\Gamma) \to L^p(\Gamma) \times \mathbb{R}^k$ is a Fredholm one with zero kernel. We denote the dependence of operators (3.22) - (3.24) and the functions defining them on Γ by notation $S_\Gamma,\ L_\Gamma$ etc.

Let us turn to the sequence of contours Γ_n that is the matter of the theorem. By the assumption of the theorem there are homomorphisms $\alpha_n:\ \Gamma \to \Gamma_n$ of the class $C^{1,\nu}(\Gamma)$ such that

$$\lim_{n\to\infty} |\alpha_n(t) - t|_{C^{1,\nu}} = 0. \tag{3.25}$$

The operation of superposition $\varphi \to \varphi\circ\alpha_n$ induces the involution operator $M \to M\circ\alpha_n$ by the rule $(M\circ\alpha_n)(\varphi\circ\alpha_n) = (M\varphi)\circ\alpha_n$ which converts the Banach space $\mathcal{L}[L^p(\Gamma_n)]$ of operators bounded in $L^p(\Gamma_n)$ to $\mathcal{L}[L^p(\Gamma)]$. The denotation $(M\circ\alpha_n)(\varphi\circ\alpha_n) = M\varphi$ for the operator $M:\ L^p(\Gamma_n) \to \mathbb{R}^k$ has a similar meaning. It is claimed that in this notation

$$|S_{\Gamma_n}\circ\alpha_n - S_\Gamma|_{\mathcal{L}} \to 0, \quad |L_{\Gamma_n}\circ\alpha_n - L_\Gamma|_{\mathcal{L}} \to 0 \tag{3.26}$$

as $n \to \infty$ by the operator norm of the corresponding spaces. Indeed let

$$q_n(t_0, t) = [\alpha_n(t) - \alpha_n(t_0)]_J^{-1} e_J[\alpha_n(t)] e_J^{-1}(t)(t - t_0)_J.$$

Then by virtue of (3.25) the sequence of matrix–functions $q_n \to 1$ in the norm $C^\nu(\Gamma \times \Gamma)$. It remains to note that in this notation

$$[(S \circ \alpha_n)\varphi](t_0) = \frac{1}{\pi i} \int_\Gamma [q_n(t_0, t)(t - t_0)_J^{-1} e_J(t) + 1]\varphi(t)|\alpha_n'(t)||dt|,$$

$$[(L_{\Gamma_n} \circ \alpha_n)\varphi]_j = \int_\Gamma g_{\Gamma_n, j}[\alpha_n(t)]|\alpha_n'(t)|\varphi(t)|dt|.$$

Now let ϕ be a $J-$ analytic function in D for which the real functions $f_n = \operatorname{Re}\phi|_{\Gamma_n}$ are uniformly bounded in the norm of the spaces $L^p(\Gamma_n)$. Let

$$\xi_j = \int_{D_0} \psi_j \operatorname{Re}\phi \, dxdy, \quad 1 \le j \le k.$$

Let $\varphi_n \in L^p(\Gamma_n)$ be defined by the equality $\phi = I_{\Gamma_n}\varphi_n$ in the domain $D_n \subseteq D$ bounded by the contour Γ_n. Then $N_{\Gamma_n}\varphi_n = 2f_n$ and $(L_{\Gamma_n}\varphi_n)_j = \xi_j$ or equivalently

$$(N_{\Gamma_n} \circ \alpha_n)\widetilde{\varphi}_n = 2f_n, \quad [(L_{\Gamma_n} \circ \alpha_n)\widetilde{\varphi}_n]_j = \xi_j,$$

where $\widetilde{\varphi}_n = \varphi_n \circ \alpha_n$. According to (3.23) relation (3.26) is also valid for the operator N therefore by virtue of Lemma 3.2 below, the sequence $\widetilde{\varphi}_n$ is bounded in $L^(\Gamma)$. Taking into account (3.26), (3.22) it follows that the sequence of functions $(S_{\Gamma_n} \circ \alpha_n)\widetilde{\varphi}_n$ and $\phi \circ \alpha_n$ are bounded in $L^p(\Gamma)$, that completes the proof of the theorem. □

Lemma 3.2. *Let X, Y be Banach spaces and $N \in \mathcal{L}(X, Y)$ be a Fredholm operator with a non-zero kernel. Let a sequence $N_n \to N$ as $n \to \infty$ in the norm of the space $\mathcal{L}(X, Y)$. If a sequence of vectors $N_n x_n$ is bounded in Y, then the sequence x_n is bounded in X.*

Proof. First let us assume that the image $\operatorname{im} N$ of the operator coincides with Y, i.e. the operator N is invertible. Then the operators N_n are also invertible for n sufficiently large and the sequence N_n^{-1} converges to N^{-1} in $\mathcal{L}(Y, X)$. So the sequence $x_n = N_n^{-1} y_n$ is bounded. In the general case by the assumption of the theorem the image $\operatorname{im} N$ is closed and $Y = Y_0 \oplus \operatorname{im} N$ for some finite-dimensional subspace Y_0. Let us choose a basis $y_1, \ldots, y_k$ of the subspace and consider the operators $\widetilde{N}, \widetilde{N}_n \in \mathcal{L}(X \times \mathbb{R}^k, Y)$ by the formula

$$\widetilde{N}(x, \lambda) = Nx + \sum\nolimits_1^k \lambda_i y_i, \quad \widetilde{N}_n(x, \lambda) = N_n x + \sum\nolimits_1^k \lambda_i y_i.$$

Then the operator $\widetilde{N}$ is bounded and the sequence $\widetilde{N}_n \to \widetilde{N}$ in the operator norm. Hence, because of $\widetilde{N}_n(x_n, 0) = y_n$, the boundedness of the sequence x_n follows. □

Let us note that an analogue of Theorem 3.3 is valid with respect to the Hölder classes $C^\mu(\overline{D})$, $0 < \mu < \nu$. It was obtained in [30] by much simpler means.

4 Solvability of boundary value problems in the Hardy class

Let us turn to matrix (1.7) of the plane elasticity theory. In this case for matrix (3.1) we have the expressions

$$(i)\quad z_J = \begin{pmatrix} x+\nu_1 y & 0 \\ 0 & x+\nu_2 y \end{pmatrix}, \quad z_J^{-1} = \begin{pmatrix} (x+\nu_1 y)^{-1} & 0 \\ 0 & (x+\nu_2 y)^{-1} \end{pmatrix},$$

and

$$(ii)\quad z_J = \begin{pmatrix} x+\nu y & y \\ 0 & x+\nu y \end{pmatrix}, \quad z_J^{-1} = \begin{pmatrix} (x+\nu y)^{-1} & y(x+\nu y)^{-2} \\ 0 & (x+\nu y)^{-1} \end{pmatrix}.$$

The simplest multi-valued $J-$analytic function allowing branching while traversing a fixed point $z=0$ is the matrix–function $L(z)=\ln z_J$, where $\ln z_J$ is the value from matrix J of the function $f(\zeta)=\ln(x+\zeta y)$ which is analytic in the upper half-plane $\operatorname{Im}\zeta>0$. It is assumed that z varies in a simply connected zero-free domain and a continuous branch of the logarithm is selected. It is easy to see that the matrix–function L satisfies equation (2.11) and its values commute with matrix J. In the case under consideration (1.7) we have the following expression for this matrix

$$\text{(i)}\ \ln z_J = \begin{pmatrix} \ln(x+\nu_1 y) & 0 \\ 0 & \ln(x+\nu_2 y) \end{pmatrix}, \quad \text{(ii)}\ \ln z_J = \begin{pmatrix} \ln(x+\nu y) & (x+\nu y)^{-1} y \\ 0 & \ln z_\nu \end{pmatrix}.$$

While anticlockwise traversing a point $z=0$ this matrix–function acquires an increment equal to $2\pi i$.

Matrices of this type may serve for describing multi-valuedness type in representations (2.12) of the solutions of the Lamé system and conjugate function (2.15) in the domain D. Let the domain D be bounded by the contour Γ which consists of connected components $\Gamma_1,\dots,\Gamma_m$ and in the case of the bounded domain D the contour Γ_m envelopes all other components. Let us choose apoint a_j inside each contour Γ_j and consider the multi-valued $J-$analytic matrix–functions $L_j(z)$, $1\le j\le m-1$ by the formula

$$2\pi i L_j(z) = \begin{cases} \ln(z-a_j)_J, & \text{if the domain is bounded,} \\ \ln(z-a_j)_J - \ln(z-a_m)_J & \text{otherwise.} \end{cases} \tag{4.1}$$

Obviously, while anticlockwise traversing the point a_k, $1\le k\le m-1$, the $J-$analytic function $L_j(z)\eta$, $\eta\in\mathbb{R}^2$, acquires an increment equal to $\delta_{kj}\eta$, where δ_{kj} is the Kronecker symbol. In particular, the solution of the Lamé system

$$u = \operatorname{Re}\ b\sum_{j=1}^{m-1} H_j(z)\eta_j, \tag{4.2}$$

becomes single-valued if and only if $\operatorname{Re} b\eta_j = 0$, $1\le j\le m-1$. We denote the class of such single-valued solutions by U_0. Because the matrix b is invertible, the dimension of the space $\{\eta\in\mathbb{R}^2,\ |\ \operatorname{Re} b\eta=0\}=2$ is equal to two and therefore $\dim U_0 = 2(m-1)$. In the same way conjugate function

$$v = \operatorname{Re} c\sum_{j=1}^{m-1} H_j(z)\eta_j, \tag{4.3}$$

becomes single-valued if and only if $\operatorname{Re} c\eta_j = 0$, $1 \le j \le m-1$. We denote the class of such single-valued functions by V_0. For the same reasons $\dim V_0 = 2(m-1)$.

The following lemma demonstrates that the suitable solution u_0 of the Lamé system, whereby the Neumann problem (2.2) is adduced to the definition (2.9), can be selected in the class U_0.

Lemma 4.1. *Let a function $g \in C(\Gamma)$ satisfy the condition*

$$\int_\Gamma g(t)|dt| = 0. \tag{4.4}$$

Then there is a unique element $u_0 \in U_0$ such that for its conjugate function v_0 the difference $g - v_0^+$ has a smooth antiderivative $f \in C^1(\Gamma)$. Moreover, the function conjugate to $u - u_0$ is a single-valued one.

Proof. By virtue of (1.11) we have the matrix equality

$$\begin{pmatrix} b & \overline{b} \\ c & \overline{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a_{21} & -a22 \end{pmatrix} \begin{pmatrix} b & \overline{b} \\ bJ & \overline{bJ} \end{pmatrix},$$

hence the matrix on its left-hand side is invertible. Thereby there are unique vectors $\eta_j \in \mathbb{C}^2$ for which

$$\operatorname{Re} b\eta_j = 0, \quad \operatorname{Re} c\eta_j = \int_{\Gamma_j} g(t)|dt|, \ 1 \le j \le m-1.$$

Let $u_0 \in U_0$ in representation (4.2) be defined by these vectors and the function v_0 be conjugate to u_0. Then the increment of v_0 along the contour Γ_j is equal to $\operatorname{Re} c\eta_j$ so

$$\int_{\Gamma_j} [g - (v_0)'_e]|dt| = 0, \quad 1 \le j \le m-1. \tag{4.5}$$

In accordance with Theorem 2.1 the function $(v_0^+)'_s$ satisfies the necessary orthogonality condition (4.4), so the equality which is similar to (4.5) is also valid for $j = m$. Hence the statement of the lemma follows directly. □

By means of the spaces U_0 and V_0 it is possible to specify the type of multi-valuedness of $J-$analytic function in representations (2.12) and (2.15).

Theorem 4.1. *Let a domain D be bounded by a contour Γ which consists of connected components $\Gamma_1, \dots, \Gamma_m$, and in the case of bounded domain the contour Γ_m envelopes all other components. Then in any single-valued solution u of the Lamé system in the domain D can be uniquely represented in the form*

$$u = \operatorname{Re} b\phi + u_0, \quad u_0 \in U_0, \tag{4.6}$$

with some single-valued $J-$analytic function ϕ. Herewith $u = 0$ implies that $u_0 = 0$ and $\phi = \eta \in \mathbb{R}^2$, $\operatorname{Re} b\eta = 0$.

Similarly any single-valued function v conjugate to some (generally speaking multi-valued) solution of the Lamé system can be uniquely represented in the form

$$v = \operatorname{Re} c\phi + v_0, \quad v_0 \in V_0, \tag{4.7}$$

with some single–valued $J-$analytic function ϕ. Herewith $v = 0$ implies that $v_0 = 0$ and $\phi = \phi_0$, where $\phi_0 = \eta \in \mathbb{R}^2$, $\operatorname{Re} c\eta = 0$ if the domain D is unbounded and $\phi_0 = \eta_0 + z_J\eta_1$, $\operatorname{Re} c\eta_0 = 0$, $\operatorname{Re} c\eta_1 = \operatorname{Re} cJ\eta_1 = 0$ otherwise.

Proof. Let us consider representation (2.12) with some multi-valued function ϕ which will denoted here by ϕ_1. Let η_j be its increment along the contour Γ_j and ϕ_0 is defined by the sum in the right-hand side of (4.2). Then the difference $\phi = \phi_1 - \phi_0$ is single-valued, so the function $\operatorname{Re} b\phi = u - \operatorname{Re} b\phi_0$ is also single-valued. Therefore the single-valued function $u_0 = \operatorname{Re} b\phi_0$ belongs to U_0, which implies representation (4.6). If in this representation $u = 0$ and ϕ_0 is the sum in the right-hand side of (4.2), then $\operatorname{Re} b(\phi + \phi_0) = 0$ and therefore the function $\phi + \phi_0 = \eta \in \mathbb{R}^2$, $\operatorname{Re} b\eta = 0$. In particular, the function ϕ_0 is single-valued. This is possible only under the condition $\eta_j = 0$, $1 \le j \le m-1$. Thereby $\phi_0 = 0$ and $\phi = \eta$.

Representation (4.7) for the conjugate function is established similarly. One just need to prove that the equality $\operatorname{Re} c\phi = 0$ is possible only for the function $\phi = \phi_0$ specified in the lemma. Differentiating this equality according to (3.4) for the first and the second partial derivatives we obtain that

$$\operatorname{Re} c\phi' = \operatorname{Re} cJ\phi' = 0, \quad \operatorname{Re} c\phi'' = \operatorname{Re} cJ\phi'' = \operatorname{Re} cJ^2\phi'' = 0.$$

Hence, by virtue of Lemma 2.2 $\phi'' = 0$ and therefore ϕ is the $J-$analytic polynomial $\eta_0 + z_J\eta_1$ of the first order where $\operatorname{Re} c\eta_0 = 0$ and $\operatorname{Re} c\eta_1 = \operatorname{Re} cJ\eta_1 = 0$. It just remains to notice that in the case of unbounded domain the function ϕ is bounded, so $\eta_1 = 0$. □

For the solutions of the Lamé system and for the functions conjugate to them the Hardy class is introduced similarly to the case of $J-$analytic functions by the condition of finiteness of corresponding norm (3.8). As in the case of harmonic functions we denote this class by $h^p(D)$. By Theorems 3.3 and 4.1 it follows directly that transformations (4.2) and (4.3) map H^p to h^p. Thereby if the solution u of the Lamé system belongs to the class h^p then the $J-$analytic function ϕ in representation (4.6) belongs to H^p and therefore the conjugate function v defined by corresponding equality (4.7), also belongs to h^p. For harmonic functions this fact is known as the Riesz theorem [12]. [12]. Let us note that by virtue of remark to Theorem 3.3. a similar result is also valid for the Hölder classes C^μ, $0 < \mu < \nu$.

Another corollary of the result under discussion is that for elements of the class h^p there are almost everywhere angular limit values defining functions belonging to $L^p(\Gamma)$. In particular, Dirichlet problem (2.1) for the Lamé system can be formulated in the Hardy class. In generalized formulation (2.9) the Neumann problem can be also considered in the Hardy class. In other words we need to find such solution $u \in h^p$ of the Lamé system that its conjugate function v is single-valued and satisfies boundary condition (2.9) with some function χ constant on connected components of Γ. For these problems considered in the Hardy class the following analogue of Theorem 2.1 holds.

Theorem 4.2. *Let a domain D be bounded by a Lyapunov contour Γ.*

Then Dirichlet problem (2.1) *is uniquely solvable in the class $h^p(D)$. As for Neumann problem* (2.9), *the homogeneous problem has the trivial solution in this class and the nonhomogeneous problem in the case of an unbounded domain is unconditionally solvable. If the domain D is bounded then this problem is solvable if and only if orthogonality condition* (2.10) *is satisfied.*

If $\Gamma \in C^{1,\nu}$ and $f \in C^\mu(\Gamma)$, $0 < \mu < \nu$, then any solution $u \in h^p(D)$ of these problems belongs to the class $C^\mu(\overline{D})$. A similar statement is also valid for the classes $C^{1,\mu}$.

Proof. By virtue of Theorem 4.1 solvability of the Dirichlet problem reduces to solvability of the problem

$$\mathrm{Re}\ b\phi^+ + u_0^+ = f \tag{4.8}$$

with respect to the pair $(\phi, u_0) \in H^p(D) \times U_0$. Therefore for $G = b$ the statements of the theorem on the smoothness of solutions of the Dirichlet problem are corollares of the similar statements of Theorem 3.2. In particular by virtue of Theorem 2.1 the homogeneous Dirichlet problem has only zero solution in the class h^p. For the same reasons any solution $\phi \in H^p$ of the homogeneous Riemann–Hilbert problem $\mathrm{Re}\, b\phi^+ = 0$ belongs to the class $C^{1,\mu}(\overline{D})$. Because it defines solution $u = \mathrm{Re}\, b\phi$ of the homogeneous Dirichlet problem by virtue of Theorem 2.1 the function $u = 0$ and so $\phi = \eta \in \mathbb{C}^2$, $\mathrm{Re}\ b\eta = 0$. Therefore the kernel dimension $\dim\ker R$ of the operator R of Riemann–Hilbert problem (3.18) with $G = b$ is equal to 2. Because the index $\mathrm{ind}\, R = 2(2-m)$, the codimension of the image $\mathrm{im}\ R$ is equal to $2(m-1)$. The functions u_0^+, $u_0 \in U_0$ do not belong to this image, because $J-$analytic function ϕ_0 corresponding to them in representation (2.12) is multi-valued (it can be single-valued only if $\phi_0 = 0$). For the same reasons the dimension of the space $U_0^+ = \{u_0^+\ u_0 \in U_0\}$ is equal to $2(m-1)$. Therefore $L^p(\Gamma) = \mathrm{im}\, R \oplus U_0^+$, that proves unconditional solvability of problem (4.8) and hence solvability of the Dirichlet problem.

Let us turn to problem (2.9). In its formulation the solution u and the function v conjugate to u must be single-valued. Therefore as above its solvability reduces to solvability of the problem

$$\mathrm{Re}\ c\phi^+ + \chi = f \tag{4.9}$$

for the pair composed of $\phi \in H^p(D)$ and a real 2−vector–function χ, which is constant at connected components of the contour Γ. Because function v in (2.9) is defined up to an additive constant, without loss of generality one can assume that the function χ vanishes at the component Γ_m of the contour Γ. We denote this class of functions by X, its dimension is obviously equal to $2(m-1)$. In the case $G = c$ the statements of the theorem on smoothness of the Neumann problem solutions is corollary of the similar statements of Theorem 3.2. In particular taking into account Theorem 2.1 we obtain that homogeneous problem (2.9) has only trivial solutions in the class h^p. For the same reasons any solution $\phi \in H^p$ of the homogeneous Riemann–Hilbert problem $\mathrm{Re}\, c\phi^+ = 0$ belongs to the class $C^{1,\mu}(\overline{D})$. Because it defines solution $u = \mathrm{Re}\, c\phi$ of the homogeneous Neumann problem by virtue Theorem 2.1 the function u is a trivial solution. For these solutions the conjugate function v is constant, so $\mathrm{Re}\ c\phi = \xi \in \mathbb{R}^2$ in the domain D.

In accordance with Theorem 4.1 the kernel dimension $\dim\ker R$ of the operator R of Riemann–Hilbert problem (3.18) with $G = c$ is equal to 2 if the domain D is unbounded and it is equal to 3 otherwise.

Since index $\operatorname{ind} R = 2(2-m)$, it follows that the codimension k of the image $\operatorname{im} R$ of the operator R is equal to $2(m-1)$ if the domain D is unbounded and it is equal to $2(m-1)+1$ otherwise. Elements $\chi \in X$ do not belong to $\operatorname{im} R$ because otherwise the function χ should be constant at the contour Γ, but in accordance with the definition of X it is possible only if $\chi = 0$. Therefore in the case of an unbounded domain D we have the decomposition $L^p(\Gamma) = \operatorname{im} R \oplus X$, which proves unconditional solvability of problem (4.9) and hence the solvability of the Neumann problem too. Let the domain D be bounded. Then condition (2.10) is necessary for solvability of the problem R. Indeed for $\phi \in C^{1,\mu}(\overline{D})$ it follows by Theorem 2.1 and the remark to formula (2.9). In the general case the statement under consideration follows by density of the class $C^{1,\mu}(\overline{D})$ in $H^p(D)$. This can be easily verified by representing ϕ by the Cauchy integral $I\varphi$ and by approximating φ in the norm $L^p(\Gamma)$ by elements of $C^{1,\mu}(\Gamma)$.

Let us denote the class of functions $f \in L^p(\Gamma)$ satisfying the condition (2.10) by $\widetilde{L}^p(\Gamma)$. Then R can be considered as the operator $\widetilde{R} : H^p(D) \to \widetilde{L}^p(\Gamma)$ and in this case the codimension of its image is equal to $2(m-1)$. As was mentioned in Section 2 when discussing condition (2.10), the space $X \subseteq \widetilde{L}^p(\Gamma)$ so $\widetilde{L}^p(\Gamma) = \operatorname{im} \widetilde{R} \oplus X$. Thereby condition (2.10) is necessary and sufficient for solvability of problem (4.9) and hence for solvability of the Neumann problem which completes the proof of the theorem. □

Along with problems (2.1) и (2.9) one may consider the Dirichlet problem in the class $h^p(D)$

$$v^+ = f \tag{4.10}$$

for conjugate functions. In this formulation v is assumed to be a single–valued function conjugate to, generally speaking, multi-valued solution u of the Lamé system.

Theorem 4.3. *Let a domain D be bounded by a Lyapunov contour Γ.*

Then in the case of a bounded domain case Dirichlet problem (4.10) is uniquely solvable in the class $h^p(D)$.

If the domain D is unbounded then the homogeneous problem has only trivial solution in this class and nonhomogeneous problem is solvable if and only if orthogonality condition (2.10) is satisfied.

If $\Gamma \in C^{1,\nu}$ and $f \in C^\mu(\Gamma)$, $0 < \mu < \nu$, then any solution $v \in h^p(D)$ of this problem belongs to the class $C^\mu(\overline{D})$. A similar statement is also valid for the classes $C^{1,\mu}$.

Proof. First let us show that the homogeneous problem

$$v^+ = 0 \tag{4.11}$$

has only zero solution in the class $C^1(\overline{D})$. If the solution of the Lamé system u, to which the function v is conjugate, is single-valued in the domain D then this fact is established in the usual way by means of the Green formula as in the case of Theorem 2.1 for the Neumann problem. In general case we need to make some changes in this argument connected with multi-valuedness of the function u. Using the notation of Theorem 4.1 we join the connected components Γ_j by disjoint smooth arcs (slits) $R_1, \dots, R_l$, so that in complement of $R = R_1 \cup \ldots \cup R_l$ the domain D can be splitted in disjoint domains $D_1, \ldots, D_n$ each of which is bounded by a simple piecewise-smooth contour. In the domain D_r it is possible to choose a single-valued branch of the function u, in the aggregate these branches define the function in open set $D \setminus R$ which we denote by u. On the slits R_k it is discontinuous. More precisely if we orient the arc R_k and denote the limit values of u at the corresponding slit sides by $u^{\pm}$ then difference $u^+ - u^-$ maintains some constant value:

$$u^+ - u^- = \xi_k \in \mathbb{R}^2 \quad \text{on } R_k. \tag{4.12}$$

This follows by the fact that the partial derivatives of the function u are continuous all over the domain D. □

Now let us consider Lamé system (1.1) in the domain D_r. Multiplying equality (1.1) by u, integrating and applying by the Green formula in view of (2.6) we obtain

$$\int_{D_r} \left[\sum_{i,j=1}^{2} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) \frac{\partial u}{\partial x_i} \right] dx_1 dx_2 = \int_{\partial D_r} u v'_e |dt|, \tag{4.13}$$

where $x_1 = x,\ x_2 = y$ and the unit tangent vector e is oriented at the contour so that the domain D_r remains on the left.

If the domain D_r is unbounded then the existence of the double integral here is provided by condition (2.3). By virtue of boundary condition (4.11) the integrals in the right-hand side of (4.13) over $\Gamma \cap \partial D_r$ vanish. Two domains D_r border each slit R_k and the corresponding vectors e_j are opposite at R_k. Summing equalities (4.13) by $1 \le r \le n$ and using (4.12) we deduce the equality

$$\int_{D} \left[\sum_{i,j=1}^{2} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) \frac{\partial u}{\partial x_i} \right] dx_1 dx_2 = \sum_{k=1}^{l} \xi_k \int_{R_k} v'_e |dt| = 0, \tag{4.14}$$

where the tangent vector e at R_k is selected in accordance with slit orientation and it is taken into account that by virtue of (4.11) the integrals over R_k vanish. It is seen from the expressions for coefficients a_{ij} in (1.1) that the block matrix $a = (a_{ij})$ is symmetric and nonnegative defined. Therefore the expression under integral in the left-hand side of (4.14) vanish. By virtue of the matrix a symmetry it follows that the right-hand sides of relations (2.6) are identically equal to zero. Therefore the function v is constant in the domain D, which with (4.11) is possible only if $v = 0$.

Further reasoning is absolutely similar to reasons used for proving Theorem 4.2 for the Neumann problem. It is just necessary to take into account that in accordance with Theorem 4.1 problem (4.10) is reduced to the problem

$$\text{Re } c\phi^+ + v_0^+ = f$$

with respect to the pair $(\phi, v_0) \in H^p(D) \times V_0$. Therefore here the role of X is played by the space $V_0^+ = \{v_0^+\ v_0 \in V_0\}$ of dimension $2(m-1)$.

It should be noted that if problem (4.10) is considered in the class of single-valued functions u and v then in addition there appear $2(m-1)$ linear independent orthogonality conditions on the right-side f which are necessary and sufficient (jointly with (2.10) in the case of bounded domain) for its solvability.

In the case of orthotropic medium analogues of Theorems 4.1 and 4.2 in the Hölder class $C^\mu(\overline{D})$ were established earlier in papers [1] where the scheme of investigation of these problems is also given for the anisotropic case. One can also obtain results similar to Theorems 4.1 and 4.2 for solutions of general elliptic systems of the second order with constant leading, and only leading, coefficients.

5 Potentials of double layer

Considering the unit exterior normal $n = n_1 + in_2$ to the Lyapunov contour Γ, we establish linkage between the point $t \in \Gamma$ and the functions

$$p_0(t,\xi) = \frac{n_1(t)\xi_1 + n_2(t)\xi_2}{|\xi|^2}, \quad q_0(t,\xi) = \frac{n_2(t)\xi_1 - n_1(t)\xi_2}{|\xi|^2}, \tag{5.1}$$

homogeneous of degree -1 relative to the variable $\xi = \xi_1 + i\xi_2$. They define the integrals

$$(P_0\varphi)(z) = \frac{1}{\pi}\int_\Gamma p_0(t, t-z)\varphi(t)|dt|, \quad (Q_0\varphi)(z) = \frac{1}{\pi}\int_\Gamma q_0(t,t-z)\varphi(t)|dt| \tag{5.2}$$

with the real density φ define, which are obviously harmonic functions in the domain D. These integrals can be also considered for $z = t_0 \in \Gamma$, in this case they are denoted by $P_0^*\varphi$ and $Q_0^*\varphi$ correspondingly. The contour Γ being a Lyapunov one, the kernel $p_0(t, t-t_0)$ has weak singularity, the second integral $(Q_0^*\varphi)(t_0)$ is realized as a singular one.

Since $i\xi(p_0 + iq_0)$ coincides with the tangent vector $e = (e_1, e_2) = i(n_1 + in_2)$, integrals (5.2) are the real and the imaginary parts of the Cauchy integral

$$(P_0\varphi)(z) + i(Q_0\varphi)(z) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)dt}{t-z}. \tag{5.3}$$

Therefore operators P_0, $Q_0 : L^p(\Gamma) \to h^p(D)$ are bounded, $p > 1$, and the formulas

$$(P_0\varphi)^+ = \varphi + P_0^*\varphi, \quad (Q_0\varphi)^+ = Q_0^*\varphi, \tag{5.4}$$

are valid for their boundary values. For the same reason for $\Gamma \in C^{1,\nu}$ these operators are bounded as $C^\mu(\Gamma) \to C^\mu(\overline{D})$ and $C^{1,\mu}(\Gamma) \to C^{1,\mu}(\overline{D})$, $0 < \mu < \nu$, are bounded. The operator P_0 is well known to be bounded as $C(\Gamma) \to C(\overline{D})$.

The integral $P_0\varphi$ in (5.2) represents the classical potential of double layer. We construct generalized potentials of double layer for the Lamé system solutions and functions conjugate to them by similar scheme in terms of the Cauchy integral for $J-$analytic functions. For this purpose in notation (3.1) we introduce homogeneous of the degree -1 matrix–functions of the variable $\xi = \xi_1 + i\xi_2$ by the formula

$$H_{kr}(\xi) = \mathrm{Im}\,[b_k(i\xi)_J\xi_J^{-1}b_r^{-1}], \quad k, r = 1, 2, \tag{5.5}$$

where for uniformity we write $b_1 = b$, $\quad b_2 = c$. They define the integral operators

$$(P_{kr}\varphi)(z) = \frac{1}{\pi}\int_\Gamma p_0(t,t-z)H_{kr}(t-z)\varphi(t)|dt|, \; z \in D, \tag{5.6}$$

and

$$(P^*_{kr}\varphi)(t_0) = \frac{1}{\pi}\int_\Gamma p_0(t, t-t_0)H_{kr}(t-t_0)\varphi(t)|dt|,\ t_0 \in \Gamma. \tag{5.7}$$

The next lemma describes the connection between the functions $P_{kr}\varphi$ and the Cauchy integral $I\varphi$ introduced in Section 3. It is convenient to introduce the class h^p of functions $w = P_{kr}\varphi$ with φ for which the right-hand side of (3.8) is finite.

Lemma 5.1. *The following equality*

$$-[\mathrm{Im}\,(b_k b_r^{-1})]Q_0\varphi + P_{kr}\varphi = 2\,\mathrm{Re}[b_k I(b_r^{-1}\varphi)], \quad k, r = 1, 2, \tag{5.8}$$

holds. In particular, the operator $P_{kr} : L^p(\Gamma) \to h^p(D)$ is bounded, $p > 1$, and the following formula

$$(P_{kr}\varphi)^+ = [\mathrm{Re}(b_k b_r^{-1})]\varphi + P^*_{kr}\varphi, k, r = 1, 2, \tag{5.9}$$

holds for the angular boundary values. The integral operators P^*_{kr} are compact in $L^p(\Gamma)$.

Proof. Taking $n = -ie$ in (5.1) we obtain the equality $i\xi(p_0+iq_0)(t,\xi)\xi = e(t)$. Because the factors p and q are real, it follows that

$$-q_0(t,\xi)\xi_J + p_0(t,\xi)(i\xi)_J = e_J(t),$$

which due to (5.5) reduces to the relation

$$-q_0(t,\xi)[\mathrm{Im}\,(b_k b_r^{-1}] + p(t,\xi)H_{kr}(\xi) = \mathrm{Im}\,[b_k \xi_J^{-1} e_J(t) b_r^{-1}].$$

First, by virtue of (5.2), (5.6) and the definition of the Cauchy integral this implies equality (5.8).

A similar equality is also valid for the integral operators with the asterisk in the notation:

$$-[\mathrm{Im}\,(b_k b_r^{-1})]Q_0^*\varphi + P^*_{kr}\varphi = \mathrm{Re}[b_k S(b_r^{-1}\varphi)]. \tag{5.10}$$

On the other hand formulas (3.7) and (5.3), applied to (5.8), produce the relation

$$-[\mathrm{Im}\,(b_k b_r^{-1})]Q_0^*\varphi + (P_{kr}\varphi)^+ = [\mathrm{Re}(b_k b_r^{-1})]\varphi + \mathrm{Re}[b_k S(b_r^{-1}\varphi].$$

Hence jointly with the previous equality, it implies (5.9).

As was mentioned above, the contour Γ being a Lyapunov one, the function $p_0(t, t-t_0)$ and (jointly with it) the matrix–functions $p_0(t, t-t_0)H_{kr}(t-t_0)$ have weak singularity. Therefore the integral operators P^*_{kr} are compact in $L^p(\Gamma)$.

In view of notation (5.5) by equality (5.8) and representations (2.12), (2.13) it directly follows that the following pairs are solutions of the Lamé system and functions conjugating to these solutions:

$$\begin{aligned} u &= P_{11}\varphi, \quad v = -[\mathrm{Im}\,(cb^{-1})]Q_0\varphi + P_{21}\varphi; \\ v &= P_{22}\varphi, \quad u = -[\mathrm{Im}\,(bc^{-1})]Q_0\varphi + P_{12}\varphi. \end{aligned} \tag{5.11}$$

Furthermore four equalities (5.9) can be written in more explicit form:

$$(P_{kk}\varphi)^+ = \varphi + P^*_{kk}\varphi, \quad k = 1,2,$$

$$(P_{12}\varphi)^+ = [\mathrm{Re}(bc^{-1})]\varphi + P^*_{kr}\varphi, \ (P_{21}\varphi)^+ = [\mathrm{Re}(cb^{-1})]\varphi + P^*_{21}\varphi.$$

In accordance with (5.11) it is natural to name the integrals $P_{11}\varphi$ and $P_{22}\varphi$ *generalized potentials of double layer* for solutions of the Lamé system and functions conjugate to them respectively.

Lemma 5.1 can be also extended to the spaces of continuous functions. □

Lemma 5.2. *Let $\Gamma \in C^{1,\nu}$, $0 < \mu < \nu$ and C^* mean any symbol from C, $C^\mu, C^{1,\mu}$. Then the operator $P_{kr} : C^*(\Gamma) \to C^*(\overline{D})$ is bounded and the operator P^*_{kr} is compact in $C^*(\Gamma)$.*

Proof. First let us consider the operators P_{kr}. The boundedness of these operators $C^\mu(\Gamma) \to C^\mu(\overline{D})$ follows by equality (5.8) and the similar properties of the operators Q and I. With respect to the classes $C^{1,\mu}$ the proof is based on the derivation formula of the Cauchy integral $\phi = I\varphi$. Let $\varphi \in C^1(\Gamma)$ and $D\varphi$ mean the derivative of φ with respect to arc length parameter of Γ counted in the positive direction. Then

$$\frac{\partial \phi}{\partial x}(z) = \frac{1}{2\pi i}\int_\Gamma [(t-z)_J^{-2} e_J(t)\varphi(t)|dt| = -\frac{1}{2\pi i}\int_\Gamma [D(t-z)_J^{-1}]\varphi(t)|dt|,$$

whence after integration by parts we come to the differentiation formulas

$$\frac{\partial (I\varphi)}{\partial x} = I(e_J^{-1} D\varphi), \quad \frac{\partial (I\varphi)}{\partial y} = I(Je_J^{-1} D\varphi). \tag{5.12}$$

Taking into account (5.3) we also have the similar formulas for the operator Q_0:

$$\frac{\partial (Q_0\varphi)}{\partial x} = \mathrm{Im}\left[\frac{1}{\pi i}\int_\Gamma \frac{(D\varphi(t)|dt|}{t-z}\right], \quad \frac{\partial (Q_0\varphi)}{\partial y} = \mathrm{Im}\left[\frac{1}{\pi}\int_\Gamma \frac{(D\varphi(t)|dt|}{t-z}\right]. \tag{5.13}$$

Applying these formulas to (5.8) we obtain the boundedness of the operators P_{kr} : $C^{1,\mu}(\Gamma) \to C^{1,\mu}(\overline{D})$.

The proof of the statement of the lemma for the spaces C is based on the estimate

$$\sup_{z\in D}\int_\Gamma |p_0(t,t-z)H_{kr}(t-z)||dt| \le M \sup_{z\in D}\int_\Gamma |p_0(t,t-z)|dt| < \infty, \tag{5.14}$$

where it is taken into account that matrix-functions homogeneous of degree 0 are bounded. Therefore the statement under consideration will be established if we show that for $\varphi \in C(\Gamma)$ the function $(P_{kr}\varphi)(z)$, $z \in D$, has a limit at fixed boundary point $t_0 \in \Gamma$. In accordance with (5.8) the operator P_{kr} transforms constant vector–functions in constants. Therefore without loss of generality we can assume that $\varphi(t_0) = 0$. If $\Gamma_0 \subseteq \Gamma$ is some neighborhood of the point t_0 then, obviously,

$$\int_{\Gamma\setminus\Gamma_0} p_0(t,t-z)H(t-z)\varphi(t)|dt| \to \int_{\Gamma\setminus\Gamma_0} p_0(t,t-t_0)H(t-t_0)\varphi(t)|dt|$$

as $t \to t_0$. On the other hand by virtue of (5.14) under the appropriate choice of Γ_0 the similar integral over Γ_0 can be made sufficiently small uniformly over z.

Turning to the operators P^*_{kr} we firstly note that under assumption $\Gamma \in C^{1,\nu}$ the function $g(t_0, t) = |t - t_0| p_0(t, t_0 - t)$ belongs to the class $C^\nu(\Gamma \times \Gamma)$ and vanishes at $t = t_0$ (we denote this class by C^ν_0). On the other hand $H_{kr}(t - t_0)$ belongs to $C^\nu(\Gamma \times \Gamma)$ as a function of two variables. This fact is valid for any sufficiently smooth even function $H(\xi)$ homogeneous of degree zero. Therefore the functions

$$P_{kr}(t_0, t) = |t - t_0| p_0(t, t_0 - t) H_{kr}(t - t_0)$$

also belong to the class C^ν_0. As was established in [29] in this case the operator

$$(P^*_{kr}\varphi)(t_0) = \frac{1}{\pi}\int_\Gamma P_{kr}(t_0, t)|t - t_0|^{-1}|dt|$$

is bounded from $C(\Gamma)$ to $C^\nu(\Gamma)$ and, in particular, compact in the spaces $C(\Gamma)$ and $C^\mu(\Gamma)$.

As for the last case of the spaces $C^{1,\mu}$ we establish preliminary differentiation formula

$$DP^*_{kr}\varphi = \widetilde{P}^*_{kr} D\varphi, \quad \varphi \in C^{1,\mu}(\Gamma), \tag{5.15}$$

where the operator $\widetilde{P}^*_{kr}$ is obtained by the substitution of $p_0(t_0, t - t_0)$ for $p_0(t, t - t_0)$ in the integrand on the left-hand side of (5.7).

The proof is based on using differentiation formulas (5.12) and (5.13). Let the operator $\widetilde{Q}^*_0$ be obtained from Q^*_0 in a similar way, i.e. by substitution of $q_0(t_0, t - t_0)$ for $q_0(t, t - t_0)$ in the integrand. Besides we set $\widetilde{S} = e_J S e_J^{-1}$ or in the explicit form

$$(\widetilde{S}\varphi)(t_0) = \frac{1}{\pi i}\int_\Gamma e_J(t_0)(t - t_0)_J^{-1}\varphi(t)|dt|, \quad t_0 \in \Gamma.$$

Then as in the proof of Lemma 5.1 we make sure that the equality similar to (5.10) is also valid for the operators under discussion:

$$[\mathrm{Im}\,(b_k b_r^{-1})]\widetilde{Q}^*_0\varphi + \widetilde{P}^*_{kr}\varphi = \mathrm{Re}[b_k \widetilde{S}(b_r^{-1}\varphi)]. \tag{5.16}$$

Let us fix a point $t_0 \in \Gamma$ and substitute partial derivatives (5.12) in the expression

$$e_1(t_0)\frac{\partial(I\varphi)}{\partial x}(z) + e_2(t_0)\frac{\partial(I\varphi)}{\partial y}(z)$$

Then passing to the limit as $z \to t_0$ and taking into account Sokhotskii–Plejmel formulas (3.7) we obtain $2D(I\varphi)^+ = D\varphi + \widetilde{S}D\varphi$. On the other hand differentiation of the Sokhotskii–Plejmel formula implies the similar equality $2D(I\varphi)^+ = D\varphi + DS\varphi$. Comparing it with the previous one we obtain the differentiation formula $DS = \widetilde{S}D$ for the singular operator S. By absolute analogy using (5.13) we obtain the equality $DQ_0 = \widetilde{Q}_0 D$ for the operator Q_0. Acting by the operator D on equality (5.10) and applying these formulas we obtain

$$[\mathrm{Im}\,(b_k b_r^{-1})]\widetilde{Q}_0 D\varphi + DP^*_{kr}\varphi = \mathrm{Re}[b_k \widetilde{S}(b_r^{-1}\varphi)].$$

Hence, jointly with (5.16), it implies (5.15).

It is easy to verify that the function $\widetilde{g}(t_0,t) = p_0(t_0, t-t_0)$ belongs to the class C_0^ν together with $g(t_0,t) = p_0(t, t-t_0)$. Therefore the operator $\widetilde{P}^*_{kr}$ is also compact in the space $C^\mu(\Gamma)$. Hence on the basis of (5.15) follows the compactness of the operator P^*_{kr} in $C^{1,\mu}(\Gamma)$. □

If the conjugate function is represented by either of the formulas in (5.11) then by formulas (2.17) we can identify the elements of the stress tensor σ. Therefore an important role is played by the explicit differentiation formulas for function v in representations (5.11). They are obtained directly from relation (5.8) of Lemma 5.1 and the Cauchy integral differentiation formulas (5.12).

Lemma 5.3. *Let $\varphi \in C^1(\Gamma)$ and $D\varphi$ be the derivative of φ with respect to the tangent direction $e = in$ at Γ. Then*

$$\frac{\partial v}{\partial x}(z) = \frac{1}{\pi}\int_\Gamma \mathrm{Im}\,[c(t-z)_J^{-1}c^{-1}](D\varphi)(t)|dt,$$

$$\frac{\partial v}{\partial y}(z) = \frac{1}{\pi}\int_\Gamma \mathrm{Im}\,[cJ(t-z)_J^{-1}c^{-1}](D\varphi)(t)|dt,$$

if $v = P_{22}\varphi$ and

$$\frac{\partial v}{\partial x}(z) = \frac{1}{\pi}\int_\Gamma \mathrm{Im}\,[c(t-z)_J^{-1}b^{-1}](D\varphi)(t)|dt,$$

$$\frac{\partial v}{\partial y}(z) = \frac{1}{\pi}\int_\Gamma \mathrm{Im}\,[cJ(t-z)_J^{-1}b^{-1}](D\varphi)(t)|dt,$$

if the function v is conjugate to $u = P_{11}\varphi$.

6 Representations by potentials of double layer

Let us consider the problem of representation of solutions of the Lamé system by potentials of double layer $u = P_{11}\varphi$. Preliminarily we describe the kernel $\ker P_{11} = \{\varphi \in L^p(\Gamma)\,|P_{11}\varphi = 0\}$.

Lemma 6.1. *Let a domain D be bounded by a contour $\Gamma \in C^{1,\nu}$ consisting of connected components $\Gamma_1, \dots, \Gamma_m$, and let in the case of a bounded domain the contour Γ_m envelope all other components.*

Then the kernel of the operator P_{11} belongs to the class $Y_1(\Gamma)$ of functions constant at the connected components of Γ and vanishing at Γ_m in the case of a bounded domain.

Proof. Let us write equality (5.8) for the operator under consideration:

$$P_{11}\varphi = 2\,\mathrm{Re}[bI(b^{-1}\varphi)]. \tag{6.1}$$

The relation $Y_1 \subseteq \ker P_{11}$ follows directly by this equality and the Cauchy formula. Conversly, let $P_{11}\varphi = 0$. In the domain D we consider the Cauchy integral $\phi = I(b^{-1}\varphi)$

and the analoguous integral in the complement $\widetilde{D} = \mathbb{C} \setminus \overline{D}$ which we denote by $\psi = \widetilde{I}(b^{-1}\varphi)$. By virtue of (3.7) for boundary values of these functions we can write:

$$\phi^+ - \psi^- = b^{-1}\varphi, \tag{6.2}$$

where it is taken into account that Γ is negatively oriented with respect to $\widetilde{D}$.

By (6.1) the assumption $P_{11}\varphi = 0$ means that $\operatorname{Re} b\phi = 0$. Therefore the function ϕ is constant in the domain D, more precisely, $b\phi = i\xi$, $\xi \in \mathbb{R}^2$. Hence on the basis of (6.2) and by real-valuedness of φ

$$\operatorname{Im} b\psi^- = \xi, \quad \varphi = -\operatorname{Re} b\psi^-. \tag{6.3}$$

Therefore the function $u_0 = (\operatorname{Im} b\psi) - \xi$ is a solution of the homogeneous Dirichlet problem for the Lamé system in connected components of the open set $\widetilde{D}$ and by virtue of Theorem 4.2 it is identically equal to zero. Therefore $\operatorname{Re} b(i\psi + b^{-1}\xi) = 0$ in $\widetilde{D}$, so ψ is constant. If the domain D is bounded that function ψ vanishes at infinity therefore its restriction on infinite connected component of $\widetilde{D}$ is equal to zero. Hence jointly with the second equation in (6.3) it follows that $\varphi \in Y_1$.

Similarly to Theorem 3.1 the following main theorem solves the problem on representations of solutions of the Lamé system by generalized potentials of double layer. In the notation of Theorem 4.1 it is convenient to set $\widetilde{U}_0 = U_0$ if the domain D is bounded and $\widetilde{U}_0 = \{u_0 + \xi,\ u_0 \in U_0, \xi \in \mathbb{R}^2\}$ otherwise. □

Theorem 6.1. *Let a domain D be bounded by a contour $\Gamma \in C^{1,\nu}$ consisting of connected components $\Gamma_1, \dots, \Gamma_m$, and let in the case of a bounded domain the contour Γ_m envelope all other components.*

Then any solution $u \in h^p(D)$, $1 < p < \infty$, of the Lamé system can be represented in the form

$$u = P_{11}\varphi + u_0 \tag{6.4}$$

with some $\varphi \in L^p(\Gamma)$ and $u_0 \in \widetilde{U}_0$, and in this representation $u = 0$ if and only if $u_0 = 0$ and $P_{11}\varphi = 0$.

If this function belongs to the class $C^*(\overline{D})$, where C^* is any of the symbols C, C^μ, $C^{1,\mu}$, $\mu < \nu$, then $\varphi \in C^*(\Gamma)$.

Proof. The image $\operatorname{im} P_{11}$ of the operator P_{11} does not intersect $\widetilde{U}_0$. Indeed, according to (6.1) the function $u = P_{11}\varphi$ can be written in the form $u = \operatorname{Re} b\phi$, where $J-$analytic function $\phi = I(b^{-1}\varphi)$ is single valued in the domain D. Therefore in expansion (4.6) of the function u the summand u_0 is equal to 0. In the case of unbounded domain D it is necessary to take into account that $u(\infty) = 0$.

By virtue of Lemma 5.1 the composition of P_{11} and the operator of the Dirichlet problem (2.1) is a Fredholm operator $1 + P_{11}^*$ with zero index, therefore by virtue of Theorem 4.2 the operator P_{11} belongs to the same type. In particular, the image $\operatorname{im} P_{11}$ is closed and its codimension coincides with the dimension of the kernel $Y_1 = \ker P_{11}$. It is easy to see that the dimensions of the spaces Y_1 and $\widetilde{U}_0$ are the same and equal to $2(m-1)$ if the domain D is bounded and equal to $2m$ otherwise. Therefore $h^p(D) = \widetilde{U}_0 \oplus \operatorname{im} P_{11}$, whence the first part of the theorem follows.

As for the last part its statement concerning $C^* = C^\mu$, $C^{1,\mu}$, by virtue of Theorem 4.2 the operator of the Dirichlet problem is acting from $C^*(D)$ to $C^*(\Gamma)$. In accordance with Lemma 6.1 the kernel $Y_1 \subseteq C^*(\Gamma)$, $\widetilde{U}_0 \subseteq C^*(D)$ and the relations $\ker P_{11} = Y_1$, $\widetilde{U}_0 \cap \operatorname{im} P_{11} = 0$ are also valid for the operator $P_{11} : C^*(\Gamma) \to C^*(D)$. Therefore, taking into account Lemma 5.2 the above is also completely valid for the space $C^*(D)$.

The proof of this statement for $C^* = C$ is based on reducing the Dirichlet problem in the space h^p to the equivalent Fredholm equation in $L^p(\Gamma)$. Let $k = \dim Y_1 = \dim \widetilde{U}_0$ and consider in the spaces $\widetilde{U}_0$ and Y_1 bases $u_1, \dots, u_k$ and $g_1, \dots, g_k$ respectively. Then by the first statement of the theorem which has already been proved the operator

$$\widetilde{P}\varphi = P_{11}\varphi + \sum\nolimits_1^k (\varphi, g_j)u_j, \quad (\varphi, g) = \int_\Gamma \varphi(t)g(t)|dt|,$$

is acting from $l^p(\Gamma)$ to $h^p(D)$ and the Dirichlet problem $u^+ = f$ can be reduced to the equivalent second order Fredholm equation

$$\varphi + P_{11}^*\varphi + \sum\nolimits_1^k (\varphi, g_j)u_j^+ = f. \tag{6.5}$$

If φ is a solution of this equation then the first pair in (5.11) defines a solution u of the Dirichlet problem and the corresponding conjugate function. According to Lemma 5.2 the operator P_{22}^* is also compact in $C(\Gamma)$. It is well known that any solution $\varphi \in L^p(\Gamma)$ of equation of this type with the right-hand side $f \in C(\Gamma)$ belongs to $C(\Gamma)$. But then by Lemma 5.2 function $u = P_{11}\varphi$ also belongs to space $C(\overline{D})$. This fact completes the proof of the theorem. □

As we can see from the proofs of Theorems 3.2 and 4.2, solvability of the Dirichlet problem for the Lamé system is reduced to a singular integral equation on Γ, which does not allow to consider this problem in the class $C(\overline{D})$ within the scope of this approach. Classical potentials of double layer are known to be constructed in terms of the fundamental matrix for the initial elliptic problem [16, 7]. For the Lamé system variants of matrices of this type were suggested in [14], but potentials of double layer constructed by means of them also reduce main boundary problems for the Lamé system to singular integral equations on the boundary. The advantage of generalized potentials of double layer $u = P_{11}\varphi$ is that they give an opportunity to reduce the problem to Fredholm equation (6.5) which is free from the drawback mentioned above. It should be noted that potentials of double layer are connected with the the fundamental matrix of the Lamé system in the form $\operatorname{Re}[(2\pi i)^{-1} b \ln z_J]$.

Let us turn to the operator P_{22} and first of all describe its kernel.

Lemma 6.2. *Under assumptions of Lemma* 6.1 *the kernel* Y_2 *of the operator* P_{22} *is finite-dimensional and satisfies the conditions*

$$Y_1 \subseteq Y_2 \subseteq C^{1,\nu-0}(\Gamma), \quad \dim Y_2 = \begin{cases} 3m-2, & D \text{ is bounded}, \\ 3m & \text{otherwise}. \end{cases} \tag{6.6}$$

Proof. We use the same scheme as in the proof of Lemma 6.1. Let us write equality (5.8) for operator under consideration:

$$P_{22}\varphi = 2\operatorname{Re}[cI(c^{-1}\varphi)] \tag{6.7}$$

and let $P_{22}\varphi = 0$. In the domain D we consider the Cauchy integral $\phi = I(c^{-1}\varphi)$ and the analogous integral in $\widetilde{D}$ which we denote by $\psi = \widetilde{I}(c^{-1}\varphi)$. For the boundary values of these functions we have the relation similar to (6.3)

$$\phi^+ - \psi^- = c^{-1}\varphi. \tag{6.8}$$

Formula (6.7) and the assumption $P_{22}\varphi = 0$ imply the equality $\operatorname{Re} c\phi = 0$, therefore in accordance with Theorem 4.1 the function ϕ is a polynomial $p(z)$ of the form

$$p(z) = \begin{cases} \eta^0 + z_J\eta^1, & D \text{ bounded}, \\ \eta^0 & \text{otherwise}, \end{cases} \tag{6.9}$$

where $\eta^j \in \mathbb{C}^2$ satisfy the conditions $\operatorname{Re} c\eta^0 = 0$, $\operatorname{Re} c\eta^1 = \operatorname{Re} cJ\eta^1 = 0$. Hence on the base of (6.8) and real-valuedness of φ

$$\operatorname{Im} c(\psi - p)^- = 0 \tag{6.10}$$

and

$$\varphi = -\operatorname{Re} c\psi^-. \tag{6.11}$$

We emphasize that in the case of a bounded domain D the function ψ vanishes at infinity. Moreover by Theorem 3.2 this function belongs to the class $C^{1,\nu-0}$ in the closure of each connected component of the open set $\widetilde{D}$. The opposite statement is also true: if for some $J-$analytic function ψ of this type there is such polynomial p of the form (6.9) that boundary condition (6.10) is satisfied, then function (6.11) belongs to Y_2. In particular, on the base of Theorem 3.2 $C^{1,\nu-0}(\overline{D})$ contains the kernel Y_2 of the operator P_{22}. We carry out describing its dimension for two types of the domain D separately.

1) Let domain D be bounded and $\widetilde{D}_j$ be a connected component of $\widetilde{D}$ having the contour Γ_j as its boundary. Then all domains $\widetilde{D}_j$ are bounded and by virtue of Theorem 4.1 the conjugate function $v = \operatorname{Im} c(\psi - p)$ is identically equal to zero in each of them. Hence the restriction of $i(\psi - p)$ to the domain $\widetilde{D}_j$ is a polynomial $p_j = \eta_j^0 + z_J\eta_j^1$ satisfying conditions (6.9), i.e. $\operatorname{Re} c\eta_j^0 =$ and $\operatorname{Re} c\eta_j^1 = \operatorname{Re} cJ\eta_j^1 = 0$.

Thus $\varphi \in Y_2$ if and only if $\varphi|_{\Gamma_j} = \operatorname{Im} cp_j, \quad 1 \le j \le m$. Since the functions $\operatorname{Im} cp_j$ form the space of dimension 3 for each j, the dimension of the space G_2 is equal to $3m$.

2) Let the domain D be unbounded, then the first $m-1$ domains $\widetilde{D}_j$ are bounded and the domain $\widetilde{D}_m$ is unbounded. In accordance with (6.9) boundary condition (6.10) in the last domain can be written in the form

$$\operatorname{Im} c(\psi - \eta^0)^-|_{\Gamma_m} = \operatorname{Im}(cz_J\eta^1)|_{\Gamma_m}. \tag{6.12}$$

Since the space $\{\eta^1,\ \operatorname{Re} c\eta^1 = \operatorname{Re} cJ\eta^1 = 0\}$ is one-dimensional, by Theorem 4.2 there exists a unique conjugate function v which belongs to class $C^{1,\nu-0}$ in the closure of domain $\widetilde{D}_m$ and which boundary value coincides with the right-hand side of (6.12). There exists a unique vector $\eta^0 \in \mathbb{C}^2$ for which $\operatorname{Re} c\eta^0 = 0$, $\operatorname{Im} c\eta^0 = -v(\infty)$. Hence the function v can be represented in the form $v = \operatorname{Im} c(\psi - \eta^0)$, where $J-$analytic function $\psi \in C^{1,\nu-0}$ vanishes at infinity. Therefore the space Y_2 consists of functions φ for which

$$\varphi|_{\Gamma_j} = \operatorname{Im} cp_j, \quad 1 \le j \le m-1, \quad \varphi|_{\Gamma_m} = \operatorname{Re} c\psi^-,$$

so its dimension is equal to $3(m-1)+1$.

By (6.6) Y_2 contains the class Y_1, defined in Lemma 6.1. Thus relations (6.6) are completely established. □

Let us turn to the problem of representation of conjugate functions by potentials $P_{22}\varphi$. Let for short $C^{1,\nu-0}$ mean the class of functions belonging to $C^{1,\mu}$ for all $\mu<\nu$. As in Theorem 6.1 we set $\widetilde{V}_0=V_0$, if the domain D is bounded and $\widetilde{V}_0=\{v_0+\xi,\ v_0\in V_0,\xi\in\mathbb{R}^2\}$ otherwise.

Theorem 6.2. *Under the assumptions of Theorem 6.1 there exists a finite-dimensional space $V\subseteq C^{1,\nu-0}(\overline{D})$ of the dimension $\dim Y_2-1$, containing the class $\widetilde{V}_0$, such that any function $v\in h^p(D)$, conjugate to some (generally speaking multi-valued) solution of the Lamé system, can be represented in the form*

$$v=P_{22}\varphi+v_0 \tag{6.13}$$

with some $\varphi\in L^p(\Gamma)$ and $v_0\in V$, and $v=0$ in this representation if and only if $v_0=0$ and $P_{22}\varphi=0$.

If this function belongs to $C^*(\overline{D})$, where C^* is any symbol of C, C^μ, $C^{1,\mu}$, $\mu<\nu$, then $\varphi\in C^*(\Gamma)$.

Proof. is absolutely similar to the proof of Theorem 6.1. Since the composition of P_{22} and the operator of Dirichlet problem (4.10) is a Fredholm operator $1+P^*_{22}$ of zero index, taking into account Theorem 4.3, we conclude that the operator P_{22} is a Fredholm one and its index is equal to 1, if the domain D is bounded, and its index is equal to zero otherwise. Therefore its image $\operatorname{im}P_{22}$ is closed and has the codimension equal to $\dim Y_2-1$. As in the proof of Theorem 6.1 $\widetilde{V}_0\cap\operatorname{im}P_{22}=0$. Thus in $h^p(D)$ there exists a subspace $V\supseteq\widetilde{V}_0$ of dimension $\dim G_2-1$, for which decomposition (6.13) is valid.

As in the case of Theorem 6.1 analogous argument can be also made for the space $C^*(D)$, where $C^*=C^\mu$, $C^{1,\mu}$, It just required to prove that the subspace V can be chosen independently of $X(D)$ in $C^{,\nu-0}(\overline{D})$. Previous arguments demonstrate that the codimension $s=\dim Y_2-1$ of the subspace $P_{22}[C^{1,\mu}(\Gamma)]\subseteq C^{1,\mu}(\overline{D})$ does not depend on the choice of $\mu<\nu$. Obviously, functions $v_1,\dots,v_l\in C^{1,\nu-0}(\overline{D})$ are linearly dependent modulo $P_{22}[C^{1,\nu-0}(\Gamma)]$ if and only if they linearly dependent modulo $C^{1,\mu}(\overline{D})$ for some $\mu<\nu$. Therefore the codimension of the subspace $P_{22}[C^{1,\nu-0}(\Gamma)]\subseteq C^{1,\nu-0}(\overline{D})$ is also equal to s and of the desired V is obvious.

As for the case $X=C$, analogously to the proof of Theorem 6.1 we set $k=\dim G_2$ and consider bases $g_1,\dots,g_k$ and $v_1,\dots,v_{k-1}$ in the spaces Y_2 и V respectively. Then on the base of Lemma 6.2 and Theorem 4.3 any element $f\in L^p(\Gamma)$ can be represented in the form

$$f=(P_{22}\varphi)^++\sum\nolimits_1^{k-1}\lambda_jv_j^++\lambda_kn,$$

with some $\varphi\in L^p(\Gamma)$ and $\lambda_j\in\mathbb{R}$, where we would remind that the function $n=(n_1,n_2)$ is the unit exterior normal. Herewith the equality $f=0$ involves $\varphi\in Y_2$ and $\lambda_1=\ldots=\lambda_k=0$. Therefore, the operator

$$N\varphi=\varphi+P^*_{22}\varphi+\sum\nolimits_1^{k-1}(g_j,\varphi)v_j^++(g_k,\varphi) \tag{6.14}$$

is invertible in space $L^p(\Gamma)$. Obviously, then it is also invertible in the space $C(\Gamma)$.

Let now $v \in C(\overline{D})$, $f = v^+$ and $\varphi = N^{-1}f \in C(\Gamma)$. Then by Lemma 5.2 the conjugate function

$$v_0 = P_{22}\varphi + \sum_1^{k-1} (g_j, \varphi)v_j \tag{6.15}$$

belongs to $C(\overline{D})$ and, obviously, $v^+ = v_0^+ + (g_k, \varphi)n$. By Theorem 4.3 both functions v^+ and v_0^+ are orthogonal to n, so indeed $(g_k, \varphi) = 0$, $v = v_0$ and decomposition (6.15) transforms into (6.6). Thereby Theorem 6.2 is established completely. □

It follows by Theorem 6.2 that the statement of Theorem 4.2 for the Neumann problem is also valid for the space $C(\overline{D})$. To reduce this problem to an integral equation on Γ it is convenient to choose the space V in Theorem 6.2 in another way. We denote by V_1 the class of conjugate functions v, whose boundary value χ is constant on the connected components Γ_j, $j = 1, ..., m-1$, of the contour Γ and vanishes on Γ_m. Since the function χ is orthogonal to n, by Theorem 4.3 the problem $v^+ = \chi$ is solvable and its solution v belongs to the class $C^{1,\nu-0}(\overline{D})$. It is clear that the solution u of the Lamé system having v as its conjugate function, is multi-valued, and it can be single-valued only if $v = 0$. Therefore in the second statement of Theorem 4.1 the space V_0 can be replaced by V_1. Defining $\widetilde{V}_1$ by V_1 similarly to the previous arguments, the choice of the space V in Theorem 6.2 can be subjected to the condition $V \supseteq \widetilde{V}_1$. Let $v_1, \ldots, v_{k-1}$ be a basis of the space V defined by this way, and the first $2(m-1)$ these elements form a basis in V_1. We denote solutions of the Lamê system corresponding to v_j, by u_j, so there are only first $2(m-1)$ of these functions, which are multi-valued. As above the operator N, defined by formula (6.14), is invertible. If for a given function f, which is orthogonal to n, we have $\varphi = N^{-1}\varphi$, then as above we make sure that conjugate function (6.15) solves the Dirichlet problem $v^+ = f$. Setting

$$\chi = \sum_{j\le 2(m-1)} (g_j, \varphi)v_j^+, \quad \widetilde{v} = \sum_{j>2(m-1)} (g_j, \varphi)v_j,$$

we obtain the equality $(P_{22}\varphi)^+ + \chi + \widetilde{v}^+ = f$. Hence in accordance with expression (5.11) we conclude for the second pair that the function

$$u = [\mathrm{Re}(bc^{-1})]Q\varphi + P_{12}\varphi + \sum_{j>2(m-1)} (g_j, \varphi)u_j$$

solves the Neumann problem $v^+ + \chi = f$.

7 The structure of matrix kernels of potentials

It is convenient to slightly modify the integrand in formula (5.6) for potentials. For this purpose let us introduce the quadratic form

$$\omega(\xi) = (\xi_1 + \nu_1\xi_2)(\xi_1 + \nu_2\xi_2), \tag{7.1}$$

which will be used in both cases (i) and (ii), in the last case it turns into $\omega(\xi) = (\xi_1 + \nu\xi_2)^2$. Setting

$$p(t, \xi) = \frac{\mathrm{Re}[n(t)\bar{\xi}]}{|\omega(\xi)|^2}, \quad G_{kr}(\xi) = \frac{|\omega(\xi)|^2}{|\xi|^2} H_{kr}(\xi), \tag{7.2}$$

the integrand $p_0 H$ in (5.6) can be written in the form pG. Obviously, the matrix–function $G(\xi)$ is homogeneous of degree 2 and it will be under investigation in the sequel.

In our notation the first pair of relations in (5.11) for the potential u of the Lamé system and for the function v conjugate to it has the form

$$u(z) = \frac{1}{\pi}\int_\Gamma p(t, t-z)G_{11}(t-z)\varphi(t)|dt|, \qquad (7.3_1)$$

$$v(z) = -\frac{1}{\pi}\int_\Gamma q_0(t, t-z)\mathrm{Im}\,(cb^{-1})\varphi(t)|dt| + \frac{1}{\pi}\int_\Gamma p(t, t-z)G_{21}(t-z)\varphi(t)|dt|,$$

and corresponding relation for the second pair can be written similarly:

$$v(z) = \frac{1}{\pi}\int_\Gamma p(t, t-z)G_{22}(t-z)\varphi(t)|dt|, \qquad (7.3_2)$$

$$u(z) = -\frac{1}{\pi}\int_\Gamma q_0(t, t-z)\mathrm{Im}\,(bc^{-1})\varphi(t)|dt| + \frac{1}{\pi}\int_\Gamma p(t, t-z)G_{12}(t-z)\varphi(t)|dt|.$$

First let us find the explicit expressions for the matrices which are under the sign of real part. Though the definition of the matrices b and c in Theorem 1.1 depends on cases (i) and (ii) of the characteristic equation in the upper half-plane, the elements of the matrices cb^{-1} and bc^{-1} can be expressed in terms of their symmetric combinations

$$(i)\ s = \nu_1 + \nu_2,\ t = \nu_1\nu_2; \quad (ii)\ s = 2\nu,\ t = \nu^2, \qquad (7.4)$$

and in this sense they do not depend on the mentioned cases. More precisely, these matrices elements are rational functions of the variables s, t and the elastic modules α_i и β_i entering (1.2).

Lemma 7.1. *Apart from the special case the numbers*

$$\begin{aligned} e_{(1)} &= \alpha_3(\alpha_3+\alpha_4) - 2\alpha_5\alpha_6 - (\alpha_2\alpha_6 - \alpha_3\alpha_5)s + [2\alpha_5^2 - \alpha_2(\alpha_3+\alpha_4)]t, \\ e_{(2)} &= \beta_4^2 - \beta_4\beta_6 s + (\beta_6^2 - 2\beta_1\beta_4)t + \beta_1\beta_4 s^2 - \beta_1\beta_6 st + \beta_1^2 t^2, \end{aligned} \qquad (7.5)$$

are not equal to zero and

$$cb^{-1} = \frac{1}{e_{(1)}}\begin{pmatrix} -e_{22} & -e_{12} \\ e_{21} & e_{11} \end{pmatrix}, \quad bc^{-1} = \frac{1}{e_{(2)}}\begin{pmatrix} -e_{11} & -e_{12} \\ e_{21} & e_{22} \end{pmatrix}, \qquad (7.6)$$

where

$$\begin{aligned} e_{11} &= (2\alpha_5\beta_4 + \alpha_3\beta_6) + (\alpha_2\beta_4 - \alpha_3\beta_1)s - (\alpha_2\beta_6 + 2\alpha_5\beta_1)t, \\ e_{12} &= -\alpha_3\beta_4 + \alpha_3\beta_6 s - \alpha_3\beta_1(s^2 - t) + (\alpha_2\beta_4 + 2\alpha_5\beta_6)t - 2\alpha_5\beta_1 st - \alpha_2\beta_1 t^2, \\ e_{21} &= [(\alpha_3+\alpha_4)\beta_4 + \alpha_6\beta_6] + (\alpha_5\beta_4 - \alpha_6\beta_1)s - [\alpha_5\beta_6 + (\alpha_3+\alpha_4)\beta_1]t, \\ e_{22} &= -\alpha_6\beta_4 + \alpha_6\beta_6 s - \alpha_6\beta_1(s^2 - t) + [\alpha_5\beta_4 + (\alpha_3+\alpha_4)\beta_6]t - (\alpha_3+\alpha_4)\beta_1 st \\ &\quad -\alpha_5\beta_1 t^2. \end{aligned}$$

In the special case

$$cb^{-1} = \frac{\delta_2}{\alpha_2^2} \begin{pmatrix} -i(\delta_1\delta_2 + \alpha_5^2) & -\alpha_2(\delta_2 + i\alpha_5) \\ \alpha_2(\delta_2 - i\alpha_5) & -i\alpha_2^2 \end{pmatrix},$$

$$bc^{-1} = \frac{1}{\delta_2^2(\delta_1 - \delta_2)} \begin{pmatrix} i\alpha_2^2 & -\alpha_2(\delta_2 + i\alpha_5) \\ \alpha_2(\delta_2 - i\alpha_5) & i(\delta_1\delta_2 + \alpha_5^2) \end{pmatrix}, \tag{7.7}$$

where δ_j are the numbers entering formula (1.6).

Let us mention that according to (1.5) in the special case all coefficients of the polynomial $e_{(1)}(s,t)$ vanish.

Proof. of the equalities (7.7) is carried out by direct verification in terms of formulas (1.13) of Theorem 1.1, therefore below we exclude the special case. Simultaneously in both cases (i) and (ii) we introduce the skew-symmetric bilinear form

$$(i)\ [g_1, g_2] = (g_1(\nu_1)g_2(\nu_2) - g_1(\nu_2)g_2(\nu_1))/(\nu_1 - \nu_2),$$

$$(ii)\ [g_1, g_2] = g_1'(\nu)g_2(\nu) - g_1(\nu)g_2'(\nu), \tag{7.8}$$

Obviously, by definition case (i) reduces to (ii) as $\nu_1 \to \nu,\ \nu_2 \to \nu$. As a numerical function of the roots ν this form is a polynomial in two variables (7.4). Let us introduce the matrix $W(g_1, g_2)$ for the polynomial pair $g_j(z),\ j = 1, 2$, by the rule

$$(i)\ W(g_1, g_2) = \begin{pmatrix} g_1(\nu_1) & g_1(\nu_2) \\ g_2(\nu_1) & g_2(\nu_2) \end{pmatrix}, \quad (ii)\ W(g_1, g_2) = \begin{pmatrix} g_1(\nu) & g_1'(\nu) \\ g_2(\nu) & g_2'(\nu) \end{pmatrix}. \tag{7.9}$$

It is easy to see that we have the following expression for its determinant

$$(i)\ \det W(g_1, g_2) = (\nu_1 - \nu_2)[g_1, g_2], \quad (ii)\ \det W(g_1, g_2) = -[g_1, g_2],$$

So this matrix invertibility is provided by the condition $[g_1, g_2] \neq 0$. If this condition is satisfied then the direct verification demonstrates that in both cases (i) and (ii) the following equality holds

$$W(f_1, f_2)[W(g_1, g_2)]^{-1} = \frac{1}{[g_1, g_2]} \begin{pmatrix} [f_1, g_2] & -[f_1, g_1] \\ [f_2, g_2] & -[f_2, g_1] \end{pmatrix}.$$

In the sequel the role of g_k is played by polynomials (1.4).

In notation (7.8) expressions (1.12) for the matrices b and c of Theorem 1.1 can be represented in the following form:

$$b = W(p_2, -p_3), \quad c = W(-q_3, q_2). \tag{7.10}$$

Hence, since by Theorem 1.1 these matrices are invertible it follows that the numbers $[p_2, p_3]$ and $[q_3, q_2]$ are not equal to zero and

$$cb^{-1} = \frac{1}{[p_3, p_2]} \begin{pmatrix} -[p_3, q_3] & -[p_2, q_3] \\ [p_3, q_2] & [p_2, q_2] \end{pmatrix},$$

$$bc^{-1} = \frac{1}{[q_3, q_2]} \begin{pmatrix} -[p_2, q_2] & -[p_2, q_3] \\ [p_3, q_2] & [p_3, q_3] \end{pmatrix}.$$

The values of the bilinear form figured here can be written in the explicit form. For this purpose it should be noted that by virtue of skew-symmetry of the form we can write

$$\Big[\sum_0^3 a_i z^i, \sum_0^3 b_j z^j\Big] = \sum_{i>j} (a_i b_j - a_j b_i)[z^i, z^j].$$

At the basic elements $g_1(z) = z^i,\ g_2(z) = z^j$ form (7.8) takes the values

$$[z^i, 1] = \begin{cases} 1, & i = 1, \\ s, & i = 2, \\ s^2 - t, & i = 3, \end{cases} \qquad [z^i, z^j] = \begin{cases} t, & i = 2, j = 1, \\ st, & i = 3, j = 1, \\ t^2, & i = 3, j = 2. \end{cases}$$

Thus

$$\Big[\sum_0^3 a_i z^i, \sum_0^3 b_j z^j\Big] = (a_1 b_0 - a_0 b_1) + (a_2 b_0 - a_0 b_2)s + (a_3 b_0 - a_0 b_3)(s^2 - t) +$$

$$+(a_2 b_1 - a_1 b_2)t + (a_3 b_1 - a_1 b_3)st + (a_3 b_2 - a_2 b_3)t^2.$$

Substituting here the coefficients of polynomials (1.4), in the notation (7.5), (7.6) we obtain: $[p_3, p_2] = e_{(1)}$, $[q_3, q_2] = e_{(2)}$ и $[p_2, q_2] = e_{11}$, $[p_2, q_3] = e_{12}$, $[p_3, q_2] = e_{21}$, $\{p_3, q_3\} = e_{22}$, that completes the proof of the lemma. □

Let us begin describing matrices G_{kr} in (7.2) starting with G_{11}. In addition to (7.1) we introduce the quadratic form

$$2\omega_1(\xi) = \nu_1 |\xi_1 + \nu_2 \xi_2|^2 + \nu_2 |\xi_1 + \nu_1 \xi_2|^2. \tag{7.11}$$

The coefficients of these forms are also expressed via the variables s, t:

$$\omega(\xi) = \xi_1^2 + s\xi_1\xi_2 + t\xi_2^2, \quad 2\omega_1(\xi) = s\xi_1^2 + 2t\xi_1\xi_2 + \bar{s}t\xi_2^2. \tag{7.12}$$

Theorem 7.1. *Apart from the special case the matrix G_{11} is given by the equality*

$$G_{11}(\xi) = \mathrm{Im}\left[\frac{1}{e_{(1)}} \begin{pmatrix} e_{(1)}\omega_1(\xi) - e\,\overline{\omega(\xi)} & -e_1\,\overline{\omega(\xi)} \\ e_2\,\overline{\omega(\xi)} & e_{(1)}\omega_1(\xi) + e\,\overline{\omega(\xi)} \end{pmatrix}\right], \tag{7.13}$$

where

$$\begin{aligned} 2e &= 2\alpha_3\alpha_6 + 4(\alpha_3 + \alpha_4)\alpha_5 t + 2\alpha_2\alpha_5 t^2 + [2\alpha_5\alpha_6 + \alpha_3(\alpha_3 + \alpha_4)]s \\ &+(\alpha_2\alpha_6 + \alpha_3\alpha_5)(s^2 - 2t) + [\alpha_2(\alpha_3 + \alpha_4) + 2\alpha_5^2]st, \\ e_1 &= \alpha_3^2 + 4\alpha_5^2 t + \alpha_2^2 t^2 + 2\alpha_3\alpha_5 s + \alpha_2\alpha_3(s^2 - 2t) + 2\alpha_2\alpha_5 st, \\ e_2 &= \alpha_6^2 + (\alpha_3 + \alpha_4)^2 t + \alpha_5^2 t^2 + \alpha_5\alpha_6(s^2 - 2t) + (\alpha_3 + \alpha_4)(\alpha_5 st + \alpha_6 s). \end{aligned}$$

In the special case

$$G_{11}(\xi) = \frac{1}{\alpha_2^3} \begin{pmatrix} \delta_1(\xi_{(1)}^2 + \delta_2^2\xi_2^2) & 0 \\ \delta_0(\xi_{(1)}^2 - \delta_1\delta_2\xi_2^2) & \delta_2(\xi_{(1)}^2 + \delta_1^2\xi_2^2) \end{pmatrix}, \tag{7.14}$$

where we use the notation $\xi_{(1)} = \alpha_2\xi_1 - \alpha_5\xi_2$, $\delta_0 = \alpha_5(\delta_2 - \delta_1)/\alpha_2$.

Proof. Simultaneously to both cases (i) and (ii) we set

$$(i)\ \Delta = \frac{1}{2}\begin{pmatrix} \nu_1 - \nu_2 & 0 \\ 0 & \nu_2 - \nu_1 \end{pmatrix}, \quad (ii)\ \Delta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and show that in this notation

$$G_{11}(\xi) = \mathrm{Im}\,[(\omega_1(\xi) + \overline{\omega(\xi)}(b\Delta b^{-1})]. \tag{7.15}$$

(i) Let

$$h(\xi, \nu) = \frac{-\xi_2 + \nu\xi_1}{\xi_1 + \nu\xi_2},$$

then

$$(-\xi_2 + \xi_1 J)(\xi_1 + \xi_2 J)^{-1} = \mathrm{diag}[h(\xi, \nu_1), h(\xi, \nu_2)]. \tag{7.16}$$

Hence by the definition of the matrix Δ

$$(-\xi_2 + \xi_1 J)(\xi_1 + \xi_2 J)^{-1} = \frac{h(\xi, \nu_1) + h(\xi, \nu_2)}{2} + \frac{h(\xi, \nu_1) - h(\xi, \nu_2)}{\nu_1 - \nu_2}\Delta.$$

It is easy to see that

$$\frac{h(\xi, \nu_1) + h(\xi, \nu_2)}{2} = \frac{\omega_0(\xi)}{\omega(\xi)}, \quad \frac{h(\xi, \nu_1) - h(\xi, \nu_2)}{\nu_1 - \nu_2} = \frac{|\xi|^2}{\omega(\xi)}, \tag{7.17}$$

where $2\omega_0(\xi) = (-\xi_2 + \nu_1\xi_1)(\xi_1 + \nu_2\xi_2) + (-\xi_2 + \nu_2\xi_1)(\xi_1 + \nu_1\xi_2)$.

Therefore,

$$|\xi|^{-2}|\omega(\xi)|^2 b[(-\xi_2 + \xi_1 J)(\xi_1 + \xi_2 J)^{-1}]b^{-1} = |\xi|^{-2}\omega_0(\xi)\overline{\omega(\xi)} + \overline{\omega(\xi)}(b\Delta b^{-1}). \tag{7.18}$$

It can be easily verified that

$$\mathrm{Im}\, h(\xi, \nu) = \frac{|\xi|^2 \mathrm{Im}\, \nu}{|\xi_1 + \nu\xi_2|^2}, \tag{7.19}$$

so by definition (7.11)

$$\mathrm{Im}\left[\frac{h(\xi, \nu_1) + h(\xi, \nu_2)}{2}\right] = \frac{|\xi|^2\omega_1(\xi)}{|\omega(\xi)|^2}.$$

Hence jointly with equality (7.17) we come to the relation

$$\mathrm{Im}\,[(\omega_0(\xi)\overline{\omega(\xi)}] = |\xi|^2 \mathrm{Im}\,[\omega_1(\xi)]$$

for the quadratic forms. Therefore in accordance with definitions (5.5), (7.2) the imaginary part of equality (7.18) coincides with (7.15).

(ii) By the definition of the matrix Δ in this case

$$(-\xi_2 + \xi_1 J)(\xi_1 + \xi_2 J)^{-1} = h(\xi, \nu)\left(1 + \frac{\xi_1}{-\xi_2 + \nu\xi_1}\Delta\right)\left(1 + \frac{\xi_2}{\xi_1 + \nu\xi_2}\Delta\right)^{-1},$$

so

$$(-\xi_2 + \xi_1 J)(\xi_1 + \xi_2 J)^{-1} = h(\xi, \nu) + \frac{|\xi|^2}{\omega(\xi)^2}\Delta.$$

Setting $\nu_k = \nu$ in the first relation (7.17), we obtain the equality

$$h(\xi, \nu) = \frac{\omega_1(\xi)}{\omega(\xi)},$$

which similarly to the previous case (i) leads to (7.15).

To describe the matrix $b\Delta b^{-1}$ entering (7.15), let us introduce the symmetric bilinear form similarly to (7.8)

$$(i)\ \{g_1, g_2\} = [g_1(\nu_1)g_2(\nu_2) + g_1(\nu_2)g_2(\nu_1)]/2, \quad (ii)\ \{g_1, g_2\} = g_1(\nu)g_2(\nu).$$

It is obvious that case (i) turns into (ii) as $\nu_1 \to \nu,\ \nu_2 \to \nu$. As a numeric function of the roots ν this form is a polynomials in two variables (7.4). The direct verification demonstrates that in notation of (7.9) for both cases (i) and (ii) the following equality holds

$$W(f_1, f_2)\Delta[W(g_1, g_2)]^{-1} = \frac{1}{[g_1, g_2]}\begin{pmatrix} \{f_1, g_2\} & -\{f_1, g_1\} \\ \{f_2, g_2\} & -\{f_2, g_1\} \end{pmatrix}.$$

Hence taking into account (7.10)

$$b\Delta b^{-1} = \frac{1}{[p_3, p_2]}\begin{pmatrix} -\{p_2, p_3\} & -\{p_2, p_2\} \\ \{p_3, p_3\} & \{p_2, p_3\} \end{pmatrix}. \tag{7.20}$$

The matrix elements can be calculated similarly to Lemma 7.1. By virtue of the form symmetry we have:

$$\left\{\sum_0^2 a_i z^i, \sum_0^2 b_j z^j\right\} = \sum_0^2 a_i b_i t^i + \sum_{i>j}(a_i b_j + a_j b_i)\{z^i, z^j\}.$$

Since

$$2\{z, 1\} = s, \quad 2\{z^2, 1\} = s^2 - 2t, \quad 2\{z^2, z\} = st,$$

and $\{z^i, z^i\} = t^i$, $1 \le i \le 2$, the previous equality can be written in the form

$$2\left\{\sum_0^2 a_i z^i, \sum_0^2 b_j z^j\right\} = 2\sum_0^2 a_i b_i t^i + (a_1 b_0 + a_0 b_1)s + (a_2 b_0 + a_0 b_2)(s^2 - 2t).$$

Substituting here the coefficients of the polynomials p from (1.4), in notation of (7.13) we obtain $e = \{p_2, p_3\},\ e_1 = \{p_2, p_2\},\ e_2 = \{p_3, p_3\}$. Hence jointly with (7.15), (7.20) and Lemma 7.1 the first part of the theorem follows.

Relations (7.14) are obtained by (7.15) by direct calculation of the matrix $b\Delta b^{-1}$ on base of equalities (1.6), (1.13) of Theorem 1.1. Herewith to simplify calculations it should be taken into account that $\delta_2(\xi_1 + \nu_k\xi_2) = \xi_{(1)} + i\delta_k\xi_2$, whence

$$\alpha_2^2\omega(\xi) = (\xi_{(1)} + i\delta_1\xi_2)(\xi_{(1)} + i\delta_2\xi_2),$$

$$2\alpha_2^3 \mathrm{Im}\,\omega_1(\xi) = \delta_1(\xi_{(1)}^2 + \delta_1^2\xi_2^2) + \delta_2(\xi_{(1)}^2 + \delta_1^2\xi_2^2).$$

□

Theorem 7.2. *The following equalities hold*

$$G_{22}(\xi) = \mathrm{Im}\begin{pmatrix} s\xi_1^2 + t\xi_1\xi_2 & t\xi_1^2 + \bar{s}t\xi_1\xi_2 \\ s\xi_1\xi_2 + t\xi_2^2 & t\xi_1\xi_2 + \bar{s}t\xi_2^2 \end{pmatrix}, \tag{7.21}$$

$$\begin{aligned} G_{21} &= \mathrm{Re}(\overline{\omega}\Omega)\mathrm{Im}\,(cb^{-1}) + G_{22}\,\mathrm{Re}(cb^{-1}), \\ G_{12} &= \mathrm{Im}\,(bc^{-1})\,\mathrm{Re}(\overline{\omega}\Omega) + \mathrm{Re}(bc^{-1})G_{22}, \end{aligned} \tag{7.22}$$

where

$$\Omega(\xi) = \begin{pmatrix} |\xi|^{-2}[(t-1)\xi_1\xi_2 + s\xi_1^2] & t^2 \\ -1 & |\xi|^{-2}[(t-1)\xi_1\xi_2 - s\xi_2^2] \end{pmatrix}.$$

Proof. Equality (7.18) is also true for the matrix c. Since the matrix d entering (1.14), commutes with Δ, the matrix $c\Delta c^{-1}$ in the right-hand side of this equality can be replaced by the matrix c_0 entering (1.14). Therefore,

$$|\xi|^{-2}|\omega(\xi)|^2 c[(-\xi_2 + \xi_1 J)(\xi_1 + \xi_2 J)^{-1}]c^{-1} = |\xi|^{-2}\omega_0(\xi)\overline{\omega(\xi)} + \overline{\omega(\xi)}(c_0\Delta c_0^{-1}). \tag{7.23}$$

Direct verification demonstrates that in both cases (i) and (ii) the product

$$c_0\Delta c_0^{-1} = \frac{1}{2}\begin{pmatrix} s & 2t \\ -2 & -s \end{pmatrix}.$$

The definition of the quadratic form ω_0 in (7.17) implies that similarly to (7.12) it can be written in the form $2\omega_0(\xi) = 2(t-1)\xi_1\xi_2 + s(\xi_1^2 - \xi_2^2)$. Hence

$$|\xi|^{-2}[\omega_0(\xi) + |\xi|^2 c_0\Delta c_0^{-1}] = \Omega(\xi). \tag{7.24}$$

Substituting this expression in (7.23) and taking the imaginary part of this equality, we obtain $G_{22} = \mathrm{Im}\,(\overline{\omega}\Omega)$ and this equality coincides with (7.21).

Similarly by definition (5.5), (7.2) we can be written as

$$G_{21}(\xi) = |\xi|^{-2}|\omega(\xi)|^2\mathrm{Im}\,[c(-\xi_2 + \xi_1 J)(\xi_1 + \xi_2 J)^{-1}c^{-1}(cb^{-1})].$$

Hence by (7.23), (7.24)

$$G_{21}(\xi) = \mathrm{Im}\,[\overline{\omega(\xi)}\Omega(\xi)(cb^{-1})].$$

Therefore follows the first formula in (7.22). The second formula is proved similarly. □

Let us note that equalities (7.22) of Theorem 7.2 should be used in combination with Lemma 7.1, herewith in the special case according to (1.6) variables (7.4) are defined by the equalities

$$s = \frac{-2\alpha_5 + i\delta_1\delta_2}{\alpha_2}, \quad t = \frac{\alpha_5^2 - \delta_1\delta_2 - i\alpha_5(\delta_1 + \delta_2)}{\alpha_2^2}. \tag{7.25}$$

Similarly to Theorem 7.2 it is also easy to obtain explicit expressions for the kernels entering Lemma 5.3. Let

$$\Omega_1(\xi) = \frac{1}{\omega(\xi)}\begin{pmatrix} \xi_1 & -t\xi_2 \\ \xi_2 & \xi_1 + s\xi_2 \end{pmatrix}, \quad \Omega_2(\xi) = \frac{1}{\omega(\xi)}\begin{pmatrix} s\xi_1 + t\xi_2 & t\xi_1 \\ -\xi_1 & t\xi_2 \end{pmatrix}. \tag{7.26}$$

and for brevity $x_1 = x,\ x_2 = y$.

Lemma 7.2. *Under assumptions of Lemma* 5.3 *the following equalities hold*

$$\frac{\partial v}{\partial x_j}(z) = \frac{1}{\pi}\int_\Gamma \operatorname{Im}\,[\Omega_j(t-z)](D\varphi)(t)|dt|, \quad j=1,2,$$

if $v = P_{22}\varphi$, *and*

$$\frac{\partial v}{\partial x_j}(z) = \frac{1}{\pi}\int_\Gamma \operatorname{Im}\,[\Omega_j(t-z)(cb^{-1})](D\varphi)(t)|dt|, \quad j=1,2,$$

if the function v *is conjugate to* $u = P_{11}\varphi$.

Proof. According to Lemma 5.2 it suffices to verify that

$$c\xi_J^{-1}c^{-1} = \Omega_1(\xi), \quad cJ\xi_J^{-1}c^{-1} = \Omega_2(\xi).$$

As in the proof of Theorem 7.2 here the matrix c can be replaced by the matrix c_0 entering Lemma 1.2(a) and these relations can be established by direct verification. □

8 Potentials of double layer in orthotropic medium

Let conditions (1.16) characterizing the orthotropic medium be satisfied. In this case according to (1.17) the characteristic equation $p_1p_2 - p_3^2 = 0$ is biquadratic that allows us to represent its roots ν in the upper half-plane explicitly [1] via the modules of elasticity. For this purpose let us introduce positive ρ and ρ_0 by the formulas

$$\rho^2 = \sqrt{\frac{\alpha_1}{\alpha_2}}, \quad \rho_0^2 = \frac{\alpha_1\alpha_2 - \alpha_4^2 + 2\alpha_3(\sqrt{\alpha_1\alpha_2} - \alpha_4)}{\alpha_2\alpha_3}. \tag{8.1}$$

Positiveness of expression in the right-hand side of the second equality follows by the condition $\alpha_4^2 < \alpha_1\alpha_2$. For the same reasons the number

$$\rho_0^2 - 4\rho^2 = \frac{(\sqrt{\alpha_1\alpha_2} - \alpha_4)(\sqrt{\alpha_1\alpha_2} - \alpha_4 - 2\alpha_3)}{\alpha_2\alpha_3} \tag{8.2}$$

has the same sign as $\sqrt{\alpha_1\alpha_2} - \alpha_4 - 2\alpha_3$.

In this notation we have the following formulas for the roots ν:

$$\begin{gathered}\nu_{1,2} = i\rho e^{\pm i\theta}, \quad 2\theta = \arccos\left[\frac{\rho_0^2 - 2\rho^2}{2\rho^2}\right], \text{ if } \rho_0 < 2\rho,\\ \nu_{1,2} = i\rho e^{\pm\tau}, \quad 2\tau = \operatorname{arch}\left[\frac{\rho_0^2 - 2\rho^2}{2\rho^2}\right], \text{ if } \rho_0 > 2\rho,\\ \nu_1 = \nu_2 = i\rho, \text{ if } \rho_0 = 2\rho.\end{gathered} \tag{8.3}$$

Indeed, let δ be the expression in square brackets in (8.3), so $\rho_0^2 = 2(\delta+1)\rho^2$. Then

$$p_1(z)p_2(z) - p_3^2(z) = \alpha_2\alpha_3(\rho^4 + 2\delta\rho^2z^2 + z^4).$$

Therefore, $\nu^2 = -\rho^2(\delta \pm \sqrt{\delta^2 - 1})$, that reduces to (8.3) after elementary manipulations.

Let us note that case (1.5_0) of the diagonalizable Lamé system corresponds to the first equality in (8.3) and in this case the expressions for the roots ν_j coincide with (1.6_0). Case (*ii*) of multiple roots corresponds to the last equality in (8.3). It is easy to see that independently of three possible cases in (8.3) we have a common expressions for variables (7.4):

$$s = i\rho_0, \ t = -\rho^2. \tag{8.4}$$

In particular, expressions (7.12) for the quadratic forms turn into

$$\omega(\xi) = \xi_1^2 - \rho^2\xi_2^2 + i\rho_0\xi_1\xi_2, \quad 2\omega_1(\xi) = -2\rho^2\xi_1\xi_2 + i\rho_0(\xi_1^2 + \rho^2\xi_2^2). \tag{8.5}$$

In the orthotropic medium the special case is defined by relations (1.5_0). The formulas for the elements $e_{(1)}, \ e_{(2)}, \ e_{ij}$ and $e, \ e_i$, entering relations (7.5), (7.6) and (7.13), are significantly simplified for this medium. Respectively the matrices G_{kr} allow quite foreseeable explicit expressions. First let us assume that $\alpha_3 + \alpha_4 \neq 0$. It is seen from (1.2) that (1.16) implies the similar condition $\beta_5 = \beta_6 = 0$. Therefore

$$e_{(1)} = \alpha_3(\alpha_3 + \alpha_4) - \alpha_2(\alpha_3 + \alpha_4)t, \quad e_{(2)} = \beta_4^2 + \beta_1\beta_4(s^2 - t) - \beta_1\beta_4 t + \beta_1^2 t^2,$$

$$e_{11} = (\alpha_2\beta_4 - \alpha_3\beta_1)s, \quad e_{22} = -(\alpha_3 + \alpha_4)\beta_1 st,$$
$$e_{12} = -\alpha_3\beta_4 - \alpha_3\beta_1(s^2 - t) + \alpha_2\beta_4 t - \alpha_2\beta_1 t^2, \quad e_{21} = (\alpha_3 + \alpha_4)\beta_4 - (\alpha_3 + \alpha_4)\beta_1 t,$$

$$2e = \alpha_3(\alpha_3 + \alpha_4)s + \alpha_2(\alpha_3 + \alpha_4)st,$$
$$e_1 = \alpha_3^2 + \alpha_2^2 t^2 + \alpha_2\alpha_3(s^2 - 2t), \quad e_2 = (\alpha_3 + \alpha_4)^2 t.$$

Since $\beta_1 = \alpha_2\alpha_3, \ \beta_4 = -\alpha_3\alpha_4$, taking into account (8.1), (8.4) after elementary manipulations we obtain:

$$\begin{gathered}
e_{(1)} = (\alpha_3 + \alpha_4)(\alpha_3 + \sqrt{\alpha_1\alpha_2}), \quad e_{(2)} = \alpha_3(\alpha_3 + \alpha_4)(\alpha_1\alpha_2 - \alpha_4^2), \\
e_{11} = -i\rho_0\alpha_2\alpha_3(\alpha_3 + \alpha_4), \quad e_{22} = i\rho_0\alpha_2\alpha_3(\alpha_3 + \alpha_4)\rho^2, \\
e_{12} = e_{21} = \alpha_3(\alpha_3 + \alpha_4)(\sqrt{\alpha_1\alpha_2} - \alpha_4), \\
2e = i\rho_0(\alpha_3 + \alpha_4)(\alpha_3 - \sqrt{\alpha_1\alpha_2}), \quad e_1 = (\alpha_3 + \alpha_4)^2, \quad e_2 = -(\alpha_3 + \alpha_4)^2\rho^2.
\end{gathered} \tag{8.6}$$

In its turn the substitution of (8.2) and (8.5), (8.6) in (7.13) reduces (7.21), (7.22) to the following formulas

$$\begin{gathered}
G_{11}(\xi) = \frac{\rho_0}{\alpha_3 + \sqrt{\alpha_1\alpha_2}} \begin{pmatrix} \rho^2(\alpha_2\xi_1^2 + \alpha_3\xi_2^2) & (\alpha_3 + \alpha_4)\xi_1\xi_2 \\ \rho^2(\alpha_3 + \alpha_4)\xi_1\xi_2 & \alpha_3\xi_1^2 + \alpha_1\xi_2^2 \end{pmatrix}, \\
G_{22}(\xi) = \rho_0 \begin{pmatrix} \xi_1^2 & \rho^2\xi_1\xi_2 \\ \xi_1\xi_2 & \rho^2\xi_2^2 \end{pmatrix}.
\end{gathered} \tag{8.7}$$

In the same way the substitution of (8.6) in (7.6) gives the expressions

$$\begin{gathered}
cb^{-1} = \frac{\alpha_3}{\alpha_3 + \sqrt{\alpha_1\alpha_2}} \begin{pmatrix} -i\rho_0\rho^2\alpha_2 & -\delta \\ \delta & -i\rho_0\alpha_2 \end{pmatrix}, \\
bc^{-1} = \frac{1}{\alpha_1\alpha_2 - \alpha_4^2} \begin{pmatrix} i\rho_0\alpha_2 & -\delta \\ \delta & i\rho_0\rho^2\alpha_2 \end{pmatrix}
\end{gathered} \tag{8.8}$$

with positive constant $\delta = \sqrt{\alpha_1\alpha_2} - \alpha_4$. Therefore by (8.4), (8.5) formulas (7.22) take the following form:

$$G_{21}(\xi) = \frac{\alpha_3\rho_0}{\alpha_3 + \sqrt{\alpha_1\alpha_2}} \begin{pmatrix} \rho^2\xi_1\xi_2 g_1(\xi) & -\rho^4\alpha_2\widehat{\xi}^2 - \delta\xi_1^2 \\ \rho^2[\alpha_2\widehat{\xi}^2 + \delta\xi_2^2] & \xi_1\xi_2 g_2(\xi) \end{pmatrix},$$
$$G_{12}(\xi) = \frac{\rho_0}{\alpha_1\alpha_2 - \alpha_4^2} \begin{pmatrix} -\xi_1\xi_2 g_1(\xi) & \rho^2[\rho^2\alpha_2\widehat{\xi}^2 - \delta\xi_2^2] \\ -\rho^2\alpha_2\widehat{\xi}^2 + \delta\xi_1^2 & -\rho^2\xi_1\xi_2 g_2(\xi) \end{pmatrix}, \qquad (8.9)$$

where for short $\widehat{\xi}^2 = \xi_1^2 - \rho^2\xi_2^2$ and

$$g_j(\xi) = \alpha_2(\rho^2+1)\widehat{\xi}^2|\xi|^{-2} + (-1)^j[\alpha_2\rho_0^2\xi_j^2|\xi|^{-2} - \delta],\ j = 1,2.$$

Let us remind that the first equality in (8.7) and equalities (8.8), (8.9) are obtained under the assumption $\alpha_3 + \alpha_4 \neq 0$. These formulas also hold in the special case $\alpha_3 + \alpha_4 = 0$. Indeed, according to (1.5_0), (1.6) in this case (7.14) reduces to

$$G_{11}(\xi) = \frac{1}{\alpha_2\alpha_3} \begin{pmatrix} \sqrt{\alpha_1\alpha_3}(\alpha_2\xi_1^2 + \alpha_3\xi_2^2) & 0 \\ 0 & \sqrt{\alpha_2\alpha_3}(\alpha_3\xi_1^2 + \alpha_1\xi_2^2) \end{pmatrix}.$$

Since $\alpha_3 + \alpha_4 = 0$ the expression for ρ_0 in (8.1) reduces to the form

$$\rho_0 = \frac{\alpha_3 + \sqrt{\alpha_1\alpha_2}}{\sqrt{\alpha_2\alpha_3}}$$

and hence follows the first equality in (8.7). The validness of formulas (8.8) and (8.9) is verified similarly.

Formulas (8.7) – (8.9) are further simplified in case (1.18) of the isotropic medium, when $\alpha_1 = \alpha_2,\ \alpha_4 = \alpha_1 - 2\alpha_3$. In this case (8.2) turn into $\rho = 1,\ \rho_0 = 2$ and $\delta = 2\alpha_3$, $g_j(\xi) = 2(-1)^j(\alpha_1 - \alpha_3)$, $\widehat{\xi}^2 = \xi_1^2 - \xi_2^2$. Therefore the formulas mentioned above take the following explicit form:

$$cb^{-1} = \frac{\alpha_3}{æ} \begin{pmatrix} -i(æ+1) & -(æ-1) \\ æ-1 & -i(æ+1) \end{pmatrix}, \quad bc^{-1} = \frac{1}{4\alpha_3} \begin{pmatrix} i(æ+1) & -(æ-1) \\ æ-1 & i(æ+1) \end{pmatrix},$$

where still $æ = (\alpha_1 + \alpha_3)/(\alpha_1 - \alpha_3)$, and

$$G_{11}(\xi) = \frac{1}{æ} \begin{pmatrix} \xi_1^2 - \xi_2^2 + æ|\xi|^2 & 2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_2^2 - \xi_1^2 + æ|\xi|^2 \end{pmatrix},$$

$$G_{22}(\xi) = 2\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix} = \begin{pmatrix} \xi_1^2 - \xi_2^2 + |\xi|^2 & 2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_2^2 - \xi_1^2 + |\xi|^2 \end{pmatrix},$$
$$G_{21}(\xi) = \frac{\alpha_3}{æ} \begin{pmatrix} -4\xi_1\xi_2 & -2æ(\xi_1^2 - \xi_2^2) - (æ-1)|\xi|^2 \\ 2(\xi_1^2 - \xi_2^2) + (æ-1)|\xi|^2 & 4\xi_1\xi_2 \end{pmatrix},$$
$$G_{12}(\xi) = \frac{1}{4\alpha_3} \begin{pmatrix} 4\xi_1\xi_2 & 2æ(\xi_1^2 - \xi_2^2) - (æ-1)|\xi|^2 \\ -2(\xi_1^2 - \xi_2^2) + (æ-1)|\xi|^2 & -4\xi_1\xi_2 \end{pmatrix}.$$

It is curious to note that the matrix G_{22} and, therefore, the class of conjugate functions in the isotropic case under consideration does not depend on the modules of elasticity.

Setting

$$G_1(\xi) = \begin{pmatrix} \xi_1^2 - \xi_2^2 & 2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_2^2 - \xi_1^2 \end{pmatrix}, \; G_2(\xi) = \begin{pmatrix} æ(\xi_1^2 - \xi_2^2) & -2\xi_1\xi_2 \\ -2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix}, \; E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we can write

$$G_{11}(\xi) = |\xi|^2 + \frac{1}{æ} G_1(\xi), \quad G_{22}(\xi) = |\xi|^2 + G_1(\xi),$$

$$G_{21}(\xi) = \frac{\alpha_3}{æ}[(æ - 1)|\xi|^2 + 2G_2(\xi)]E, \quad G_{12}(\xi) = \frac{1}{4\alpha_3}[(æ - 1)|\xi|^2 - 2G_2(\xi)]E,$$

So we have

$$P_{11} = P_0 + \frac{1}{æ} P_1, \quad P_{22} = P_0 + P_1,$$

$$P_{21} = \frac{\alpha_3}{æ}[(æ - 1)P_0 + 2P_2]E, \quad P_{12} = \frac{1}{4\alpha_3}[(æ - 1)P_0 - 2P_2]E,$$

where the operators P_1, P_2 act by the formulas

$$(\mathrm{P}_j\varphi)(z) = \frac{1}{\pi}\int_\Gamma p(t, t - z) G_j(t - z)\varphi(t)|dt|, \quad p(t, \xi) = \frac{n_1(t)\xi_1 + n_2(t)\xi_2}{|\xi|^4}.$$

We take into account that in the case under consideration we have one multiple root $\nu = i$ and so $|\omega(\xi)|^2 = |\xi|^4$. Thus equalities (7.3_1) and (7.3_2) take the form

$$u = P_0\varphi + \frac{1}{æ} P_1\varphi, \quad v = \frac{\alpha_3}{æ}[(æ + 1)Q_0\varphi + (æ - 1)P_0\widetilde{\varphi} + 2P_2\widetilde{\varphi}],$$

and

$$v = P_0\varphi + P_1\varphi, \quad u = \frac{1}{4\alpha_3}[(æ + 1)Q_0\varphi + (æ - 1)P_0\widetilde{\varphi} - 2P_2\widetilde{\varphi}],$$

respectively, where $\widetilde{\varphi} = (-\varphi_2, \varphi_1)$.

By means of Lemma 7.2 we can also write explicitly the expressions for partial derivatives of the conjugate function. In the case under consideration matrices (7.26) take the form

$$\Omega_1(\xi) = \frac{1}{(\xi_1 + i\xi_2)^2} \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 + 2i\xi_2 \end{pmatrix}, \quad \Omega_2(\xi) = \frac{1}{(\xi_1 + i\xi_2)^2} \begin{pmatrix} 2i\xi_1 + \xi_2 & \xi_1 \\ -\xi_1 & \xi_2 \end{pmatrix}.$$

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