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SHARP INEQUALITY OF JACKSON–STECHKIN TYPE AND WIDTHS OF FUNCTIONAL CLASSES IN THE SPACE L_2

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Abstract. For classes of differentiable periodic functions, defined by means of generalized moduli of continuity $\Omega_m(f,t)$, satisfying the condition

$$\left(\int_0^h \Omega_m^{2/m}(f^{(r)},t)dt\right)^{m/2} \leq \Phi(h),$$

where $m \in \mathbb{N}, \quad r \in \mathbb{Z}_+, h > 0$ and Φ is a given majorant, under certain restrictions on the majorant, the exact values of various n-widths in the space L_2 are calculated.

1 Setting of extremal problems

Let $L_2 =: L_2[0,2\pi]$ be the space of all real-valued Lebesgue measurable 2π-periodic functions f with finite norm

$$\|f\| := \|f\|_{L_2} = \left(\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx\right)^{1/2}.$$

By $\mathcal{T}_{2n-1}$ we denote the space of all trigonometric polynomials

$$T_{n-1}(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n-1}(\alpha_k \cos kx + \beta_k \sin kx)$$

of order not exceeding $n-1$. It is well known that for an arbitrary function $f \in L_2$ with the Fourier series

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx)$$

the value of its best approximation by elements of the space $\mathcal{T}_{2n-1}$ is equal to

$$E_{n-1}(f) = \inf\left\{\|f - T_{n-1}\| : T_{n-1}(x) \in \mathcal{T}_{2n-1}\right\}$$
$$= \|f - S_{n-1}(f)\| = \left\{\sum_{k=n}^{\infty} \rho_k^2(f)\right\}^{1/2}, \tag{1.1}$$

where

$$S_{n-1}(f,x) = \sum_{k=1}^{n-1} (a_k(f)\cos kx + b_k(f)\sin kx)$$

is the partial sum of order $n-1$ of the Fourier series of the function f and $\rho_k^2 = a_k^2(f) + b_k^2(f)$.

By the modulus of continuity of order m ($m \in \mathbb{N}$) of a function $f \in L_2$ we mean the quantity

$$\omega_m(f,t) := \sup\left\{\|\Delta_h^m(f\cdot)\| : |h| \le t\right\}, \tag{1.2}$$

where

$$\Delta_h^m(f,x) = \sum_{k=0}^{m} (-1)^{m-k}\binom{m}{k} f(x+kh)$$

is the difference of order m of the function f with step h at the point x.

By $L_2^{(r)}(r \in \mathbb{Z}_+; L_2^0 \equiv L_2)$ we denote the space of all 2π-periodic functions $f \in L_2$ whose derivatives of order $(r-1)$ are absolutely continuous and the derivatives of order r are such that $f^{(r)} \in L_2$.

By inequalities of Jackson–Stechkin type in a normed space X inequalities are meant in which the best approximation $E_{n-1}(f)_X$ of the function $f \in X$ by a finite-dimensional subspace $\mathfrak{N}_n \subset X$ is estimated via the modulus of continuity of the function itself or its derivative:

$$E_{n-1}(f)_X \le \mathcal{X} n^{-r} \omega_m\left(f^{(r)}, \frac{t}{n}\right)_X, \tag{1.3}$$

where $t > 0$, $f^{(r)} \in X$, $r \in \mathbb{Z}_+$, $f^0 \equiv f$, $m \in \mathbb{N}$, and $\mathcal{X}$ is independent of f and n.

Pointing at importance of studying the problem of minimization of the constant $\mathcal{X}$ in inequality (1.3), the authors of the monograph [2] emphasize that “the interest to sharp constants, which arose in connection with inequalities of Jackson–Stechkin type, could be not that justified if each new case did not require implementing new ideas and methods, which later appeared to be useful for solving other extremal problems”.

When solving extremal problems of the approximation theory for periodic differentiable functions $f \in L_2$, related to finding the sharp constant in inequality (1.3), mathematicians considered various extremal characteristics. (See, for example, [1], [3], [4], [6]–[10], [12]–[16].) In particular, in [13] the following extremal characteristic

$$\mathcal{X}_{m,n,r}(h) = \sup\left\{ n^r E_{n-1}(f)\left(\int_0^h \omega_m^{2/m}(f^{(r)},t)dt\right)^{-m/2} : f \in L_2^{(r)},\ f^{(r)} \ne const\right\},$$

was considered, where $m,n\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $0<h\le\pi/n$, and it was proved that

$$\mathcal{X}_{m,n,r}(h)=\left\{\frac{n}{2(nh-\sin nh)}\right\}^{m/2}. \tag{1.4}$$

Equality (1.4) implies in particular for $m=1$ the result due to L.V. Taikov [10].

Sometimes, when solving similar extremal problems of the approximation theory for periodic differentiable functions $f\in L_2$, it is more convenient to use the following characteristic equivalent to quantity (1.2)

$$\Omega_m(f,t)_2=\left\{\frac{1}{t^m}\int_0^t\cdots\int_0^t\left\|\Delta_{\overline{h}}^m f(\cdot)\right\|^2dh_1\cdots dh_m\right\}^{1/2},\ t>0, \tag{1.5}$$

where $\overline{h}=(h_1,h_2,\cdots,h_m)$,

$$\Delta_{\overline{h}}^m=\Delta_{h_1}^1\circ\cdots\circ\Delta_{h_m}^1,\quad \Delta_{h_j}^1f=f(\cdot+h_j)-f(\cdot),\quad j=\overline{1,m}.$$

(See, for example, [12], [14]). In order to obtain an analogue of equality (1.4) we introduce the following extremal characteristic for the modulus of continuity (1.5)

$$\mathcal{K}_{m,n,r}(h)=\sup\left\{n^rE_{n-1}(f)\left(\int_0^h\Omega_m^{2/m}(f^{(r)},t)dt\right)^{-m/2};\ f\in L_2^{(r)},\ f^{(r)}\ne const\right\} \tag{1.6}$$

and we shall prove the following statement.

Theorem 1.1. *Let $m,n\in\mathbb{N}$, $r\in\mathbb{Z}_+$ and $0<h\le\pi/n$. Then the following equalities*

$$\mathcal{K}_{m,n,r}(h)=\left\{\frac{n}{2(nh-Si(nh))}\right\}^{m/2}, \tag{1.7}$$

hold, where $Si(t)=\int_0^t\frac{\sin x}{x}dx$ is the integral sine.

Proof. Indeed, $f\in L_2^{(r)}$ and

$$f(x)\sim\frac{1}{2}a_0(f)+\sum_{k=1}^{\infty}(a_k(f)\cos kx+b_k(f)\sin kx)$$

is the Fourier series for f, by direct calculations we get

$$\Omega_m^2\left(f^{(r)},t\right)=2^m\sum_{k=1}^{\infty}k^{2r}\rho_k^2\left(1-\frac{\sin kt}{kt}\right)^m, \tag{1.8}$$

where $\rho_k^2=\rho_k^2(f)=a_k^2(f)+b_k^2(f)$, $k\in\mathbb{N}$. By Hölder's inequality for sums and by formulas (1.1) and (1.8), we have

$$E_{n-1}^2(f)-\sum_{k=n}^{\infty}\rho_k^2\frac{\sin kt}{kt}=\sum_{k=n}^{\infty}\rho_k^2\left(1-\frac{\sin kt}{kt}\right)$$

$$= \sum_{k=n}^{\infty} \rho_k^{2(1-1/m)} \cdot \rho_k^{2/m} \left(1 - \frac{\sin kt}{kt}\right) \leq \left(\sum_{k=n}^{\infty} \rho_k^2\right)^{1-1/m} \left(\sum_{k=n}^{\infty} \rho_k^2 \left(1 - \frac{\sin kt}{kt}\right)^m\right)^{1/m}$$

$$\leq \left(E_{n-1}(f)\right)^{1-1/m} \cdot \frac{1}{2n^{2r/m}} \cdot \Omega_m^{2/m}\left(f^{(r)}; t\right).$$

This implies that

$$E_{n-1}^2(f) \leq \sum_{k=n}^{\infty} \rho_k^2 \frac{\sin kt}{kt} + \left(E_{n-1}^2(f)\right)^{1-1/m} \cdot \frac{1}{2n^{2r/m}} \cdot \Omega_m^{2/m}\left(f^{(r)}; t\right).$$

By integrating this inequality in t from 0 to h and by using the definition of the integral sine, we get

$$hE_{n-1}^2(f) \leq \sum_{k=n}^{\infty} \rho_k^2 \frac{Si(kh)}{k} + \left(E_{n-1}^2(f)\right)^{1-1/m} \cdot \frac{1}{2n^{2r/m}} \cdot \int_0^h \Omega_m^{2/m}\left(f^{(r)}; t\right) dt. \tag{1.9}$$

Dividing both parts of inequality (1.9) by $h > 0$ and taking into account that the function $Si(x)/x$ is non-increasing on $[0, \infty)$ [15], we have

$$\max\left\{\frac{Si(kh)}{kh} : k \geq n\right\} = \frac{Si(nh)}{nh}, \quad 0 < nh \leq \pi.$$

Therefore from inequality (1.9) it follows that

$$\left(1 - \frac{Si(nh)}{nh}\right)^{m/2} \cdot E_{n-1}(f) \leq \frac{1}{(2h)^{m/2}} \cdot \frac{1}{n^r} \left(\int_0^h \Omega_m^{2/m}\left(f^{(r)}; t\right) dt\right)^{m/2}. \tag{1.10}$$

Inequality (1.10) implies that

$$\frac{n^r \cdot E_{n-1}(f)}{\left(\int_0^h \Omega_m^{2/m}\left(f^{(r)}, t\right) dt\right)^{m/2}} \leq \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2},$$

hence, taking into account the definition of quantity (1.6), we obtain the following estimate above

$$\mathcal{K}_{m,n,r}(h) \leq \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}. \tag{1.11}$$

In order to prove equality (1.7), it suffices to consider the function $f_0(x) = \cos nx \in L_2$ for which

$$E_{n-1}(f_0) = 1,\ \Omega_m^2\left(f_0^{(r)}; t\right) = 2^m n^{2r}\left(1 - \frac{\sin nt}{nt}\right)^m,\ 0 < nt \leq \pi, \tag{1.12}$$

hence we obtain the estimate below

$$\mathcal{K}_{m,n,r}(h) \geq \frac{n^r \cdot E_{n-1}(f_0)}{\left(\int\limits_0^h \Omega_m^{2/m}\left(f_0^{(r)}, t\right) dt\right)^{m/2}} = \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}. \tag{1.13}$$

Comparing inequalities (1.11) and (1.13), we get equality (1.7). □

As a corollary of Theorem 1.1 we get the following statement.

Theorem 1.2. *For each $m, n \in \mathbb{N}$, and $r \in \mathbb{Z}_+$ the following inequality*

$$\frac{1}{2^{m/2}} \leq \sup\left\{\frac{n^r E_{n-1}(f)}{\Omega_m(f^{(r)}, \pi/n)} : f \in L_2^{(r)},\ f^{(r)} \neq const\right\} \leq \left\{\frac{\pi}{2(\pi - Si(\pi))}\right\}^{m/2}. \tag{1.14}$$

holds. The lower estimate in (1.14) *is attained for the function* $f_0(x) = \cos nx \in L_2^{(r)}$.

Proof. Indeed, taking into account that the modulus of continuity $\Omega_m(f^{(r)}, t)$ is an increasing function, by (1.10) we get

$$\left(1 - \frac{Si(nh)}{nh}\right)^{m/2} \cdot E_{n-1}(f) \leq \frac{1}{2^{m/2}} \cdot \frac{1}{n^r} \cdot \Omega_m\left(f^{(r)}, h\right). \tag{1.15}$$

Taking $h = \pi/n$ in (1.15) we can get for any $f \in L_2^{(r)}$ the estimate above

$$\frac{n^r E_{n-1}(f)}{\Omega_m(f^{(r)}, \pi/n)} \leq \left\{\frac{\pi}{2(\pi - Si(\pi))}\right\}^{m/2}. \tag{1.16}$$

On the other hand, by using the function $f_0(x) = \cos nx \in L_2^{(r)}$ and taking into account equality (1.12), we get the estimate below.

$$\sup_{\substack{f \in L_2^{(r)} \\ f^{(r)} \neq const}} \frac{n^r E_{n-1}(f)}{\Omega_m(f^{(r)}, \pi/n)} \geq \frac{n^r E_{n-1}(f_0)}{\Omega_m(f_0^{(r)}, \pi/n)} = \frac{1}{2^{m/2}}. \tag{1.17}$$

Estimates (1.16) and (1.17) imply the desired inequality (1.14). □

2 Widths of classes of functions

First we recall the notions and definitions required for for formulation of further results.

Let $S = \{x,\ \|x\| \leq 1\}$ be the unit ball in L_2, $\mathfrak{N}$ a convex centrally symmetric subset of L_2, $\Lambda_n \subset L_2$ an n-dimensional subspace , $\Lambda^n \subset L_2$ a subspace of codimension n, $\mathcal{L} : L_2 \to \Lambda_n$ a continuous linear operator, and $\mathcal{L}^\perp : L_2 \to \Lambda_n$ be the continuous linear projection operator.

The quantities

$$b_n(\mathfrak{N}, L_2) = \sup\left\{\sup\left\{\varepsilon > 0;\ \varepsilon S \cap \Lambda_{n+1} \subset \mathfrak{N}\right\} : \Lambda_{n+1} \subset L_2\right\},$$

$$d_n(\mathfrak{N}, L_2) = \inf\left\{\sup\left\{\inf\left\{\|f-\varphi\| : \varphi \in \Lambda_n\right\} : f \in \mathfrak{N}\right\} : \Lambda_n \subset L_2\right\},$$

$$\lambda_n(\mathfrak{N}, L_2) = \inf\left\{\inf\left\{\sup\left\{\|f-\mathcal{L}f\| : f \in \mathfrak{N}\right\} : \mathcal{L}L_2 \subset \Lambda_n\right\} : \Lambda_n \subset L_2\right\},$$

$$d^n(\mathfrak{N}, L_2) = \inf\left\{\sup\left\{\|f\| : f \in \mathfrak{N} \cap \Lambda^n\right\} : \Lambda^n \subset L_2\right\},$$

$$\Pi_n(\mathfrak{N}, L_2) = \inf\left\{\inf\left\{\sup\left\{\|f-\mathcal{L}^{\perp}f\| : f \in \mathfrak{N}\right\} : \mathcal{L}^{\perp}L_2 \subset \Lambda_n\right\} : \Lambda_n \subset L_2\right\}$$

are called the Bernshtein, Kolmogorov, linear, Gelfand and projection n-widths in the space L_2. Since L_2 is a Hilbert space, the following relations for the above n-widths hold [11], [5]:

$$b_n(\mathfrak{N}, L_2) \le d^n(\mathfrak{N}, L_2) \le d_n(\mathfrak{N}, L_2) = \lambda_n(\mathfrak{N}, L_2) = \Pi_n(\mathfrak{N}, L_2). \tag{2.1}$$

Let also

$$E_{n-1}(\mathfrak{N}) := \sup\{E_{n-1}(f) : f \in \mathfrak{N}\}.$$

We shall call a majorant any continuous increasing function $\Phi(t)$ on $[0,\infty)$ such that $\Phi(0)=0$. The set of all majorants we denote by $\mathfrak{M}$. By $\mathfrak{M}_k$ with $k \in \mathbb{N}$ we denote the set of all majorants $\Phi \in \mathfrak{M}$, satisfying the following conditions:

1) $t_1^{-k}\Phi(t_1) < t_2^{-k}\Phi(t_2)$, if $0 < t_1 < t_2 < \infty$;

2) $\lim\limits_{t\to 0_+} t^{-k}\Phi(t) = 0$.

For $m \in \mathbb{N}$, $r \in \mathbb{Z}_+$ and $h > 0$ we introduce the following classes of functions:

$$W^{(r)}(\Omega_m, h) := \left\{ f \in L_2^{(r)} : \int\limits_0^h \Omega_m^{2/m}\left(f^{(r)}, t\right) dt \le 1 \right\},$$

$$W_1^{(r)}(\Omega_m, \Phi) := \left\{ f \in L_2^{(r)} : \left(\int\limits_0^h \Omega_m^{2/m}\left(f^{(r)}, t\right) dt \right)^{m/2} \le \Phi(h) \right\},$$

where $\Phi \in \mathfrak{M}_1$. Following paper [14], we denote by t_* the value of the argument of the function $\sin t/t$, at which it attains the minimal value on $[0,\infty)$. Clearly t_* is the minimal positive root of the equation $t = \tan t$, $4,49 < t_* < 4,51$. Let

$$\left(1 - \frac{\sin t}{t}\right)_* := \left\{1 - \frac{\sin t}{t}, \text{ if } 0 \le t \le t_*;\ 1 - \frac{\sin t_*}{t_*}, \text{ if, } t \ge t_*\right\}.$$

Theorem 2.1. *Let $nh \le t_*$. then the following equalities hold*

$$\gamma_{2n}\left(W^{(r)}(\Omega_m, h); L_2\right) = \gamma_{2n-1}\left(W^{(r)}(\Omega_m, h); L_2\right)$$

$$= E_{n-1}\left(W^{(r)}(\Omega_m, h)\right) = \frac{1}{n^r}\left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}, \tag{2.2}$$

where $\gamma_n(\cdot)$ is any of the n-widths mentioned above.

Proof. By inequality (1.10) we get that for arbitrary function $f \in L_2^{(r)}$

$$E_{n-1}(f) \le \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2} \cdot \frac{1}{n^r}\left(\int_0^h \Omega_m^{2/m}(f^{(r)},t)dt\right)^{m/2} \tag{2.3}$$

hence, taking into account the definition of the class $W^{(r)}(\Omega_m,h)$ and also relations (2.1), we have

$$\gamma_{2n}\left(W^{(r)}(\Omega_m,h);L_2\right) \le \gamma_{2n-1}\left(W^{(r)}(\Omega_m,h);L_2\right)$$

$$\le E_{n-1}\left(W^{(r)}(\Omega_m,h)\right) \le \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}\frac{1}{n^r}. \tag{2.4}$$

For obtaining an estimate below we consider the ball of trigonometric polynomials

$$S_{2n+1} := \left\{T_n \in \mathcal{T}_{2n+1} : \|T_n\| \le \frac{1}{n^r}\left(\frac{n}{2(nh - Si(nh))}\right)^{m/2}\right\}$$

and, by using the following inequality proved in [12]

$$\Omega_m\left(T_n^{(r)},\tau\right) \le 2^{m/2}n^r\left(1 - \frac{\sin n\tau}{n\tau}\right)_*^{m/2} \cdot \|T_n\|, \tag{2.5}$$

we shall show that the ball S_{2n+1} is contained in the class $W^{(r)}(\Omega_m,h)$. Indeed, taking into accopu8nt that $nh \le t_*$, by (2.5) we get

$$\int_0^h \Omega_m^{2/m}\left(T_n^{(r)},\tau\right)d\tau \le 2n^{2r/m}\int_0^h\left(1 - \frac{\sin n\tau}{n\tau}\right)_* d\tau\ \|T_n\|^{2/m}$$

$$\le n^{2r/m} \cdot \frac{2(nh - Si(nh))}{n} \cdot \frac{n}{2(nh - Si(nh))} \cdot \frac{1}{n^{2r/m}} = 1,$$

which proves the inclusion $S_{2n+1} \subset W^{(r)}(\Omega_m,h)$.

By the definition of the Bernshtein n-width and inequalities (2.1), we get

$$\gamma_{2n}\left(W^{(r)}(\Omega_m,h);L_2\right) \ge b_{2n}\left(W^{(r)}(\Omega_m,h);L_2\right)$$

$$\ge b_{2n}(S_{2n+1},L_2) \ge \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}\frac{1}{n^r}. \tag{2.6}$$

Comparing inequalities (2.4) and (2.6), we get the desired equality (2.2). □

Theorem 2.1 implies the following statements.

Corollary 2.1. *Under the assumptions of Theorem* 2.1 *for all* $n \in \mathbb{N}$

$$\sup\left\{|a_n(f)|,\,|b_n(f)| : f \in W^{(r)}(\Omega_m,h)\right\} = \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}\frac{1}{n^r},$$

where $a_n(f)$ *and* $b_n(f)$ *are the cosine Fourier coefficients, the sine Fourier coefficients respectively, of the function* f.

Proof. Without loss of generality we shall give the proof for the coefficients $a_n(f)$. Taking into account the orthogonality of the function $\cos nx$ and the partial Fourier sum $S_{n-1}(f,x)$, we have

$$a_n(f) = \frac{1}{\pi}\int_0^{2\pi}(f(x) - S_{n-1}(f,x))\cos nx dx.$$

By the Cauchy-Bunyakovsky inequality and relation (2.2), we get

$$\sup\left\{|a_n(f)| :\ f \in W^{(r)}(\Omega_m, h)\right\} \le \sup\left\{\|f - S_{n-1}(f)\| :\ f \in W^{(r)}(\Omega_m, h)\right\}$$

$$= E_{n-1}\left(W^{(r)}(\Omega_m, h)\right) = \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}\frac{1}{n^r}. \tag{2.7}$$

In order to obtain an estimate below we consider the function

$$g_0(x) = \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}\frac{1}{n^r}\cos nx.$$

Elementary calculations show that $g_0(x) \in W^{(r)}(\Omega_m, h)$. On the other hand we have

$$\sup\left\{|a_n(f)| :\ f \in W^{(r)}(\Omega_m, h)\right\} \ge |a_n(g_0)| = \left\{\frac{n}{2(nh - Si(nh))}\right\}^{m/2}\frac{1}{n^r}. \tag{2.8}$$

The statement of Corollary 2.1 follows by comparing inequalities (2.7) and (2.8). □

Theorem 2.2. *Let for all $0 < t < \infty$ and $n \in \mathbb{N}$ the majorant $\Phi \in \mathfrak{M}_1$ satisfy the condition*

$$\left(\frac{\Phi(t)}{\Phi(\pi/n)}\right)^{2/m} \ge \frac{\int_0^{nt}(1 - \frac{\sin\tau}{\tau})_* d\tau}{\pi - Si(\pi)}. \tag{2.9}$$

Then for all $m, n \in \mathbb{N}$ and $r \in \mathbb{Z}_+$

$$\gamma_{2n-1}\left(W_1^{(r)}(\Omega_m, \Phi); L_2\right) = \gamma_{2n}\left(W_1^{(r)}(\Omega_m, \Phi); L_2\right)$$

$$= E_{n-1}\left(W_1^{(r)}(\Omega_m, \Phi)\right) = \frac{1}{n^r}\left\{\frac{n}{2(\pi - Si(\pi))}\cdot\Phi\left(\frac{\pi}{n}\right)\right\}^{m/2}, \tag{2.10}$$

where $\gamma_n(\cdot)$ is any of the n-widths mentioned above.

The set of majorants satisfying condition (2.9) *is not empty.*

Proof. By setting in inequality (2.3) $h = \pi/n$ and using the definition of the class $W_1^{(r)}(\Omega_m, \Phi)$, we get that for any function $f \in W_1^{(r)}(\Omega_m, \Phi)$

$$E_{n-1}(f) \le \frac{1}{n^r}\left\{\frac{n}{2(\pi - Si(\pi))}\cdot\Phi\left(\frac{\pi}{n}\right)\right\}^{m/2}. \tag{2.11}$$

By using inequality (2.11) and taking into account relations (2.1), we get the following estimate above for all n-widths mentioned above

$$\gamma_{2n}\left(W_1^{(r)}(\Omega_m,\Phi);L_2\right)\le\gamma_{2n-1}\left(W_1^{(r)}(\Omega_m,\Phi);L_2\right)$$

$$\le d_{2n-1}\left(W_1^{(r)}(\Omega_m,\Phi);L_2\right)\le E_{n-1}\left(W_1^{(r)}(\Omega_m,\Phi)\right)$$

$$\le\frac{1}{n^r}\left\{\frac{n}{2(\pi-Si(\pi))}\cdot\Phi\left(\frac{\pi}{n}\right)\right\}^{m/2}. \tag{2.12}$$

For obtaining an estimate below for all n-widths mentioned above we consider the ball

$$S^*_{2n+1}:=\left\{T_n\in\mathcal{T}_{2n+1}:\ \|T_n\|\le n^{-r}\left(\frac{n}{2(\pi-Si(\pi))}\cdot\Phi\left(\frac{\pi}{n}\right)\right)^{m/2}\right\}.$$

Further, using inequality (2.5) and taking into account relations (2.9), we shall prove that $S^*_{2n+1}\subset W^{(r)}(\Omega_m,\Phi)$. Indeed, for any trigonometric polynomial $T_n\in S^*_{2n+1}$

$$\int_0^h\Omega_m^{2/m}(T_n^{(r)},t)dt\le 2n^{2r/m}\|T_n\|^{2/m}\int_0^h\left(1-\frac{\sin nt}{nt}\right)_*dt$$

$$=2n^{2r/m}\|T_n\|^{2/m}\frac{1}{n}\int_0^{nh}\left(1-\frac{\sin t}{t}\right)_*dt$$

$$\le 2n^{2r/m}n^{-2r/m}\frac{n}{2(\pi-Si(\pi))}\cdot\Phi(\pi/n)\cdot\frac{1}{n}\int_0^{nh}\left(1-\frac{\sin t}{t}\right)_*dt$$

$$=\frac{\Phi(\pi/n)}{\pi-Si(\pi)}\int_0^{nh}\left(1-\frac{\sin t}{t}\right)_*dt\le\Phi(h).$$

Consequently, the ball S^*_{2n+1} is contained in the class $W^{(r)}(\Omega_m,\Phi)$. By the definition of the Bernshtein n-width and by relations (2.1) we get the following estimate below

$$\gamma_{2n}(W^{(r)}(\Omega_m,\Phi);L_2)\ge b_{2n}\left(W^{(r)}(\Omega_m,\Phi);L_2\right)$$

$$\ge b_{2n}(S^*_{2n+1},L_2)\ge n^{-r}\left\{\frac{n}{2(\pi-Si(\pi))}\cdot\Phi(\pi/n)\right\}^{m/2}. \tag{2.13}$$

Comparing inequalities (2.12) and (2.13) we get inequality (2.10).

Let us show that the function $\Phi_*(t)=t^{\alpha m/2}$, with

$$\alpha=\frac{\pi}{\pi-Si(\pi)},\quad 2,42<\alpha<2,44, \tag{2.14}$$

satisfies condition (2.9). Substituting Φ_* in (2.9), we get the inequality

$$\left(\frac{nt}{\pi}\right)^\alpha \geq \frac{\int\limits_0^{nt} \left(1-\frac{\sin\tau}{\tau}\right)_* d\tau}{\pi - Si(\pi)},$$

which is to be proved. Setting $\mu = nt,\ 0 \leq t < \infty$, we rewrite the last inequality in the following equivalent form

$$\mu^\alpha \geq \frac{\pi^\alpha \int\limits_0^{\mu} \left(1-\frac{\sin\tau}{\tau}\right)_* d\tau}{\pi - Si(\pi)}, \quad 0 \leq \mu < \infty. \tag{2.15}$$

By considering the auxiliary function

$$\beta(\mu) = \mu^\alpha - \pi^\alpha(\pi - Si(\pi))^{-1} \int\limits_0^{\mu} \left(1-\frac{\sin\tau}{\tau}\right)_* d\tau \tag{2.16}$$

we shall show that $\beta(\mu) \geq 0$ for all $\mu \in [0, +\infty)$. We shall consider the following three cases:

1) $0 \leq \mu \leq \pi$; 2) $\pi \leq \mu \leq t_*$; 3) $t_* \leq \mu < \infty$.

Let $0 \leq \mu \leq \pi$. We expand the function $\beta(\mu)$ in a neighbourhood of zero:

$$\beta(\mu) = \mu^\alpha \left(1 - \frac{\pi^\alpha}{18(\pi - Si(\pi))} \cdot O(\mu^{3-\alpha})\right).$$

This equality implies that for sufficiently small $\mu > 0$ the function $\beta(\mu)$ is positive. Assuming the contrary, we shall prove that on the interval $(0, \pi)$ the function $\beta(\mu)$ does not change sign.

To this end assume that there exists a point $\xi \in (0, \pi)$ at which the function $\beta(\mu)$ changes sign. Since $\beta(0) = \beta(\pi) = 0$, by the Rolle Theorem we conclude that the derivative

$$\beta'(\mu) = \alpha\mu^{\alpha-1} - \frac{\pi^\alpha}{\pi - Si(\pi)}\left(1 - \frac{\sin\mu}{\mu}\right)$$

$$= \frac{1}{\mu}\left(\alpha\mu^\alpha - \frac{\pi^\alpha}{\pi - Si(\pi)}(\mu - \sin\mu)\right) := \frac{\beta_1(\mu)}{\mu} \tag{2.17}$$

should have at least two different roots on the interval $(0, \pi)$.

Clearly the same refers to the function $\beta_1(\mu)$, which we shall investigate next. Taking into account formulas (2.14) and (2.17), we deduce that $\beta_1(0) = \beta_1(\pi) = 0$. Consequently, the derivative

$$\beta_1'(\mu) = \alpha^2\mu^{\alpha-1} - \frac{\pi^\alpha}{\pi - Si(\pi)}(1 - \cos\mu) \tag{2.18}$$

by the Rolle Theorem should have at least three different roots on the interval $(0, \pi)$. Since also $\beta_1'(0) = 0$, by a similar argument the second derivative

$$\beta_1''(\mu) = \alpha^2(\alpha-1)\mu^{\alpha-2} - \frac{\pi^\alpha}{\pi - Si(\pi)} \sin\mu \tag{2.19}$$

should have at least three different roots on the interval $(0, \pi)$. Since $\beta_1''(0) = 0$, the same conclusion on existence of at least three different roots on the interval $(0, \pi)$ can be derived also for the third derivative

$$\beta_1'''(\mu) = \alpha^2(\alpha-1)(\alpha-2)\mu^{\alpha-3} - \frac{\pi^\alpha}{\pi - Si(\pi)} \cos\mu. \tag{2.20}$$

By (2.14) and (2.20) $\beta_1'''(\mu)$ is the difference of two functions, one of which is convex and another one is concave. From geometric point of view it is clear that the function $\beta_1'''(\mu)$ cannot have more than two zeros on the interval $(0, \pi)$. This contradiction proves the validity of inequality (2.15) on the closed interval $0 \le \mu \le \pi$.

Next, let $\pi \le \mu \le t_*$. Assume to the contrary that there exists at least one point $\xi \in (\pi, t_*)$, at which the function $\beta(\mu)$ changes sign. Since $\beta(\pi) = 0$, by the Rolle Theorem and (2.17), the function $\beta'(\mu)$ and, hence, the function $\beta_1'(\mu)$ should have at least one zero on the interval (π, t_*). By (2.19) it follows that for any $u \in [\pi, t_*]$ the function $\beta_1''(u) > 0$. By using formulas (2.14) and (2.18), one can prove that

$$\beta_1'(\pi) = \pi^{\alpha-1}\left(\alpha^2 - \frac{2\pi}{\pi - Si(\pi)}\right) > 0.$$

Consequently, the function $\beta_1'(\mu)$ is positive and increasing on the closed interval $[\pi, t_*]$. Since $\beta_1(\pi) = 0$, by the above argument follows that on $(\pi, t_*]$ the function $\beta_1(\mu)$ should be positive and increasing. The obtained contradiction proves the validity of inequality (2.15) on the closed interval $(\pi, t_*]$.

Finally, consider the case $t_* \le \mu < \infty$. By using (2.16), we write

$$\beta(\mu) = \mu^\alpha - \frac{\pi^\alpha}{\pi - Si(\pi)}\left(\mu\left(1 - \frac{\sin t_*}{t_*}\right) + \sin t_* - Si(t_*)\right). \tag{2.21}$$

Hence

$$\beta'(\mu) = \alpha\mu^{\alpha-1} - \frac{\pi^\alpha}{\pi - Si(\pi)}\left(1 - \frac{\sin t_*}{t_*}\right). \tag{2.22}$$

Since, by elementary calculations, it follows that $\beta'(t_*) > 0$, hence by (2.22) $\beta'(\mu) > 0$ for all $\mu \in [t_*, \infty)$. By (2.21) we have

$$\beta(t_*) = \pi^\alpha\left(\left(\frac{t_*}{\pi}\right)^\alpha - \frac{t_* - Si(t_*)}{\pi - Si(\pi)}\right) > 0.$$

By the above arguments about $\beta'(\mu)$, we conclude that the function $\beta(\mu)$ is positive and increasing on $[t_*, \infty)$. Hence on the set $t_* \le \mu < \infty$ inequality (2.15) is also valid. $\square$

Corollary 2.2. *Under the assumptions of Theorem 4 for all $n \in \mathbb{N}$*

$$\gamma_{2n}\left(W^{(r)}(\Omega_m, \Phi_*); L_2\right) = \gamma_{2n-1}\left(W^{(r)}(\Omega_m, \Phi_*); L_2\right)$$

$$= E_{n-1}\left(W^{(r)}(\Omega_m, \Phi_*)\right) = \left\{\frac{\pi^{\pi/(\pi - Si(\pi))}}{2(\pi - Si(\pi))}\right\}^{m/2} \cdot n^{-(r+mSi(\pi))/2(\pi - Si(\pi))}.$$

Corollary 2.3. *Under the assumptions of Theorem 4 for all $n \in \mathbb{N}$*

$$\sup\left\{|a_n(f)|,\, |b_n(f)| :\, f \in W^{(r)}(\Omega_m, L_2)\right\} = n^{-r}\left\{\frac{n}{2(\pi - Si(\pi))} \cdot \Phi\left(\frac{\pi}{n}\right)\right\}^{m/2}.$$

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