ISSN 2077–9879

Eurasian Mathematical Journal

2014, Volume 5, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Short communications

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 5, Number 1 (2014), 135 – 140

ON VOLTERRA RELATIVELY COMPACT PERTURBATIONS OF THE LAPLACE OPERATOR

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Communicated by M. Otelbaev

Key words: Laplace operator, Volterra operator, correct restrictions of the maximal operator, correct extensions of the minimal operator, relatively compact perturbations.

AMS Mathematics Subject Classification: 35P05, 58J50.

Abstract. We formulate the result on existence of a relatively compact perturbation of the Laplace operator which possesses the Volterra property.

1 Introduction

In a Hilbert space H, we consider a linear operator L with the domain $D(L)$, and with the range $R(L)$. By the kernel of the operator L we mean the set

$$\operatorname{Ker} L = \big\{ f \in D(L) : \ Lf = 0 \big\}.$$

Definition 1.1. *An operator L is called a restriction of an operator L_1, and L_1 is called an extension of an operator L, briefly $L \subset L_1$, if:*
1) $D(L) \subset D(L_1)$,
2) $Lf = L_1 f$ for all f from $D(L)$.

Definition 1.2. *A linear closed operator L_0 in a Hilbert space H is called minimal if $\overline{R(L_0)} \neq H$ and there exists a bounded inverse operator L_0^{-1} on $R(L_0)$.*

Definition 1.3. *A linear closed operator $\widehat{L}$ in a Hilbert space H is called maximal if $R(\widehat{L}) = H$ and $Ker\,\widehat{L} \neq \{0\}$.*

Definition 1.4. *A linear closed operator L in a Hilbert space H is called correct if there exists a bounded inverse operator L^{-1} defined on all of H.*

Definition 1.5. *We say that a correct operator L in a Hilbert space H is a correct extension of minimal operator L_0 (correct restriction of maximal operator $\widehat{L}$) if $L_0 \subset L$ $(L \subset \widehat{L})$.*

Definition 1.6. *We say that a correct operator L in a Hilbert space H is a boundary correct extension of a minimal operator L_0 with respect to a maximal operator $\widehat{L}$ if L is simultaneously a correct restriction of the maximal operator $\widehat{L}$ and a correct extension of the minimal operator L_0, that is, $L_0 \subset L \subset \widehat{L}$.*

At the beginning of the 1980s, M. Otelbaev and his disciples proved abstract theorems (see [9, 7]), which allow us to describe all correct extensions of a minimal operator by using one correct extension in terms of an inverse operator. Similarly, all possible correct restrictions of a maximal operator were described. For convenience, we present the conclusions of these theorems.

Let $\widehat{L}$ be a maximal linear operator in a Hilbert space H, L be a correct restriction of the operator $\widehat{L}$, and K be an arbitrary linear bounded operator in H satisfying the following condition:

$$R(K) \subset \operatorname{Ker} \widehat{L}. \tag{1.1}$$

Then the operator L_K^{-1} defined by the formula

$$L_K^{-1} f = L^{-1} f + K f, \tag{1.2}$$

describes the inverse operators of all possible correct restrictions L_K of the maximal operator $\widehat{L}$.

Let L_0 be a minimal operator in a Hilbert space H, L be a correct extension of the minimal operator L_0, and K be a linear bounded operator in H satisfying the conditions

a) $R(L_0) \subset \operatorname{Ker} K$,

b) $\operatorname{Ker}(L^{-1} + K) = \{0\}$,

then the operator L_K^{-1} defined by formula (1.2) describes the inverse operators of all possible correct extensions L_K of the minimal operator L_0.

Let L be any known boundary correct extension of the minimal operator L_0. The existence of at least one boundary correct extension L was proved by Vishik in [12]. Let K be a linear bounded operator in H satisfying the conditions

a) $R(L_0) \subset \operatorname{Ker} K$,

b) $R(K) \subset \operatorname{Ker} \widehat{L}$,

then the operator L_K^{-1} defined by formula (1.2) describes the inverse operators of all possible boundary correct extensions L_K of the minimal operator L_0.

Definition 1.7. *A bounded operator A in a Hilbert space H is called quasinilpotent if its spectral radius is zero, that is, the spectrum consists of the single point zero.*

Definition 1.8. *An operator A in a Hilbert space H is called a Volterra operator if A is compact and quasinilpotent.*

Definition 1.9. *A correct restriction L of a maximal operator $\widehat{L}$ ($L \subset \widehat{L}$), a correct extension L of a minimal operator L_0 ($L_0 \subset L$) or a boundary correct extension L of a minimal operator L_0 with respect to a maximal operator $\widehat{L}$ ($L_0 \subset L \subset \widehat{L}$), will be called Volterra if the inverse operator L^{-1} is a Volterra operator.*

We denote by $\mathfrak{S}_\infty(H,H_1)$ the set of all linear compact operators acting from a Hilbert space H to a Hilbert space H_1. If $T\in\mathfrak{S}_\infty(H,H_1)$, then T^*T is a non-negative self-adjoint operator in $\mathfrak{S}_\infty(H)\equiv\mathfrak{S}_\infty(H,H)$ and, moreover, there is a non-negative unique self-adjoint root $|T|=(T^*T)^{1/2}$ in $\mathfrak{S}_\infty(H)$. The eigenvalues $\lambda_n(|T|)$ numbered, taking into account their multiplicity, form a monotonically converging to zero sequence of non-negative numbers. These numbers are usually called *s-numbers* of the operator T and denoted by $s_n(T)$, $n\in\mathbb{N}$. We denote by $\mathfrak{S}_p(H,H_1)$ the set of all compact operators $T\in\mathfrak{S}_\infty(H,H_1)$, for which

$$|T|_p^p=\sum_{j=1}^\infty s_j^p(T)<\infty,\quad 0<p<\infty.$$

Obviously, if rank rank $|T|=r<\infty$, then $s_n(T)=0$, for $n=r+1,r+2,\dots$. Operators of finite rank certainly belong to the classes $\mathfrak{S}_p(H,H_1)$ for all $p>0$.

In a recent paper by the author (see [2]) a wide class was described of correct restrictions and extensions for the Laplace operator, which are not Volterra. It is noticed that in the one-dimensional case (see [4]) the Volterra property for the case of Sturm-Liouville equations depends on the behaviour of the lower terms. Then the following question arises: Can lower terms help to ensure the Volterra property for the Laplace operator?

The present paper is devoted to the study of this question.

In the Hilbert space $L_2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^m$ with infinitely smooth boundary $\partial\Omega$, let us consider the minimal L_0 and maximal $\widehat{L}$ operators generated by the Laplace operator

$$-\Delta u=-\left(\frac{\partial^2u}{\partial x_1^2}+\frac{\partial^2u}{\partial x_2^2}+\cdots+\frac{\partial^2u}{\partial x_m^2}\right). \tag{1.3}$$

The closure L_0 in the space $L_2(\Omega)$ of the Laplace operator (1.3) with the domain $C_0^\infty(\Omega)$ is called the *minimal operator corresponding to the Laplace operator.*

The operator $\widehat{L}$, adjoint to the minimal operator L_0 corresponding to the Laplace operator is called the *maximal operator corresponding to the Laplace operator* (see [6]). Note that

$$D(\widehat{L})=\{u\in L_2(\Omega):\ \widehat{L}u=-\Delta u\in L_2(\Omega)\}.$$

Denote by L_D the operator, corresponding to the Dirichlet problem with the domain

$$D(L_D)=\{u\in W_2^2(\Omega):\ u|_{\partial\Omega}=0\}.$$

Then, by virtue of (1.2), the inverse operators L^{-1} to all possible correct restrictions of the maximal operator $\widehat{L}$, corresponding to the Laplace operator (1.3) have the following form:

$$u\equiv L^{-1}f=L_D^{-1}f+Kf, \tag{1.4}$$

where, by virtue of (1.1), K is an arbitrary linear operator bounded in $L_2(\Omega)$ with

$$R(K)\subset\operatorname{Ker}\widehat{L}=\{u\in L_2(\Omega):\ -\Delta u=0\}.$$

Then the direct operator L is determined from the following problem:

$$\widehat{L}u \equiv -\Delta u = f, \quad f \in L_2(\Omega), \tag{1.5}$$

$$D(L) = \{u \in D(\widehat{L}) : \ (I - K\widehat{L})u|_{\partial\Omega} = 0\}, \tag{1.6}$$

where I is the unit operator in $L_2(\Omega)$.

The operators $(L^*)^{-1}$, corresponding to the operators L^*

$$v \equiv (L^*)^{-1}g = L_D^{-1}g + K^*g, \tag{1.7}$$

describe the inverses of all possible correct extensions of the minimal operator L_0 if and only if K satisfies the condition (see [3]):

$$\mathrm{Ker}(L_D^{-1} + K^*) = \{0\}.$$

Note that the last condition is equivalent to the following: $\overline{D(L)} = L_2(\Omega)$. If the operator K in (1.4), satisfies one more additional condition

$$KR(L_0) = \{0\},$$

then the operator L, corresponding to problem (1.5)-(1.6), will turn out to be a boundary correct extension.

Now we state the main result.

2 Main results

A linear operator acting in a separable Hilbert space H is called complete if the system of its root vectors corresponding to nonzero eigenvalues is complete in H. By a weak perturbation of a complete compact self-adjoint operator A we mean an operator $A(I+C)$, where C is a compact operator such that the operator $I + C$ is continuously invertible.

Let the operator K in (1.4) be compact and positive in $L_2(\Omega)$. We denote by $\{\mu_n\}_1^\infty$ and $\{\lambda_n\}_1^\infty$ eigenvalues of the operator K and L^{-1} respectively numbered in descending order taking into account their multiplicities. We need the following

Lemma 2.1. *If the eigenvalues $\{\mu_n\}_1^\infty$ of the operator K from the representation* (1.4) *satisfy the condition*

$$\lim_{n\to\infty} \mu_{2n}\mu_n^{-1} = 1, \tag{2.1}$$

then the eigenvalues $\{\lambda_n\}_1^\infty$ of the operator L^{-1} in (1.4) *also have the property*

$$\lim_{n\to\infty} \lambda_{2n}\lambda_n^{-1} = 1.$$

In the proof of Lemma 2.1 some results of works [1, p. 16], [11, p. 41] and [5, p. 52] are applied.

So, the operator L^{-1}, corresponding to problem (1.5)-(1.6), satisfies all assumptions of Theorem A from the work of Macaev and Mogul'ski which states:

Theorem A (Macaev, Mogul'ski [8]) *Let the eigenvalues* $\{\tilde{\lambda}_i\}_1^\infty$ *of a complete positive compact operator* A *numbered in descending order taking into account their multiplicities satisfy the condition*

$$\lim_{n\to\infty} \tilde{\lambda}_{2n}\tilde{\lambda}_n^{-1} = 1.$$

Then there is a weak perturbation of the operator A *which is a Volterra operator.*

Theorem 2.1. *Let the operator* K *in problem* (1.5)-(1.6) *be compact and positive in* $L_2(\Omega)$ *and let its eigenvalues* $\{\mu_j\}_1^\infty$ *satisfy the condition* (2.1). *Then there exists a compact operator* S *in* $L_2(\Omega)$ *such that the relatively compact perturbation of the Laplace operator* L_S *defined by*

$$\begin{cases} \widehat{L}_S u = -\Delta u + S(-\Delta u) = f, \quad f \in L_2(\Omega), \\ D(L_S) = \{u \in D(\widehat{L}) : \ (I - K\widehat{L})u|_{\partial\Omega} = 0\}, \end{cases} \tag{2.2}$$

is a correct Volterra boundary problem.

Idea of the proof. By the assumptions of Theorem 2.1 it follows that the direct operator L determined from problem (1.5)-(1.6) is positive since its inverse operator L^{-1} of form (1.4) is a positive and compact operator in $L_2(\Omega)$.

Using Theorem A we obtain that there is a Volterra weak perturbation $L_S^{-1} = L^{-1}(I + S_1)$ for the operator L^{-1}. By definition the operator S_1 is compact and $I + S_1$ is continuously invertible. We denote

$$(I + S_1)^{-1} = I + S,$$

where S is a compact operator in $L_2(\Omega)$. Then problem (2.2) defines the direct operator L_S. Note that

$$D(L_S) = D(L),$$

and that L_S is a relatively compact perturbation of the Laplace operator $\widehat{L}$. □

If we consider $\operatorname{Ker}\widehat{L}$ as an independent Hilbert space, the existence of the operator K satisfying the conditions of Theorem 2.1 follows from the general theory (see [10]).

For example, the operator K which satisfies the conditions of Theorem 2.1 may have the eigenvalues μ_n with the asymptotics

$$\mu_n \approx 1/ln(n+2), \quad \text{as} \quad n \to \infty.$$

Note that in this case, the operator K is compact but not in the Schatten classes.

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Received: 10.02.2014

EURASIAN MATHEMATICAL JOURNAL
2014 – Том 5, № 1 – Астана: ЕНУ. – 141 с.
Подписано в печать 26.03.2014 г. Тираж – 120 экз.

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Дизайн: К. Булан

Отпечатано в типографии ЕНУ имени Л.Н.Гумилева

Свидетельство о постановке на учет печатного издания
Министерства культуры и информации Республики Казахстан
№ 10330 – Ж от 25.09.2009 г.