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INFINITENESS OF THE NUMBER OF EIGENVALUES EMBEDDED IN THE ESSENTIAL SPECTRUM OF A 2×2 OPERATOR MATRIX

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Abstract. In the present paper a 2×2 block operator matrix $\mathbf{H}$ is considered as a bounded self-adjoint operator in the direct sum of two Hilbert spaces. The structure of the essential spectrum of $\mathbf{H}$ is studied. Under some natural conditions the infiniteness of the number of eigenvalues is proved, located inside, in the gap or below the bottom of the essential spectrum of $\mathbf{H}$.

1 Introduction

Perturbation problems for operators with embedded eigenvalues are generally challenging since the embedded eigenvalues cannot be separated from the rest of the spectrum. Embedded eigenvalues occur in many applications arising in physics. In quantum mechanics, for instance, eigenvalues of the energy operator correspond to energy bound states that can be attained by the underlying physical system. If such an eigenvalue is embedded in the continuous spectrum, it is of fundamental importance to determine whether it, and therefore corresponding bound state, persists upon perturbing the potential. Alternatively, embedded eigenvalues in inverse scattering problems correspond to soliton-type structures for the original integrable problems whose robustness under perturbations is therefore again determined by the fate of the embedded eigenvalues.

It is well known [5, 17] that the Schrödinger operator $-\triangle + V(x)$ does not have eigenvalues embedded in its continuous spectrum if $V(x)$ is integrable. However, the Wigner-von Neumann example [17] shows that the Schrödinger operator with the potential

$$V(x) := q(x) \cos wx, \quad w > 0,$$

where for some $\alpha > 0$

$$q^{(j)}(x) = O(x^{-\alpha-j}) \quad \text{as} \quad x \to \infty, \quad j = 1, 2, 3,$$

has eigenvalues embedded in the continuous spectrum.

In [2], S. Albeverio describes how one can construct potentials that lead to any finite number of bound states having preassigned energies. In [18], M.M. Skriganov constructed potentials that have a countable number of positive eigenvalues for the one-dimensional Schrödinger operator. In this constructions, the methods of the inverse scattering problem play an important role. The existence of a potential with infinite number of eigenvalues in the continuous spectrum for a discrete Schrödinger operator is demonstrated in [12]. The existence of a multi-dimensional generalized Friedrichs model with a given number of eigenvalues located within the continuous spectrum is proved in [1]. The infiniteness of the number of eigenvalues in the gap of the essential spectrum for the three-particle discrete Schrödinger operator was proven in [8], and a formula for the number of its eigenvalues in an arbitrary interval outside of the essential spectrum was obtained in [9]. In the paper [10], it is shown that the discrete spectrum of the three-particle Schrödinger operator on a one-dimensional lattice is infinite in the case in which the masses m_α and m_β of two particles are infinite.

It is remarkable that the above mentioned operators describe systems with a conserved finite number of particles in continuous space and on a lattice. However, in both cases, there exist problems with a non-conserved number of particles that are more interesting in a certain sense. Such problems occur in solid state physics, quantum field theory and statistical physics. Systems with a non-conserved finite number of particles in continuous space were considered in [7, 21]. Usually the Hamiltonians describing such type of systems in both cases can be expressed as block operator matrices.

In the present paper we consider the 2×2 block operator matrix $\mathbf{H}$ in the direct sum of Hilbert spaces. We describe the structure of the essential spectrum of $\mathbf{H}$. We find conditions for the infiniteness of the number of eigenvalues located inside, in the gap and below of the bottom of the essential spectrum of $\mathbf{H}$, respectively.

We note that such type of operator matrices were considered in [11, 13, 20] and only its essential spectrum was investigated. This paper is a continuation of those papers.

The plan of the paper is as follows. Section 1 is a general introduction. In Section 2, the operator matrix $\mathbf{H}$ is introduced and the main results of the present paper are formulated. In Section 3, we recall some spectral properties of the corresponding Friedrichs models. In Section 4, we study the structure of the essential spectrum of $\mathbf{H}$. In Section 5, an asymptotic formula for the number of negative eigenvalues of H_{22} (diagonal entry of $\mathbf{H}$) is obtained. In Section 6, we prove the infiniteness of the number of eigenvalues of $\mathbf{H}$ lying inside, in the gap or below the bottom of its essential spectrum.

We adopt the following conventions throughout the present paper. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of all positive integers, integers, real, and complex numbers respectively. We denote by $\mathbb{T}^3$ the three-dimensional torus (the first Brillouin zone, i.e., dual group of $\mathbb{Z}^3$), the cube $(-\pi, \pi]^3$ with appropriately identified sides equipped with its Haar measure. The torus $\mathbb{T}^3$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space $\mathbb{R}^3$ modulo $(2\pi\mathbb{Z})^3$.

Denote by $\sigma(\cdot)$, $\sigma_{\rm ess}(\cdot)$ and $\sigma_{\rm disc}(\cdot)$, respectively, the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator. In what follows we deal with the operators in various spaces of vector-valued functions. They will be denoted by bold letters and will be written in the matrix form.

For each $\delta > 0$, the notation $U_\delta(p_0)$ is used for the $\delta-$neighborhood

$$\{p \in \mathbb{T}^3 : |p - p_0| < \delta\}$$

of the point $p_0 \in \mathbb{T}^3$.

2 Block operator matrix and main results

Let $L_2(\mathbb{T}^3)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^{\rm s}((\mathbb{T}^3)^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^3)^2$. Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_1 := L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 := L_2^{\rm s}((\mathbb{T}^3)^2)$, that is, $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. The Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are one-particle and two-particle subspaces of a bosonic Fock space $\mathcal{F}_{\rm s}(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$, respectively.

We consider the block operator matrix $\mathbf{H}$ acting in the Hilbert space $\mathcal{H}$ given by

$$\mathbf{H} := \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$$

with the entries $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i,\ \ i,j = 1,2:$

$$(H_{11}f_1)(p) = u(p)f_1(p), \quad (H_{12}f_2)(p) = \sqrt{\lambda}\int_{\mathbb{T}^3} v(s)f_2(p,s)ds,$$

$$(H_{22}f_2)(p,q) = w(p,q)f_2(p,q) - \mu\int_{\mathbb{T}^3} f_2(p,s)ds - \mu\int_{\mathbb{T}^3} f_2(s,q)ds,$$

where $f_i \in \mathcal{H}_i,\ i = 1,2;\ \mu,\lambda > 0;\ u(\cdot)$ is a positive valued continuous function on $\mathbb{T}^3$, the function $v(\cdot)$ is real-valued analytic on $\mathbb{T}^3$ and the function $w(\cdot,\cdot)$ is defined by

$$w(p,q) := l_1\varepsilon(p) + l_2\varepsilon(p+q) + l_1\varepsilon(q),$$

with $l_1, l_2 > 0$ and

$$\varepsilon(q) := \sum_{i=1}^{3}(1 - \cos(2q^{(i)})),\ q = (q^{(1)}, q^{(2)}, q^{(3)}) \in \mathbb{T}^3.$$

Here H_{12}^* denotes the adjoint operator to H_{12} and

$$(H_{12}^*f_1)(p,q) = \frac{\sqrt{\lambda}}{2}(v(p)f_1(q) + v(q)f_1(p)), \quad f_1 \in \mathcal{H}_1.$$

Under these assumptions the operator $\mathbf{H}$ is bounded and self-adjoint in $\mathcal{H}$.

We note that the operators H_{12} and H_{12}^* are called annihilation, creation operators respectively.

Set $\mathcal{H}_0 := \mathbb{C}$. To formulate the main results of the paper we introduce a family of bounded self-adjoint operators (Friedrichs models operators) $\widehat{\mathbf{h}}_{\mu\lambda}(p)$, $p \in \mathbb{T}^3$ which act in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\widehat{\mathbf{h}}_{\mu\lambda}(p) := \begin{pmatrix} h_{00}(p) & h_{01} \\ h_{01}^* & h_{11}(p) \end{pmatrix},$$

where the operators $h_{ii}(p) : \mathcal{H}_i \to \mathcal{H}_i$, $i = 0, 1$ and $h_{01} : \mathcal{H}_1 \to \mathcal{H}_0$ are defined as

$$h_{00}(p)f_0 = u(p)f_0, \quad h_{01}f_1 = \sqrt{\frac{\lambda}{2}} \int_{\mathbb{T}^3} v(s)f_1(s)ds,$$

$$h_{11}(p) = h^0_{11}(p) - v, \quad (h^0_{11}(p)f_1)(q) = w(p,q)f_1(q), \quad (vf_1)(q) = \mu \int_{\mathbb{T}^3} f_1(s)ds.$$

The following theorem [14] describes the location of the essential spectrum of $\mathbf{H}$.

Theorem 2.1. *The essential spectrum of* $\mathbf{H}$ *satisfies the equality*

$$\sigma_{\text{ess}}(\mathbf{H}) = \bigcup_{p \in \mathbb{T}^3} \sigma_{\text{disc}}(\widehat{\mathbf{h}}_{\mu\lambda}(p)) \cup [0; M], \quad M := \frac{9}{2}(2l_1 + l_2). \tag{2.1}$$

Moreover, the set $\sigma_{\text{ess}}(\mathbf{H})$ *is a union of at most four intervals.*

Throughout this paper we assume the following additional assumption that the real-valued analytic function $v(\cdot)$ satisfies the condition

$$\int_{\mathbb{T}^3} v(s)g(p,s)ds = 0 \tag{2.2}$$

for any function $g \in L_2^{\text{s}}((\mathbb{T}^3)^2)$, which is periodical on each variable with period π.

Note that the functions $v(p) = \sum\limits_{i=1}^{3} c_i \cos p^{(i)}$ and $v(p) = \sum\limits_{i=1}^{3} c_i \cos p^{(i)} \cos(2p^{(i)})$, where c_i, $i = 1, 2, 3$ are arbitrary real numbers, satisfies condition (2.2). Indeed, for $v(p) = \sum\limits_{i=1}^{3} c_i \cos p^{(i)}$, we have

$$\int_{\mathbb{T}^3} v(s)g(p,s)ds = \int_{\mathbb{T}^3} v(s+\bar{\pi})g(p,s+\bar{\pi})ds = -\int_{\mathbb{T}^3} v(s)g(p,s)ds, \quad \bar{\pi} = (\pi, \pi, \pi),$$

which yields equality (2.2).

Under condition (2.2) the discrete spectrum of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ coincides (see Lemma 3.1 below) with the union of discrete spectra of the operators

$$h_\mu(p) := h_{11}(p),\ p \in \mathbb{T}^3 \quad \text{and} \quad \mathbf{h}_\lambda(p) := \begin{pmatrix} h_{00}(p) & h_{01} \\ h^*_{01} & h^0_{11}(p) \end{pmatrix},\ p \in \mathbb{T}^3.$$

It follows by the definition of the operators $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$ that their structure is simpler than that of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$.

We introduce the following points of $\mathbb{T}^3$:

$$p_1 := (0,0,0),\ p_2 := (\pi,0,0),\ p_3 := (0,\pi,0),\ p_4 := (0,0,\pi),$$

$$p_5 := (\pi,\pi,0),\ p_6 := (\pi,0,\pi),\ p_7 := (0,\pi,\pi),\ p_8 := (\pi,\pi,\pi).$$

It is easy to verify that the function $w(\cdot,\cdot)$ has non-degenerate minimum at the points $(p_i, p_j) \in (\mathbb{T}^3)^2$, $i, j = \overline{1,8}$; where $\overline{1,n} = 1, \ldots, n$. Therefore, for any $p \in \mathbb{T}^3$ the integral

$$I(p) := \int_{\mathbb{T}^3} \frac{v^2(s)ds}{w(p,s)}$$

is finite.

The Lebesgue dominated convergence theorem and the equality $I(p_1)=I(p_i)$, $i=\overline{2,8}$ yield

$$I(p_1)=\lim_{p\to p_i} I(p_i),\ i=\overline{1,8},$$

and hence the function $I(\cdot)$ is continuous on $\mathbb{T}^3$. Therefore there exist points $\theta_0,\theta_1\in\mathbb{T}^3$ such that

$$\min_{p\in\mathbb{T}^3} I(p)=I(\theta_1) \quad \text{and} \quad \max_{p\in\mathbb{T}^3} I(p)=I(\theta_0).$$

From now on we assume that the function $u(\cdot)$ has a minimum at $p=\theta_0$ and a maximum at $p=\theta_1$, and introduce the following notations:

$$\mu_0:=(l_1+l_2)\left(\int_{\mathbb{T}^3}\frac{ds}{\varepsilon(s)}\right)^{-1}, \quad \lambda_k:=2u(\theta_k)(I(\theta_k))^{-1}, \quad k=0,1;$$

$$a_\lambda:=\min\Big\{\bigcup_{p\in\mathbb{T}^3}\sigma_{\rm disc}(\mathbf{h}_\lambda(p))\cap(-\infty;0]\Big\}, \quad b_\lambda:=\max\Big\{\bigcup_{p\in\mathbb{T}^3}\sigma_{\rm disc}(\mathbf{h}_\lambda(p))\cap(-\infty;0]\Big\},$$

for $\lambda>\lambda_0$.

The structure of the essential spectrum of $\mathbf{H}$ can be precisely described as in the following theorem.

Theorem 2.2. *Let $\mu=\mu_0$. Then the following assertions hold.*

1) *If $\lambda\in(0;\lambda_0]$, then $(-\infty;M]\cap\sigma_{\rm ess}(\mathbf{H})=[0;M]$;*

2) *If $\lambda\in(\lambda_0;\lambda_1]$, then $(-\infty;M]\cap\sigma_{\rm ess}(\mathbf{H})=[a_\lambda;M]$ and $a_\lambda<0$;*

3) *If $\lambda\in(\lambda_1;+\infty)$, then $(-\infty;M]\cap\sigma_{\rm ess}(\mathbf{H})=[a_\lambda;b_\lambda]\cup[0;M]$ and $a_\lambda<b_\lambda<0$.*

Let us denote by $\tau_{\rm ess}(A)$ the bottom of the essential spectrum $\sigma_{\rm ess}(A)$ of a bounded self-adjoint operator A and by $N(A;z)$ the number of eigenvalues of A lying below the point z, $z<\tau_{\rm ess}(A)$.

It is clear that $h_\mu(p_1)\equiv h_\mu(p_i)$, $i=\overline{2,8}$. Note that [16] the operator $h_\mu(p)$ with $\mu=\mu_0$ is strictly positive for any $p\in\mathbb{T}^3\setminus\{p_1,\ldots,p_8\}$, and thus the operator $h_\mu(p_1)$ corresponding to the value p_1 of p is the unique operator whose spectrum attains the bottom of the essential spectrum of H_{22}. Moreover, $\tau_{\rm ess}(H_{22})=0$ for $\mu=\mu_0$.

The main results of the present paper as follows.

Theorem 2.3. *For $\mu=\mu_0$ the operator H_{22} has infinitely many negative eigenvalues $E_1,\ldots,E_n,\ldots$, such that $\lim_{n\to\infty}E_n=0$, and the function $N(H_{22};\cdot)$ obeys the relation*

$$\lim_{z\to-0}\frac{N(H_{22};z)}{|\log|z||}=\mathcal{U}_0, \quad 0<\mathcal{U}_0<\infty. \tag{2.3}$$

It is easy to see that the infiniteness of the cardinality of the negative discrete spectrum of H_{22} directly follows by the positivity of $\mathcal{U}_0$.

For $n\in\mathbb{N}$ denote by $f_2^{(n)}$ the eigenfunction corresponding to the eigenvalue E_n of H_{22} with $\mu=\mu_0$.

Theorem 2.4. *Let $\mu = \mu_0$. Then for any $\lambda \geq 0$ the numbers $E_1, \ldots, E_n, \ldots$ are eigenvalues of $\mathbf{H}$ and the corresponding eigenfunction has the form: $f^{(n)} = (0, f_2^{(n)})$, $n \in \mathbb{N}$. Moreover,*

1) if $\lambda \in (0; \lambda_0]$, then the set $\{E_n : n \in \mathbb{N}\}$ is located below the bottom of the essential spectrum of $\mathbf{H}$;

2) if $\lambda \in (\lambda_0; \lambda_1]$, then the countable subset of $\{E_n : n \in \mathbb{N}\}$ is located in the essential spectrum of $\mathbf{H}$;

3) if $\lambda \in (\lambda_1; +\infty)$, then the countable subset of $\{E_n : n \in \mathbb{N}\}$ is located in the gap of the essential spectrum of $\mathbf{H}$.

Since $\lim\limits_{\lambda \to \lambda_1 + 0} b_\lambda = 0$, it follows from assertions 3) of Theorems 2.2 and 2.4 that for any given finite number $k \in \mathbb{N}$ there exists $\lambda' \in (\lambda_1; +\infty)$ such that for $\mu = \mu_0$ and $\lambda = \lambda'$ the set $\{E_n : n \in \mathbb{N}\} \cap [a_\lambda; b_\lambda]$ consists of k elements.

3 Spectral properties of the Friedrichs model $\widehat{\mathbf{h}}_{\mu\lambda}(p)$

In this section we study spectral properties of the Friedrichs model $\widehat{\mathbf{h}}_{\mu\lambda}(p)$, which plays a crucial role in the study of the spectral properties of H_{22} and $\mathbf{H}$.

3.1 Spectrum of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$

Let the operator $\mathbf{h}^0(p)$ acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\mathbf{h}^0(p) := \begin{pmatrix} 0 & 0 \\ 0 & h_{11}^0(p) \end{pmatrix}.$$

The perturbation $\widehat{\mathbf{h}}_{\mu\lambda}(p) - \mathbf{h}^0(p)$ of the operator $\mathbf{h}^0(p)$ is a self-adjoint operator of rank at most 3, and thus, according to the Weyl theorem, the essential spectrum of the operator $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ coincides with the essential spectrum of $\mathbf{h}^0(p)$. It is clear that $\sigma_{\mathrm{ess}}(\mathbf{h}^0(p)) = [m(p); M(p)]$, where the numbers $m(p)$ and $M(p)$ are defined by

$$m(p) := \min_{q \in \mathbb{T}^3} w(p, q), \quad M(p) := \max_{q \in \mathbb{T}^3} w(p, q).$$

This yields $\sigma_{\mathrm{ess}}(\widehat{\mathbf{h}}_{\mu\lambda}(p)) = [m(p); M(p)]$.

For any fixed μ, $\lambda > 0$ and $p \in \mathbb{T}^3$ we define the functions

$$\Delta_1(\mu, p\,; z) := 1 - \mu \int_{\mathbb{T}^3} \frac{ds}{w(p, s) - z}, \quad \Delta_2(\lambda, p\,; z) := u(p) - z - \frac{\lambda}{2} \int_{\mathbb{T}^3} \frac{v^2(s) ds}{w(p, s) - z},$$

that are regular in $\mathbb{C} \setminus [m(p); M(p)]$; these functions are the Fredholm determinants associated with the operators $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$ respectively.

The following lemma describes the relation between the eigenvalues of the operators $h_\mu(p)$, $\mathbf{h}_\lambda(p)$ and $\widehat{\mathbf{h}}_{\mu\lambda}(p)$.

Lemma 3.1. *The number $z \in \mathbb{C} \setminus [m(p); M(p)]$ is an eigenvalue of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ if and only if z is an eigenvalue of at least one of the operators $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$.*

Proof. Suppose $(f_0, f_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1$ is an eigenvector of the operator $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ associated with the eigenvalue $z \in \mathbb{C} \setminus [m(p); M(p)]$. Then f_0 and f_1 satisfy the following system of equations

$$\begin{cases} (u(p) - z)f_0 + \sqrt{\frac{\lambda}{2}} \int_{\mathbb{T}^3} v(s) f_1(s) ds = 0 \\ \sqrt{\frac{\lambda}{2}} v(q) f_0 + (w(p,q) - z) f_1(q) - \mu \int_{\mathbb{T}^3} f_1(s) ds = 0. \end{cases} \tag{3.1}$$

Since for any $z \in \mathbb{C} \setminus [m(p); M(p)]$ and $q \in \mathbb{T}^3$ the relation $w(p,q) - z \neq 0$ holds for all $p \in \mathbb{T}^3$, from the second equation of (3.1) for f_1 we have

$$f_1(q) = \frac{\mu C_{f_1}}{w(p,q) - z} - \sqrt{\frac{\lambda}{2}} \frac{v(q) f_0}{w(p,q) - z}, \tag{3.2}$$

where

$$C_{f_1} = \int_{\mathbb{T}^3} f_1(s) ds. \tag{3.3}$$

Substituting expression (3.2) for f_1 into the first equation of system (3.1) and equality (3.3), and then using condition (2.2), we conclude that the system of equations (3.1) has a nontrivial solution if and only if the system of equations

$$\begin{cases} \Delta_2(\lambda, p\,; z) f_0 = 0 \\ \Delta_1(\mu, p\,; z) C_{f_1} = 0 \end{cases}$$

has a nontrivial solution, i.e., if the condition $\Delta_1(\mu, p\,; z)\Delta_2(\lambda, p\,; z) = 0$ is satisfied.

It is clear [16] ([15]) that the number $z \in \mathbb{C} \setminus [m(p); M(p)]$ is an eigenvalue of $h_\mu(p)$ ($\mathbf{h}_\lambda(p)$) if and only if $\Delta_1(\mu, p\,; z) = 0$ ($\Delta_2(\lambda, p\,; z) = 0$). □

By Lemma 3.1 it follows that

$$\sigma_{\text{disc}}(h_\mu(p)) = \{z \in \mathbb{C} \setminus [m(p); M(p)] : \Delta_1(\mu, p\,; z) = 0\};$$

$$\sigma_{\text{disc}}(\mathbf{h}_\lambda(p)) = \{z \in \mathbb{C} \setminus [m(p); M(p)] : \Delta_2(\lambda, p\,; z) = 0\}$$

and

$$\sigma_{\text{disc}}(\widehat{\mathbf{h}}_{\mu\lambda}(p)) = \sigma_{\text{disc}}(h_\mu(p)) \cup \sigma_{\text{disc}}(\mathbf{h}_\lambda(p)). \tag{3.4}$$

In the following lemma we precisely describe the dependence of the negative eigenvalues of $\mathbf{h}_\lambda(p)$ on the parameter $\lambda > 0$.

Lemma 3.2. 1) *Let $\lambda \in (0; \lambda_0]$. Then for any $p \in \mathbb{T}^3$ the operator $\mathbf{h}_\lambda(p)$ has no negative eigenvalues;*

2) *Let $\lambda \in (\lambda_0; \lambda_1]$. Then there exists a non empty open set $D_\lambda \subset \mathbb{T}^3$ such that $D_\lambda \neq \mathbb{T}^3$, and for any $p \in D_\lambda$ the operator $\mathbf{h}_\lambda(p)$ has a unique negative eigenvalue and for any $p \in \mathbb{T}^3 \setminus D_\lambda$ the operator $\mathbf{h}_\lambda(p)$ has no negative eigenvalues;*

3) *Let $\lambda > \lambda_1$. Then for any $p \in \mathbb{T}^3$ the operator $\mathbf{h}_\lambda(p)$ has a unique negative eigenvalue.*

Proof. First we prove part 2). Let $\lambda \in (\lambda_0; \lambda_1]$. By the definition of the numbers λ_i, $i = 0, 1$ it follows that $\min_{p \in \mathbb{T}^3} \Delta_2(\lambda, p\,; 0) < 0$ and $\max_{p \in \mathbb{T}^3} \Delta_2(\lambda, p\,; 0) \geq 0$ for any $\lambda \in (\lambda_0; \lambda_1]$. Since the function $I(\cdot)$ has minimum at $p = \theta_1$ and maximum at $p = \theta_0$, and the function $u(\cdot)$ has minimum at $p = \theta_0$ and maximum at $p = \theta_1$, it is clear that

$$\min_{p \in \mathbb{T}^3} \Delta_2(\lambda, p\,; 0) = \Delta_2(\lambda, \theta_0\,; 0) \quad \text{and} \quad \max_{p \in \mathbb{T}^3} \Delta_2(\lambda, p\,; 0) = \Delta_2(\lambda, \theta_1\,; 0).$$

Therefore $\Delta_2(\lambda, \theta_0\,; 0) < 0$ and $\Delta_2(\lambda, \theta_1\,; 0) \geq 0$ for any $\lambda \in (\lambda_0; \lambda_1]$.

We introduce the notation: $D_\lambda := \{p \in \mathbb{T}^3 : \Delta_2(\lambda, p\,; 0) < 0\}$. Then it is obvious that D_λ is a non-empty open set and $D_\lambda \neq \mathbb{T}^3$.

For any $\lambda > 0$ and $p \in \mathbb{T}^3$ the function $\Delta_2(\lambda, p\,; \cdot)$ is continuous and decreasing on $(-\infty; 0]$ and $\lim_{z \to -\infty} \Delta_2(\lambda, p\,; z) = +\infty$. Then for any $p \in D_\lambda$ there exists a unique point $e_\lambda(p) \in (-\infty; 0)$ such that $\Delta_2(\lambda, p\,; e_\lambda(p)) = 0$. By Lemma 3.1 for any $p \in \mathbb{T}^3$ the point $e_\lambda(p)$ is the unique negative eigenvalue of the operator $\mathbf{h}_\lambda(p)$.

For any $p \in \mathbb{T}^3 \setminus D_\lambda$ and $z < 0$ we have $\Delta_2(\lambda, p\,; z) > \Delta_2(\lambda, p\,; 0) \geq 0$. Hence by Lemma 3.1 for each $p \in \mathbb{T}^3 \setminus D_\lambda$ the operator $\mathbf{h}_\lambda(p)$ has no negative eigenvalues.

If $\lambda \in (0; \lambda_0]$ (respectively $\lambda \in (\lambda_1; +\infty)$), then $D_\lambda = \emptyset$ (respectively $D_\lambda = \mathbb{T}^3$) and the above analysis leads again to the case 1) (respectively 3)). The straightforward details are omitted. □

3.2 Threshold energy expansion for the Fredholm determinant $\Delta_1(\mu, p\,; z)$

First we remark that $\Delta_1(\mu, p_1\,; 0) = \Delta_1(\mu, p_i\,; 0)$, $i = \overline{2, 8}$. Moreover, by the definition of μ_0 one can see that $\Delta_1(\mu, p_1\,; 0) = 0$ if and only if $\mu = \mu_0$.

The following expansion plays an important role in the proof of Theorem 2.3.

Lemma 3.3. *The following decomposition*

$$\Delta_1(\mu_0, p\,; z) = \frac{8\pi^2 \mu_0}{(l_1 + l_2)^{3/2}} \sqrt{\frac{l_1^2 + 2 l_1 l_2}{l_1 + l_2} |p - p_i|^2 - \frac{z}{2}} + O(|p - p_i|^2) + O(|z|)$$

holds for all $|p - p_i| \to 0$, $i = \overline{1, 8}$ *and* $z \to -0$.

Remark 1. A similar lemma for the two-body discrete Schrödinger operator was proven in [3] for $\varepsilon(q) = \sum_{i=1}^{3} (1 - \cos q^{(i)})$.

Proof of Lemma 3.3. Let us sketch the main idea of the proof. Take a sufficiently small $\delta > 0$ such that $U_\delta(p_i) \cap U_\delta(p_j) = \emptyset$ for all $i \neq j$, $i, j = \overline{1, 8}$.

Set

$$\mathbb{T}_\delta := \mathbb{T}^3 \setminus \bigcup_{j=1}^{8} U_\delta(p_j).$$

Using the additivity of the integral we rewrite the function $\Delta_1(\mu_0, \cdot\,; \cdot)$ as

$$\Delta_1(\mu_0, p\,; z) = 1 - \mu_0 \sum_{j=1}^{8} \int_{U_\delta(p_j)} \frac{ds}{w(p, s) - z} - \mu_0 \int_{\mathbb{T}_\delta} \frac{ds}{w(p, s) - z}. \tag{3.5}$$

Since the function $w(\cdot,\cdot)$ has non-degenerate minimum at the points (p_i, p_j), $i, j = \overline{1,8}$, we can argue as in [4]. It is easy to show that

$$\int_{U_\delta(p_j)} \frac{ds}{w(p,s)-z} = \int_{U_\delta(p_j)} \frac{ds}{w(p_i,s)} - \frac{\pi^2}{(l_1+l_2)^{3/2}} \sqrt{\frac{l_1^2+2l_1l_2}{l_1+l_2}|p-p_i|^2 - \frac{z}{2}} + O(|p-p_i|^2) + O(|z|);$$

$$\int_{\mathbb{T}_\delta} \frac{ds}{w(p,s)-z} = \int_{\mathbb{T}_\delta} \frac{ds}{w(p_i,s)} + O(|p-p_i|^2) + O(|z|)$$

as $|p-p_i| \to 0$ for $i = \overline{1,8}$ and $z \to -0$. Substituting the last two expressions to the equality (3.5) we obtain

$$\Delta_1(\mu_0, p\,; z) = \Delta_1(\mu_0, p_i\,; 0) + \frac{8\pi^2\mu_0}{(l_1+l_2)^{3/2}} \sqrt{\frac{l_1^2+2l_1l_2}{l_1+l_2}|p-p_i|^2 - \frac{z}{2}} + O(|p-p_i|^2) + O(|z|)$$

as $|p-p_i| \to 0$ for $i = \overline{1,8}$ and $z \to -0$. Now the equality $\Delta_1(\mu_0, p_i\,; 0) = 0$ completes the proof of Lemma 3.3. □

Corollary 3.1. *For some $C_1, C_2, C_3 > 0$ and $\delta > 0$ the following inequalities hold*

1) $C_1|p-p_i| \le \Delta_1(\mu_0, p\,; 0) \le C_2|p-p_i|$, $p \in U_\delta(p_i)$, $i = \overline{1,8}$;

2) $\Delta_1(\mu_0, p\,; 0) \ge C_3$, $p \in \mathbb{T}_\delta$.

Proof. The Lemma 3.3 yields assertion 1) for some positive numbers C_1, C_2. The positivity and continuity of the function $\Delta_1(\mu_0, \cdot\,; 0)$ on the compact set $\mathbb{T}_\delta$ imply assertion 2). □

At the end of this section we prove one more assertion.

Lemma 3.4. *There exist $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that*

1) $C_1(|p-p_i|^2+|q-p_j|^2) \le w(p,q) \le C_2(|p-p_i|^2+|q-p_j|^2)$, $(p,q) \in U_\delta(p_i)\times U_\delta(p_j)$, $i, j = \overline{1,8}$;

2) $w(p,q) \ge C_3$, $(p,q) \in \mathbb{T}_\delta^2$.

Proof. The expansion

$$w(p,q) = 2((l_1+l_2)|p-p_i|^2 + 2l_2(p-p_i, q-p_j) + (l_1+l_2)|q-p_j|^2) + O(|p-p_i|^4) + O(|q-p_j|^4)$$

as $|p-p_i|$, $|q-p_j| \to 0$, for $i, j = \overline{1,8}$ implies that there exist $C_1, C_2, C_3 > 0$ and $\delta > 0$ satisfying both inequalities of the lemma. □

4 Essential spectrum of H

In this section using Theorem 2.1 and the assertions proved in Section 3, we prove Theorem 2.2.

First we recall that by Theorem 2.1 and equality (3.4) we have

$$\sigma_{\rm ess}(\mathbf{H}) = \bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(h_\mu(p)) \cup \bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(\mathbf{h}_\lambda(p)) \cup [0;M]. \tag{4.1}$$

Proof of Theorem 2.2. It was shown in [16] that if $\mu=\mu_0$ then

$$\bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(h_\mu(p)) \cup [0;M] = [0;M]. \tag{4.2}$$

Hence by equality (4.1) it suffices to study the structure of the set

$$\bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(\mathbf{h}_\lambda(p)) \cup [0;M].$$

We consider the following three cases.

1) Let $\lambda \in (0;\lambda_0]$. Then by Lemma 3.2 it follows that for any $p\in\mathbb{T}^3$ the operator $\mathbf{h}_\lambda(p)$ has no negative eigenvalues, that is,

$$\bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(\mathbf{h}_\lambda(p)) \cap (-\infty;0) = \emptyset.$$

Then equalities (2.1), (4.1) and (4.2) complete the proof of assertion 1) of Theorem 2.2.

2) Let $\lambda\in(\lambda_0;\lambda_1]$. Then by assertion 2) of Lemma 3.2 there exists a non-empty open set $D_\lambda\subset\mathbb{T}^3$ such that $D_\lambda\neq\mathbb{T}^3$, and for any $p\in D_\lambda$ the operator $\mathbf{h}_\lambda(p)$ has a unique negative eigenvalue $e_\lambda(p)$. Since the function $u(\cdot)$ is continuous, $v(\cdot)$ and $w(\cdot,\cdot)$ are analytic on its domains, the function $e_\lambda : p\in D_\lambda \to e_\lambda(p)$ is continuous on D_λ.

Since for any $p\in\mathbb{T}^3$ the operator $\mathbf{h}_\lambda(p)$ is bounded and $\mathbb{T}^3$ is a compact set, there exists a positive number C_λ such that $\sup\limits_{p\in\mathbb{T}^3}\|\mathbf{h}_\lambda(p)\|\le C_\lambda$ and for any $p\in\mathbb{T}^3$ we have

$$\sigma(\mathbf{h}_\lambda(p)) \subset [-C_\lambda;C_\lambda]. \tag{4.3}$$

For any $q\in\partial D_\lambda = \{p\in\mathbb{T}^3 : \Delta_2(\lambda,p\,;0)=0\}$ there exist $\{p_n\}\subset D_\lambda$ such that $p_n\to q$ as $n\to\infty$. If we set $e_\lambda^{(n)} = e_\lambda(p_n)$, then by Lemma 3.2 for any $n\in\mathbb{N}$ the inequality $e_\lambda^{(n)}<0$ holds and from (4.3) we get $\{e_\lambda^{(n)}\}\subset[-C_\lambda;0)$. Without loss of generality we assume that $e_\lambda^{(n)}\to e_\lambda^{(0)}$ as $n\to\infty$ for some $e_\lambda^{(0)}\in[-C_\lambda;0]$ (otherwise we would have to take a subsequence).

By the continuity of the function $\Delta_2(\lambda,\cdot\,;\cdot)$ in $\mathbb{T}^3\times(-\infty;0]$ and $p_n\to q$ and $e_\lambda^{(n)}\to e_\lambda^{(0)}$ as $n\to\infty$ it follows that

$$0=\lim_{n\to\infty}\Delta_2(\lambda,p_n\,;e_\lambda^{(n)}) = \Delta_2(\lambda,q\,;e_\lambda^{(0)}).$$

Since for any $\lambda>0$ and $p\in\mathbb{T}^3$ the function $\Delta_2(\lambda,p\,;\cdot)$ is decreasing in $(-\infty;0]$ and $q\in\partial D_\lambda$ we see that $\Delta_2(\lambda,q\,;e_\lambda^{(0)})=0$ if and only if $e_\lambda^{(0)}=0$.

Now for $q \in \partial D_\lambda$ we define

$$e_\lambda(q) = \lim_{p' \to q,\, p' \in D_\lambda} e_\lambda(p') = 0.$$

Since the function $e_\lambda(\cdot)$ is continuous on the compact set $D_\lambda \cup \partial D_\lambda$ and $e_\lambda(q) = 0$ for all $q \in \partial D_\lambda$ we conclude that $\mathrm{Im} e_\lambda = [a_\lambda; 0]$ and $a_\lambda < 0$, where $\mathrm{Im} e_\lambda$ denotes an image of the function $e_\lambda(\cdot)$.

Hence the set

$$\bigcup_{p \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(\mathbf{h}_\lambda(p)) \cap (-\infty; 0]$$

coincides with the set $\mathrm{Im} e_\lambda = [a_\lambda; 0]$. Then equalities (2.1), (4.1) and (4.2) complete the proof of assertion 2) of Theorem 2.2.

3) Let $\lambda > \lambda_1$. Then by Lemma 3.2 for all $p \in \mathbb{T}^3$ the operator $\mathbf{h}_\lambda(p)$ has a unique negative eigenvalue $e_\lambda(p)$. Since the function $e_\lambda : p \in D_\lambda \to e_\lambda(p)$ is continuous on $\mathbb{T}^3$ the range $\mathrm{Im} e_\lambda$ of the function $e_\lambda(\cdot)$ is a connected closed subset of $(-\infty; 0)$, that is, $\mathrm{Im} e_\lambda = [a_\lambda; b_\lambda]$ with $b_\lambda < 0$ and hence

$$\bigcup_{p \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(\mathbf{h}_\lambda(p)) \cap (-\infty; 0] = [a_\lambda; b_\lambda].$$

Then again equalities (2.1), (4.1) and (4.2) complete the proof of assertion 3) of Theorem 2.2. □

5 Asymptotics for the number of negative eigenvalues of H_{22}

In this section first we review the corresponding Birman-Schwinger principle for the operator H_{22} and next we derive asymptotic relation (2.3) for the number of negative eigenvalues of H_{22}.

The Birman-Schwinger principle. For a bounded self-adjoint operator A acting in a Hilbert space $\mathcal{R}$, we define the number $n(\gamma, A)$ as follows

$$n(\gamma, A) := \sup\{\dim F : (Au, u) > \gamma,\, u \in F \subset \mathcal{R},\, ||u|| = 1\}.$$

Here $n(\gamma, A) = \infty$, if $\gamma < \max \sigma_{\mathrm{ess}}(A)$; if $n(\gamma, A) < \infty$, then it is equal to the number of eigenvalues of A which are greater than γ.

By the definition of $N(A; z)$, we have

$$N(H_{22}, z) = n(-z, -H_{22}),\; -z > -\tau_{\mathrm{ess}}(H_{22}).$$

Note that for any $\mu > 0$, $p \in \mathbb{T}^3$ and $z < \tau_{\mathrm{ess}}(H_{22})$ we have $\Delta_1(\mu, p\,; z) \geq 0$ and hence there exists its positive square root.

In our analysis of the discrete spectrum of H_{22} the crucial role is played by the compact integral operator $T_\mu(z)$, $z < \tau_{\mathrm{ess}}(H_{22})$ acting on $L_2(\mathbb{T}^3)$ with the kernel

$$\frac{\mu}{\sqrt{\Delta_1(\mu, p\,; z)}\sqrt{\Delta_1(\mu, q\,; z)}(w(p, q) - z)}.$$

The following lemma is a realization of the well-known Birman-Schwinger principle for the operator H_{22} (see [3, 8, 9, 15, 16, 19]).

Lemma 5.1. *For any $z < \tau_{\rm ess}(H_{22})$ the operator $T_\mu(z)$ is compact and continuous in z and*

$$N(H_{22}, z) = n(1, T_\mu(z)).$$

Proof of Theorem 2.3. Let us recall some results of [19] which are important in our work. Set $\sigma := L_2(\mathbb{S}^2)$, where $\mathbb{S}^2$ stands for the unit sphere in $\mathbb{R}^3$, and let $S_{\mathbf{r}}$, $\mathbf{r} > 0$, be the integral operator on $L_2((0,\mathbf{r}),\sigma)$ with the kernel

$$S(y,t) := \frac{1}{4\pi^2}\frac{(l_1+l_2)^2}{\sqrt{l_1^2+2l_1l_2}}\frac{1}{(l_1+l_2)\cosh y + l_2 t},$$

where $y = x - x'$, $x, x' \in (0,\mathbf{r})$, $t = \langle \xi, \eta\rangle$, $\xi, \eta \in \mathbb{S}^2$.

Let $\widehat{S}(\theta)$, $\theta \in \mathbb{R}$, be the integral operator on σ whose kernel is of the form

$$\widehat{S}(\theta,t) := \frac{1}{4\pi^2}\frac{(l_1+l_2)^2}{l_1^2+2l_1l_2}\frac{\sinh[\theta \arccos \frac{l_2}{l_1+l_2}t]}{\sinh(\pi\theta)},$$

and depends on the inner product $t = \langle \xi, \eta\rangle$ of the arguments $\xi, \eta \in \mathbb{S}^2$. For $\gamma > 0$, define

$$U(\gamma) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} n(\gamma, \widehat{S}(\theta))d\theta.$$

This function was studied in detail in [19]; where it was used in showing the existence of the Efimov effect. In particular, as it was shown in [19], the function $U(\cdot)$ is continuous in $\gamma > 0$, and the limit

$$\lim_{\mathbf{r}\to 0}\frac{1}{2}\mathbf{r}^{-1} n(\gamma, S_{\mathbf{r}}) = U(\gamma) \tag{5.1}$$

exists and the number $U(1)$ is positive.

Theorem 2.3 can be derived by using a perturbation argument based on the following lemma. □

Lemma 5.2. *Let $A(z) = A_0(z)+A_1(z)$, where $A_0(z)$ ($A_1(z)$) is compact and continuous for $z < 0$ (for $z \le 0$). Assume that the limit*

$$\lim_{z\to -0} f(z)\, n(\gamma, A_0(z)) = l(\gamma)$$

exists and $l(\cdot)$ is continuous in $(0;+\infty)$ for some function $f(\cdot)$, where $f(z) \to 0$ as $z \to 0$. Then the same limit exists for $A(z)$ and

$$\lim_{z\to -0} f(z)\, n(\gamma, A(z)) = l(\gamma).$$

For the proof of Lemma 5.2, we refer to Lemma 4.9 in [19].

Since the function $U(\cdot)$ is continuous with respect to γ, it follows by Lemma 5.2 that any perturbation of $A_0(z)$ treated in Lemma 5.2 (which is compact and continuous up to $z = 0$) does not contribute to asymptotic relation (2.3).

Let $T(\delta; |z|)$ be an integral operator in $L_2(\mathbb{T}^3)$ with the kernel

$$\frac{(l_1+l_2)^{3/2}}{16\pi^2}\sum_{i,j=1}^{8}\frac{\chi_\delta(p-p_i)\chi_\delta(q-p_j)(m|p-p_i|^2+|z|/2)^{-\frac14}(m|q-p_j|^2+|z|/2)^{-\frac14}}{(l_1+l_2)|p-p_i|^2+2l_2(p-p_i,q-p_j)+(l_1+l_2)|q-p_j|^2+|z|/2},$$

where $m := (l_1^2+2l_1l_2)/(l_1+l_2)$ and $\chi_\delta(\cdot)$ is the characteristic function of the domain $U_\delta(\mathbf{0})$, $\mathbf{0} := (0,0,0) \in \mathbb{T}^3$. The operator $T(\delta; |z|)$ is called singular part of $T_{\mu_0}(z)$.

Lemma 5.3. *For any $z \le 0$ and small $\delta > 0$ the difference $T_{\mu_0}(z) - T(\delta; |z|)$ belongs to the Hilbert-Schmidt class and is continuous with respect to $z \le 0$.*

Proof. Applying Corollary 3.1 and Lemma 3.4 we obtain that there exist $C_1, C_2 > 0$ such that the kernel of the operator $T_{\mu_0}(z) - T(\delta; |z|)$ can be estimated by the square-integrable function

$$C_1 + C_2 \sum_{i,j=1}^{8} \frac{|p-p_i|^{-\frac{1}{2}} + |q-p_j|^{-\frac{1}{2}} + 1}{|p-p_i|^2 + (p-p_i, q-p_j) + |q-p_j|^2}.$$

Hence, the operator $T_{\mu_0}(z) - T(\delta; |z|)$ belongs to the Hilbert-Schmidt class for all $z \le 0$. In combination with the continuity of the kernel of the operator with respect to $z < 0$, this implies the continuity of $T_{\mu_0}(z) - T(\delta; |z|)$ with respect to $z \le 0$. □

The following theorem is fundamental for the proof of the asymptotic relation (2.3).

Theorem 5.1. *We have the relation*

$$\lim_{|z|\to 0} \frac{n(\gamma, T(\delta; |z|))}{|\log|z||} = U(\gamma), \quad \gamma > 0. \tag{5.2}$$

Proof. The subspace of functions f, supported by the set $\bigcup\limits_{i=1}^{8} U_\delta(p_i)$ is invariant with respect to the operator $T(\delta; |z|)$.

Let $T_0(\delta; |z|)$ be the restriction of the integral operator $T(\delta; |z|)$ to the subspace $L_2(\bigcup\limits_{i=1}^{8} U_\delta(p_i))$, that is, integral operator in $L_2(\bigcup\limits_{i=1}^{8} U_\delta(p_i))$ with the kernel $T_0(\delta; |z|; \cdot, \cdot)$ defined on $\bigcup\limits_{i=1}^{8} U_\delta(p_i) \times \bigcup\limits_{j=1}^{8} U_\delta(p_j)$ as

$$T_0(\delta; |z|; p, q) :=$$
$$\frac{(l_1+l_2)^{3/2}}{16\pi^2} \frac{(m|p-p_i|^2 + |z|/2)^{-\frac{1}{4}} (m|q-p_j|^2 + |z|/2)^{-\frac{1}{4}}}{(l_1+l_2)|p-p_i|^2 + 2l_2(p-p_i, q-p_j) + (l_1+l_2)|q-p_j|^2 + |z|/2},$$

$(p, q) \in U_\delta(p_i) \times U_\delta(p_j)$ for $i, j = \overline{1, 8}$.

Since $L_2(\bigcup\limits_{i=1}^{8} U_\delta(p_i)) = \bigoplus\limits_{i=1}^{8} L_2(U_\delta(p_i))$, we can express the integral operator $T_0(\delta; |z|)$ as the block operator matrix $\mathbf{T}_0(\delta; |z|)$ acting on $\bigoplus\limits_{i=1}^{8} L_2(U_\delta(p_i))$ as

$$\mathbf{T}_0(\delta; |z|) := \begin{pmatrix} T_0^{(1,1)}(\delta; |z|) & \dots & T_0^{(1,8)}(\delta; |z|) \\ \vdots & \ddots & \vdots \\ T_0^{(8,1)}(\delta; |z|) & \dots & T_0^{(8,8)}(\delta; |z|) \end{pmatrix},$$

where $T_0^{(i,j)}(\delta; |z|) : L_2(U_\delta(p_j)) \to L_2(U_\delta(p_i))$ is an integral operator with the kernel $T_0(\delta; |z|; p, q)$, $(p, q) \in U_\delta(p_i) \times U_\delta(p_j)$ for $i, j = \overline{1, 8}$.

Set

$$L_2^{(8)}(U_r(\mathbf{0})) := \{\varphi = (\varphi_1, \cdots, \varphi_8) : \varphi_i \in L_2(U_r(\mathbf{0})),\ i = \overline{1,8}\}.$$

It is easy to show that $\mathbf{T}_0(\delta; |z|)$ is unitarily equivalent to the operator $\mathbf{T}_1(r)$, $r = |z|^{-\frac{1}{2}}$, acting on $L_2^{(8)}(U_r(\mathbf{0}))$ as

$$\mathbf{T}_1(r) := \begin{pmatrix} T_1^{(1,1)}(r) & \dots & T_1^{(1,1)}(r) \\ \vdots & \ddots & \vdots \\ T_1^{(1,1)}(r) & \dots & T_1^{(1,1)}(r) \end{pmatrix},$$

where $T_1^{(1,1)}(r)$ is the integral operator on $L_2(U_r(\mathbf{0}))$ with the kernel

$$\frac{(l_1+l_2)^{3/2}}{16\pi^2} \frac{(m|p|^2+1/2)^{-\frac{1}{4}}(m|q|^2+1/2)^{-\frac{1}{4}}}{(l_1+l_2)|p|^2+2l_2(p,q)+(l_1+l_2)|q|^2+1/2}.$$

The equivalence is realized by the unitary dilation

$$\mathbf{B}_r := \operatorname{diag}\{B_r^{(1,1)}, \dots, B_r^{(8,8)}\} : \bigoplus_{i=1}^{8} L_2(U_\delta(p_i)) \to L_2^{(8)}(U_r(\mathbf{0})),$$

Here the operator $B_r^{(i,i)} : L_2(U_\delta(p_i)) \to L_2(U_r(\mathbf{0}))$, $i = \overline{1,8}$ acts as

$$(B_r^{(i,i)} f)(p) = \left(\frac{r}{\delta}\right)^{-\frac{3}{2}} f(\frac{\delta}{r} p + p_i).$$

Denote by $\mathbf{A}_r$ and $\mathbf{C}$ the 8×1 and 1×8 matrices of the form

$$\mathbf{A}_r := \begin{pmatrix} T_1^{(1,1)}(r) \\ \vdots \\ T_1^{(1,1)}(r) \end{pmatrix}, \quad \mathbf{C} := (I, \dots, I),$$

respectively, where I is the identity operator on $L_2(U_r(\mathbf{0}))$.

It is well known that if T_1, T_2 are bounded operators and $\gamma \neq 0$ is an eigenvalue of T_1T_2, then γ is an eigenvalue for T_2T_1 as well of the same algebraic and geometric multiplicities (see *e.g.* [6]). Therefore, $n(\gamma, \mathbf{A}_r\mathbf{C}) = n(\gamma, \mathbf{C}\mathbf{A}_r)$, $\gamma > 0$. Direct calculation shows that $\mathbf{T}_1(r) = \mathbf{A}_r\mathbf{C}$ and $\mathbf{C}\mathbf{A}_r = 8T_1^{(1,1)}(r)$. So, $n(\gamma, \mathbf{T}_1(r)) = n(\gamma, 8T_1^{(1,1)}(r))$, $\gamma > 0$.

Further, replacing

$$(m|p|^2+1/2)^{\frac{1}{4}}, \quad (m|q|^2+1/2)^{\frac{1}{4}} \quad \text{and} \quad (l_1+l_2)|p|^2+2l_2(p,q)+(l_1+l_2)|q|^2+1/2$$

by the expressions

$$(m|p|^2)^{\frac{1}{4}}(1-\chi_1(p))^{-1}, \quad (m|q|^2)^{\frac{1}{4}}(1-\chi_1(q))^{-1} \quad \text{and} \quad (l_1+l_2)|p|^2+2l_2(p,q)+(l_1+l_2)|q|^2,$$

respectively, we obtain the integral operator $T_2(r)$. The error $8T_1^{(1,1)}(r) - T_2(r)$ is a Hilbert-Schmidt operator continuous up to $z = 0$.

Using the dilation

$$M : L_2(U_r(\mathbf{0}) \setminus U_1(\mathbf{0})) \to L_2((0,\mathbf{r}),\sigma), \quad (Mf)(x,w) = e^{3x/2} f(e^x w),$$

where $\mathbf{r} = \frac{1}{2}|\log|z||$, $x \in (0,\mathbf{r})$, $w \in \mathbb{S}^2$, one can see that the operator $T_2(r)$ is unitarily equivalent to the integral operator $S_{\mathbf{r}}$.

Since the difference of the operators $S_{\mathbf{r}}$ and $T(\delta;|z|)$ is compact (up to unitary equivalence) and hence, since $\mathbf{r} = 1/2|\log|z||$, we obtain the equality

$$\lim_{|z|\to 0} \frac{n(\gamma, T(\delta;|z|))}{|\log|z||} = \lim_{\mathbf{r}\to 0} \frac{1}{2}\mathbf{r}^{-1} n(\gamma, S_{\mathbf{r}}), \quad \gamma > 0.$$

Now (5.1) completes the proof of Theorem 5.1. □

Proof of Theorem 2.3. Using Lemmas 5.2, 5.3 and Theorem 5.1 we have

$$\lim_{|z|\to 0} \frac{n(1, T_{\mu_0}(z))}{|\log|z||} = U(1).$$

Taking into account the last equality and Lemma 5.1, and setting $\mathcal{U}_0 := U(1)$ we complete the proof of Theorem 2.3. □

6 Infiniteness of the number of eigenvalues embedded in the essential spectrum of H

Proof of Theorem 2.4. Let $\mu = \mu_0$. By Theorem 2.3 the operator H_{22} has infinitely many negative eigenvalues $E_1, \ldots, E_n, \ldots$, accumulating at zero. Let $f_2^{(1)}, \ldots, f_2^{(n)}, \ldots$ be the corresponding eigenfunctions.

Denote by $\mathcal{L}_0$ the subspace of all eigenfunctions of H_{22}, corresponding to the negative eigenvalues. We show that $H_{12}\Big|_{\mathcal{L}_0} = 0$. Let f_2 be the eigenfunction of H_{22} corresponding to the eigenvalue $z < 0$, that is, $H_{22} f_2 = z f_2$ or

$$f_2(p,q) = \frac{\mu}{w(p,q) - z}[\varphi(p) + \varphi(q)], \tag{6.1}$$

where

$$\varphi(p) = \int_{\mathbb{T}^3} f(p,s)ds. \tag{6.2}$$

Substituting expression (6.1) for f_2 in equality (6.2), we obtain

$$\varphi(p) = \int_{\mathbb{T}^3} \frac{\mu}{w(p,s) - z}[\varphi(p) + \varphi(s)]ds$$

or

$$\varphi(p) = \frac{\mu}{\Delta_1(\mu, p\,; z)} \int\limits_{\mathbb{T}^3} \frac{\varphi(s)ds}{w(p,s) - z}.$$

This implies that $\varphi(\cdot)$ is a periodic function of each variable with period π. Therefore, the function $f_2(\cdot,\cdot)$, defined by (6.1), is a periodic function in each six variables with period π. Hence this function satisfies condition (2.2):

$$\int_{\mathbb{T}^3} v(s) f_2(p,s) ds = 0,$$

that is, $H_{12} f_2 = 0$ for any $f_2 \in \mathcal{L}_0$.

By this, in particular, it follows that $H_{12} f_2^{(n)} = 0$ for any $n \in \mathbb{N}$. Therefore the numbers $E_1, \ldots, E_n, \ldots$ are eigenvalues of $\mathbf{H}$ and the corresponding eigenvectors have the form: $f^{(n)} = (0, f_2^{(n)})$, $n \in \mathbb{N}$.

If $\lambda \in (0; \lambda_0]$, then by Theorem 2.2 we have $\min \sigma_{\text{ess}}(\mathbf{H}) = 0$. In this case the set $\{E_n : n \in \mathbb{N}\}$ is located below the bottom of the essential spectrum of $\mathbf{H}$ and $\lim\limits_{n\to\infty} E_n = 0$. Let $\lambda \in (\lambda_0; \lambda_1]$. Then Theorem 2.2 implies that $\sigma_{\text{ess}}(\mathbf{H}) = [a_\lambda; M]$ with $a_\lambda < 0$. Hence, the countable part of the set $\{E_n : n \in \mathbb{N}\}$ is located in the essential spectrum of $\mathbf{H}$. If $\lambda > \lambda_1$, then $\sigma_{\text{ess}}(\mathbf{H}) = [a_\lambda; b_\lambda] \cup [0; M]$, $b_\lambda < 0$. It means that the countable subset of the set $\{E_n : n \in \mathbb{N}\}$ located in $(b_\lambda; 0)$. $\square$

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