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THE REAL AND COMPLEX TECHNIQUES IN HARMONIC ANALYSIS FROM THE POINT OF VIEW OF COVARIANT TRANSFORM

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Dedicated to Professor S.V. Rogosin on the occasion of his 60th birthday

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Abstract. This paper reviews complex and real techniques in harmonic analysis. We describe the common source of both approaches rooted in the covariant transform generated by the affine group.

1 Introduction

There are two main approaches in harmonic analysis on the real line. The real variables technique uses various maximal functions, dyadic cubes and, occasionally, the Poisson integral [40]. The complex variable technique is based on the Cauchy integral and fine properties of analytic functions [36, 37].

Both methods seem to have clear advantages. The real variable technique:

i. does not require an introduction of the imaginary unit for a study of real-valued harmonic functions of a real variable (Occam's Razor: among competing hypotheses, the one with the fewest assumptions should be selected);

ii. allows a straightforward generalization to several real variables.

By contrast, access to the beauty and power of analytic functions (e.g., Möbius transformations, factorisation of zeroes, etc. [31]) is the main reason to use the complex variable technique. A posteriori, a multidimensional analytic version was also discovered [34], it is based on the monogenic Clifford-valued functions [3].

Therefore, propensity for either techniques becomes a personal choice of a researcher. Some of them prefer the real variable method, explicitly cleaning out any reference to analytic or harmonic functions [40, Chapter III, p. 88]. Others, e.g. [32, 6], happily combine the both techniques. However, the reasons for switching between two methods at particular places may look mysterious.

The purpose of the present paper is to revise the origins of the real and complex variable techniques. Thereafter, we describe the common group-theoretical root of both. Such a unification deepens our understanding of both methods and illuminates their interaction.

Remark 1.1. In this paper, we consider only examples which are supported by the affine group Aff of the real line. In the essence, Aff is the semidirect product of the group of dilations acting on the group of translations. Thus, our consideration can be generalized to the semidirect product of dilations and homogeneous (nilpotent) Lie groups, cf. [11, 27]. Other important extensions are the group $\mathrm{SL}_2(\mathbb{R})$ and associated hypercomplex algebras, see Remarks 3.3, 4.4 and [22, 26, 25]. However, we do not aim here to a high level of generality, it can be developed in subsequent works once the fundamental issues are sufficiently clarified.

2 Two approaches to harmonic analysis

As a starting point of our discussion, we provide a schematic outline of complex and real variables techniques in the one-dimensional harmonic analysis. The application of complex analysis may be summarised in the following sequence of principal steps:

Integral transforms. For a function $f \in L_p(\mathbb{R})$, we apply the Cauchy or Poisson integral transforms:

$$[\mathcal{C}f](x+\mathrm{i}y) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{f(t)}{t-(x+\mathrm{i}y)}\, dt\,, \tag{2.1}$$

$$[\mathcal{P}f](x,y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}\, f(t)\, dt\,. \tag{2.2}$$

An equivalent transformation on the unit circle replaces the Fourier series $\sum_k c_k e^{\mathrm{i}kt}$ by the Taylor series $\sum_{k=0}^{\infty} c_k z^k$ in the complex variable $z = re^{\mathrm{i}t}$, $0 \le r < 1$. It is used for the Abel summation of trigonometric series [42, § III.6]. Some other summations methods are in use as well [33].

Domains. Above integrals (2.1)–(2.2) map the domain of functions from the real line to the upper half-plane, which can be conveniently identified with the set of complex numbers having a positive imaginary part. The larger domain allows us to inspect functions in greater details.

Differential operators. The image of integrals (2.1) and (2.2) consists of functions, belonging to the kernel of the Cauchy–Riemann operator $\partial_{\bar{z}}$, Laplace operator Δ respectively, i.e.:

$$\partial_{\bar{z}} = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\,, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,. \tag{2.3}$$

Such functions have numerous nice properties in the upper half-plane, e.g. they are infinitely differentiable, which make their study interesting and fruitful.

Boundary values and SIO. To describe properties of the initial function f on the real line we consider the boundary values of $[\mathcal{C}f](x+\mathrm{i}y)$ or $[\mathcal{P}f](x,y)$, i.e. their limits as $y \to 0$ in some sense. The Sokhotsky–Plemelj formula provides the boundary value of the Cauchy integral [35, (2.6.6)]:

$$[\mathcal{C}f](x,0) = \frac{1}{2}f(x) + \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \frac{f(t)}{t-x}\,dt. \tag{2.4}$$

The last term is a singular integral operator defined through the principal value in the Cauchy sense:

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \frac{f(t)}{t-x}\,dt = \lim_{\varepsilon\to 0}\frac{1}{2\pi\mathrm{i}}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{f(t)}{t-x}\,dt\,. \tag{2.5}$$

For the Abel summation the boundary values are replaced by the limit as $r \to 1^-$ in the series $\sum_{k=0}^{\infty} c_k (re^{\mathrm{i}t})^k$.

Hardy space. Sokhotsky–Plemelj formula (2.4) shows, that the boundary value $[\mathcal{C}f](x,0)$ may be different from $f(x)$. The vector space of functions $f(x)$ such that $[\mathcal{C}f](x,0) = f(x)$ is called the Hardy space on the real line [36, A.6.3].

Summing up this scheme: we replace a function (distribution) on the real line by a nicer (analytic or harmonic) function on a larger domain—the upper half-plane. Then, we trace down properties of the extensions to its boundary values and, eventually, to the initial function.

The real variable approach does not have a clearly designated path in the above sense. Rather, it looks like a collection of interrelated tools, which are efficient for various purposes. To highlight similarity and differences between real and complex analysis, we line up the elements of the real variable technique in the following way:

Hardy–Littlewood maximal function is, probably, the most important component [31, § VIII.B.1], [40, Chapter 2], [14, § I.4], [4] of this technique. The maximal function f^M is defined on the real line by the equality:

$$f^M(t) = \sup_{a>0}\left\{\frac{1}{2a}\int_{t-a}^{t+a} |f(x)|\,dx\right\}. \tag{2.6}$$

Domain is not apparently changed, the maximal function f^M is again defined on the real line. However, an efficient treatment of the maximal functions requires consideration of tents [40, § II.2], which are parametrised by their vertices, i.e. points (a,b), $a>0$, of the upper half-plane. In other words, we repeatedly need values of all integrals $\frac{1}{2a}\int_{t-a}^{t+a} |f(x)|\,dx$, rather than the single value of the supremum over a.

Littlewood–Paley theory [6, § 3] and associated dyadic squares technique [14, Chapter VII, Theorem 1.1], [40, § IV.3] as well as stopping time argument [14, Chapter VI, Lemma 2.2] are based on bisection of a function's domain into two equal parts.

SIO is a natural class of bounded linear operators in $L_p(\mathbb{R})$. Moreover, maximal operator $M : f \to f^M$ (2.6) and singular integrals are closely related [40, Chapter I].

Hardy space can be defined in several equivalent ways from previous notions. For example, it is the class of such functions that their images under maximal operator (2.6) or singular integral (2.5) belong to $L_p(\mathbb{R})$ [40, Chapter III].

The following discussion will line up real variable objects along the same axis as complex variables. We will summarize this in Table 1.

3 Affine group and its representations

It is hard to present harmonic analysis and wavelets without touching the affine group one way or another, e.g. through the *doubling condition* on the measure, cf. [41]. Unfortunately, many sources only mention the group and do not use it explicitly. On the other hand, it is equally difficult to speak about the affine group without a reference to results in harmonic analysis: two theories are intimately intertwined. In this section we collect fundamentals of the affine group and its representations, which are not yet a standard background of an analyst.

Let $G = \mathrm{Aff}$ be the $ax + b$ (or the *affine*) group [2, § 8.2], which is represented (as a topological set) by the upper half-plane $\{(a, b) \mid a \in \mathbb{R}_+,\ b \in \mathbb{R}\}$. The group law is:

$$(a, b) \cdot (a', b') = (aa', ab' + b). \tag{3.1}$$

As any other group, Aff has the *left regular representation* by shifts on functions $\mathrm{Aff} \to \mathbb{C}$:

$$\Lambda(a, b) : f(a', b') \mapsto f_{(a,b)}(a', b') = f\left(\frac{a'}{a}, \frac{b' - b}{a}\right). \tag{3.2}$$

A left invariant measure on Aff is $dg = a^{-2}\, da\, db$, $g = (a, b)$. By the definition, the left regular representation (3.2) acts by unitary operators on $L_2(\mathrm{Aff}, dg)$. The group is not unimodular and a right invariant measure is $a^{-1}\, da\, db$.

There are two important subgroups of the $ax + b$ group:

$$A = \{(a, 0) \in \mathrm{Aff} \mid a \in \mathbb{R}_+\} \quad \text{and} \quad N = \{(1, b) \in \mathrm{Aff} \mid b \in \mathbb{R}\}. \tag{3.3}$$

An isometric representation of Aff on $L_p(\mathbb{R})$ is given by the formula:

$$[\rho_p(a, b)\, f](x) = a^{-\frac{1}{p}}\, f\left(\frac{x - b}{a}\right). \tag{3.4}$$

Here, we identify the real line with the subgroup N or, even more accurately, with the homogeneous space Aff/N [9, § 2]. This representation is known as *quasi-regular* for its similarity with (3.2). The action of the subgroup N in (3.4) reduces to shifts, the subgroup A acts by dilations.

Remark 3.1. The $ax+b$ group definitely escapes Occam's Razor in harmonic analysis, cf. the arguments against the imaginary unit in the Introduction. Indeed, shifts are required to define convolutions on $\mathbb{R}^n$, and an *approximation of the identity* [40, § I.6.1] is a convolution with the dilated kernel. The same scaled convolutions define the fundamental *maximal functions*, see [40, § III.1.2] cf. Example 16 below. Thus, we can avoid usage of the upper half-plane $\mathbb{C}_+$, but the same set will anyway re-invent itself in the form of the $ax+b$ group.

The representation (3.4) in $L_2(\mathbb{R})$ is reducible and the space can be split into irreducible subspaces. Following the philosophy presented in the Introduction to the paper [28, § 1] we give the following

Definition 3.1. *For a representation ρ of a group G in a space V, a* generalized Hardy space *H is an ρ-irreducible (or ρ-primary, as discussed in Section 7) subspace of V.*

Example 1. Let $G = \mathrm{Aff}$ and the representation ρ_p be defined in $V = L_p(\mathbb{R})$ by (3.4). Then the classical Hardy spaces $H_p(\mathbb{R})$ are ρ_p-irreducible, thus are covered by the above definition.

Some ambiguity in picking the Hardy space out of all (well, two, as we will see below) irreducible components is resolved by the traditional preference.

Remark 3.2. We have defined the Hardy space completely in terms of representation theory of $ax+b$ group. The traditional descriptions, via the Fourier transform or analytic extensions, will be corollaries in our approach, see Proposition 3.1 and Example 12.

Remark 3.3. It is an interesting and important observation, that the Hardy space in $L_p(\mathbb{R})$ is invariant under the action of a larger group $\mathrm{SL}_2(\mathbb{R})$, the group of 2×2 matrices with real entries and determinants equal to 1, the group operation coincides with the multiplication of matrices. The $ax+b$ group is isomorphic to the subgroup of the upper-triangular matrices in $\mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{SL}_2(\mathbb{R})$ has an isometric representation in $L_p(\mathbb{R})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ f(x) \ \mapsto \ \frac{1}{|a-cx|^{\frac{2}{p}}}\, f\left(\frac{dx-b}{a-cx}\right), \tag{3.5}$$

which produces quasi-regular representation (3.4) by the restriction to upper-triangular matrices. The Hardy space $H_p(\mathbb{R})$ is invariant under the above action as well. Thus, $\mathrm{SL}_2(\mathbb{R})$ produces a refined version in comparison with the harmonic analysis of the $ax+b$ group considered in this paper. Moreover, as representations of the $ax+b$ group are connected with complex numbers, the structure of $\mathrm{SL}_2(\mathbb{R})$ links all three types of hypercomplex numbers [22] [26, § 3.3.4] [25, § 3], see also Remark 4.4.

To clarify a decomposition of $L_p(\mathbb{R})$ into irreducible subspaces of representation (3.4) we need another realization of this representation. It is called *co-adjoint* and is related to the *orbit method* of Kirillov [17, § 4.1.4] [12, § 6.7.1]. Again, this isometric representation can be defined on $L_p(\mathbb{R})$ by the formula:

$$[\hat{\rho}_p(a,b)\, f](\lambda) = a^{\frac{1}{p}}\, \mathrm{e}^{-2\pi \mathrm{i} b\lambda} f(a\lambda). \tag{3.6}$$

Since $a > 0$, there is an obvious decomposition into invariant subspaces of $\hat{\rho}_p$:

$$L_p(\mathbb{R}) = L_p(-\infty, 0) \oplus L_p(0, \infty). \tag{3.7}$$

It is possible to demonstrate, that these components are irreducible. This decomposition has a spatial nature, i.e., the subspaces have disjoint supports. Each half-line can be identified with the subgroup A or with the homogeneous space Aff/N.

The restrictions $\hat{\rho}_p^+$ and $\hat{\rho}_p^-$ of the co-adjoint representation $\hat{\rho}_p$ to invariant subspaces (3.7) for $p = 2$ are not unitary equivalent. Any irreducible unitary representation of Aff is unitary equivalent either to $\hat{\rho}_2^+$ or $\hat{\rho}_2^-$. Although there is no intertwining operator between $\hat{\rho}_p^+$ and $\hat{\rho}_p^-$, the map:

$$J: \ L_p(\mathbb{R}) \to L_p(\mathbb{R}): \ f(\lambda) \mapsto f(-\lambda), \tag{3.8}$$

has the property

$$\hat{\rho}_p^-(a, -b) \circ J = J \circ \hat{\rho}_p^+(a, b) \tag{3.9}$$

which corresponds to the outer automorphism $(a, b) \mapsto (a, -b)$ of Aff.

As was already mentioned, for the Hilbert space $L_2(\mathbb{R})$, representations (3.4) and (3.6) are unitary equivalent, i.e., there is a unitary intertwining operator between them. We may guess its nature as follows. The eigenfunctions of the operators $\rho_2(1, b)$ are $\mathrm{e}^{2\pi \mathrm{i}\omega x}$ and the eigenfunctions of $\hat{\rho}_2(1, b)$ are $\delta(\lambda - \omega)$. Both sets form "continuous bases" of $L_2(\mathbb{R})$ and the unitary operator which maps one to another is the Fourier transform:

$$\mathcal{F} : f(x) \mapsto \hat{f}(\lambda) = \int_{\mathbb{R}} e^{-2\pi \mathrm{i}\lambda x} f(x)\, dx. \tag{3.10}$$

Although, the above arguments were informal, the intertwining property $\mathcal{F}\rho_2(a, b) = \hat{\rho}_2(1, b)\mathcal{F}$ can be directly verified by the appropriate change of variables in the Fourier transform. Thus, cf. [36, Lemma A.6.2.2]:

Proposition 3.1. *The Fourier transform maps irreducible invariant subspaces H_2 and $H_2^\perp$ of* (3.4) *to irreducible invariant subspaces $L_2(0, \infty) = \mathcal{F}(H_2)$ and $L_2(-\infty, 0) = \mathcal{F}(H_2^\perp)$ of co-adjoint representation* (3.6)*. In particular, $L_2(\mathbb{R}) = H_2 \oplus H_2^\perp$.*

Reflection J (3.8) anticommutes with the Fourier transform: $\mathcal{F}J = -J\mathcal{F}$. Thus, J also interchange the irreducible components ρ_p^+ and ρ_p^- of quasi-regular representation (3.4) according to (3.9).

Summing up, the unique rôle of the Fourier transform in harmonic analysis is based on the following facts from the representation theory. The Fourier transform

- intertwines shifts in quasi-regular representation (3.4) to operators of multiplication in co-adjoint representation (3.6);
- intertwines dilations in (3.4) to dilations in (3.6);
- maps the decomposition $L_2(\mathbb{R}) = H_2 \oplus H_2^\perp$ into spatially separated spaces with disjoint supports;
- anticommutes with J, which interchanges ρ_2^+ and ρ_2^-.

Armed with this knowledge we are ready to proceed to harmonic analysis.

4 Covariant transform

We make an extension of the wavelet construction defined in terms of group representations. See [16] for a background in the representation theory, however, the only treated case in this paper is the $ax + b$ group.

Definition 4.1. *[23, 25]. Let ρ be a representation of a group G in a space V and F be an operator acting from V to a space U. We define a* covariant transform $\mathcal{W}_F^\rho$ *acting from V to the space $L(G, U)$ of U-valued functions on G by the formula:*

$$\mathcal{W}_F^\rho : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \qquad v \in V,\ g \in G. \tag{4.1}$$

The operator F will be called a fiducial operator *in this context (cf. the fiducial vector in [30]).*

We may drop the sup/subscripts from $\mathcal{W}_F^\rho$ if the functional F and/or the representation ρ are clear from the context.

Remark 4.1. We do not require that the fiducial operator F be linear. Sometimes the positive homogeneity, i.e. $F(tv) = tF(v)$ for $t > 0$, alone can be already sufficient, see Example 5.

Remark 4.2. It looks like the usefulness of the covariant transform is in the reverse proportion to the dimension of the space U. The covariant transform encodes properties of v in a function $\mathcal{W}_F^\rho v$ on G, which is a scalar-valued function if $\dim U = 1$. However, such a simplicity is not always possible. Moreover, the paper [27] gives an important example of a covariant transform which provides a simplification even in the case $\dim U = \dim V$.

We start the list of examples with the classical case of the group-theoretical wavelet transform.

Example 2. [38, 10, 20, 2, 30, 10]. Let V be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and ρ be a unitary representation of a group G in the space V. Let $F : V \to \mathbb{C}$ be the functional $v \mapsto \langle v, v_0 \rangle$ defined by a vector $v_0 \in V$. The vector v_0 is often called the *mother wavelet* in areas related to signal processing, the *vacuum state* in the quantum framework, etc.

In this set-up, transformation (4.1) is the well-known expression for a *wavelet transform* [2, (7.48)] (or *representation coefficients*):

$$\mathcal{W} : v \mapsto \tilde{v}(g) = \left\langle \rho(g^{-1})v, v_0 \right\rangle = \left\langle v, \rho(g)v_0 \right\rangle, \qquad v \in V,\ g \in G. \tag{4.2}$$

The family of the vectors $v_g = \rho(g)v_0$ is called *wavelets* or *coherent states*. The image of (4.2) consists of scalar valued functions on G.

This scheme is typically carried out for a square integrable representation ρ with v_0 being an admissible vector [38, 10, 2, 13, 5, 7], i.e. satisfying the condition:

$$0 < \|\tilde{v}_0\|^2 = \int_G |\langle v_0, \rho_2(g)v_0 \rangle|^2 \, dg < \infty. \tag{4.3}$$

In this case the wavelet (covariant) transform is a map into the square integrable functions [7] with respect to the left Haar measure on G. The map becomes an isometry if v_0 is properly scaled. Moreover, we are able to recover the input v from its wavelet transform through the reconstruction formula, which requires an admissible vector as well, see Example 7 below. The most popularized case of the above scheme is provided by the affine group.

Example 3. For the $ax+b$ group, representation (3.4) is square integrable for $p=2$. Any function v_0, such that its Fourier transform $\hat{v}_0(\lambda)$ satisfies

$$\int_0^\infty \frac{|\hat{v}_0(\lambda)|^2}{\lambda}\, d\lambda < \infty, \tag{4.4}$$

is admissible in the sense of (4.3) [2, § 12.2]. The *continuous wavelet transform* is generated by representation (3.4) acting on an admissible vector v_0 in expression (4.2). The image of a function from $L_2(\mathbb{R})$ is a function on the upper half-plane square integrable with respect to the measure $a^{-2}\, da\, db$. There are many examples [2, § 12.2] of useful admissible vectors, say, the *Mexican hat* wavelet: $(1-x^2)e^{-x^2/2}$. For sufficiently regular $\hat{v}_0$ admissibility (4.4) of v_0 follows by a weaker condition

$$\int_{\mathbb{R}} v_0(x)\, dx = 0. \tag{4.5}$$

We dedicate Section 8 to isometric properties of this transform.

However, square integrable representations and admissible vectors do not cover all interesting cases.

Example 4. For the above $G = \mathrm{Aff}$ and representation (3.4), we consider the operators $F_\pm : L_p(\mathbb{R}) \to \mathbb{C}$ defined by:

$$F_\pm(f) = \frac{1}{\pi \mathrm{i}} \int_{\mathbb{R}} \frac{f(x)\, dx}{\mathrm{i} \mp x}. \tag{4.6}$$

In $L_2(\mathbb{R})$ we note that $F_+(f) = \langle f, c\rangle$, where $c(x) = \frac{1}{\pi \mathrm{i}} \frac{1}{\mathrm{i}+x}$. Computing the Fourier transform $\hat{c}(\lambda) = \chi_{(0,+\infty)}(\lambda)\, e^{-\lambda}$, we see that $\bar{c} \in H_2(\mathbb{R})$. Moreover, $\hat{c}$ does not satisfy admissibility condition (4.4) for representation (3.4).

Then, covariant transform (4.1) is Cauchy integral (2.1) from $L_p(\mathbb{R})$ to the space of functions $\tilde{f}(a,b)$ such that $a^{-\frac{1}{p}}\tilde{f}(a,b)$ is in the Hardy space on the upper/lower half-plane $H_p(\mathbb{R}^2_\pm)$ [36, § A.6.3]. Due to inadmissibility of $c(x)$, the complex analysis become decoupled from the traditional wavelet theory.

Many important objects in harmonic analysis are generated by inadmissible mother wavelets like (4.6). For example, the functionals $P = \frac{1}{2}(F_+ + F_-)$ and $Q = \frac{1}{2\mathrm{i}}(F_+ - F_-)$ are defined by kernels:

$$p(x) = \frac{1}{2\pi \mathrm{i}}\left(\frac{1}{\mathrm{i}-x} - \frac{1}{\mathrm{i}+x}\right) = \frac{1}{\pi}\frac{1}{1+x^2}, \tag{4.7}$$

$$q(x) = -\frac{1}{2\pi}\left(\frac{1}{\mathrm{i}-x} - \frac{1}{\mathrm{i}+x}\right) = -\frac{1}{\pi}\frac{x}{1+x^2} \tag{4.8}$$

which are *Poisson kernel* (2.2) and the *conjugate Poisson kernel* [15, § 4.1] [14, § III.1] [31, Chapter 5] [36, § A.5.3], respectively. Another interesting non-admissible vector is the *Gaussian* e^{-x^2}.

Example 5. A step in a different direction is a consideration of non-linear operators. Take again the $ax+b$ group and its representation (3.4). We define F to be a homogeneous (but non-linear) functional $V \to \mathbb{R}_+$:

$$F_m(f) = \frac{1}{2}\int_{-1}^{1} |f(x)|\, dx. \tag{4.9}$$

Covariant transform (4.1) becomes:

$$[\mathcal{W}_p^m f](a,b) = F(\rho_p(\tfrac{1}{a}, -\tfrac{1}{b})f) = \frac{1}{2}\int_{-1}^{1} \left| a^{\frac{1}{p}} f\,(ax+b)\right|\, dx = \frac{a^{\frac{1}{q}}}{2}\int_{b-a}^{b+a} |f\,(x)|\, dx, \tag{4.10}$$

where $\frac{1}{p}+\frac{1}{q}=1$, as usual. We will see its connections with the Hardy–Littlewood maximal functions in Example 16.

Since linearity has clear advantages, we may prefer to reformulate the last example using linear covariant transforms. The idea is similar to the representation of a convex function as an envelope of linear ones, cf. [14, Chapter I, Lemma 6.1]. To this end, we take a collection $\mathbf{F}$ of linear fiducial functionals and, for a given function f, consider the set of all covariant transforms $\mathcal{W}_F f$, $F \in \mathbf{F}$.

Example 6. Let us return to the setup of the previous Example for $G = \mathrm{Aff}$ and its representation (3.4). Consider the unit ball B in $L_\infty[-1,1]$. Then, any $\omega \in B$ defines a bounded linear functional F_ω on $L_1(\mathbb{R})$:

$$F_\omega(f) = \frac{1}{2}\int_{-1}^{1} f(x)\,\omega(x)\, dx = \frac{1}{2}\int_{\mathbb{R}} f(x)\,\omega(x)\, dx. \tag{4.11}$$

Of course, $\sup_{\omega\in B} F_\omega(f) = F_m(f)$ with F_m from (4.9) and for all $f \in L_1(\mathbb{R})$. Then, for the non-linear covariant transform (4.10) we have the following expression in terms of the linear covariant transforms generated by F_ω:

$$[\mathcal{W}_1^m f](a,b) = \sup_{\omega\in B}\, [\mathcal{W}_1^\omega f](a,b). \tag{4.12}$$

The presence of supremum is the price to pay for such a "linearization".

Remark 4.3. The above construction is not much different to the *grand maximal function* [40, § III.1.2]. Although, it may look like a generalisation of covariant transform, grand maximal function can be realised as a particular case of Definition 4.1. Indeed, let $M(V)$ be a subgroup of the group of all invertible isometries of a metric space V. If ρ represents a group G by isometries of V then we can consider the group $\tilde{G}$ generated by all finite products of $M(V)$ and $\rho(g)$, $g \in G$ with the straightforward action $\tilde{\rho}$ on V. The grand maximal functions is produced by the covariant transform for the representation $\tilde{\rho}$ of $\tilde{G}$.

Remark 4.4. It is instructive to compare action (3.5) of the large $\mathrm{SL}_2(\mathbb{R})$ group on the mother wavelet $\frac{1}{x+\mathrm{i}}$ for the Cauchy integral and the principal case $\omega(x) = \chi_{[-1,1]}(x)$ (the characteristic function of $[-1,1]$) for functional (4.11). The wavelet $\frac{1}{x+\mathrm{i}}$ is an eigenvector for all matrices $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, which form the one-parameter compact subgroup $K \subset \mathrm{SL}_2(\mathbb{R})$. The respective covariant transform (i.e., the Cauchy integral) maps functions to the homogeneous space $\mathrm{SL}_2(\mathbb{R})/K$, which is the upper half-plane with the Möbius (linear-fractional) transformations of complex numbers [22] [26, § 3.3.4] [25, § 3]. By contrast, the mother wavelet $\chi_{[-1,1]}$ is an eigenvector for all matrices $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, which form the one-parameter subgroup $A \in \mathrm{SL}_2(\mathbb{R})$. The covariant transform (i.e., the averaging) maps functions to the homogeneous space $\mathrm{SL}_2(\mathbb{R})/A$, which can be identified with a set of double numbers with corresponding Möbius transformations [22, § 3.3.4] [26] [25, § 3]. Conformal geometry of double numbers is suitable for real variables technique, in particular, tents [40, § II.2] make a Möbius-invariant family.

5 The contravariant transform

Define the left action Λ of a group G on a space of functions over G by:

$$\Lambda(g) : f(h) \mapsto f(g^{-1}h). \tag{5.1}$$

For example, in the case of the affine group it is (3.2). An object invariant under the left action Λ is called *left invariant*. In particular, let L and L' be two left invariant spaces of functions on G. We say that a pairing $\langle\cdot,\cdot\rangle : L \times L' \to \mathbb{C}$ is *left invariant* if

$$\langle \Lambda(g)f, \Lambda(g)f' \rangle = \langle f, f' \rangle , \qquad \text{for all} \qquad f \in L,\ f' \in L',\ g \in G. \tag{5.2}$$

Remark 5.1. i. We do not require the pairing to be linear in general, in some cases it is sufficient to have only homogeneity, see Example 9.

ii. If the pairing is invariant on space $L \times L'$ it is not necessarily invariant (or even defined) on large spaces of functions.

iii. In some cases, an invariant pairing on G can be obtained from an *invariant functional* l by the formula $\langle f_1, f_2 \rangle = l(f_1 f_2)$.

For a representation ρ of G in V and $w_0 \in V$, we construct a function $w(g) = \rho(g)w_0$ on G. We assume that the pairing can be extended in its second component to this V-valued functions. For example, such an extension can be defined in the weak sense.

Definition 5.1. *[23, 25]. Let $\langle\cdot,\cdot\rangle$ be a left invariant pairing on $L \times L'$ as above, let ρ be a representation of G in a space V, we define the function $w(g) = \rho(g)w_0$ for $w_0 \in V$ such that $w(g) \in L'$ in a suitable sense. The* contravariant transform $\mathcal{M}^\rho_{w_0}$ *is a map $L \to V$ defined by the pairing:*

$$\mathcal{M}^\rho_{w_0} : f \mapsto \langle f, w \rangle , \qquad \textit{where } f \in L. \tag{5.3}$$

We can drop out sup/subscripts in $\mathcal{M}^{\rho}_{w_0}$ as we did for $\mathcal{W}^{\rho}_{F}$.

Example 7 (Haar paring). The most used example of an invariant pairing on $L_2(G, d\mu) \times L_2(G, d\mu)$ is the integration with respect to the Haar measure:

$$\langle f_1, f_2 \rangle = \int_G f_1(g) f_2(g)\, dg. \tag{5.4}$$

If ρ is a square integrable representation of G and w_0 is an admissible vector, see Example 2, then this pairing can be extended to $w(g) = \rho(g)w_0$. The contravariant transform is known in this setup as the *reconstruction formula*, cf. [2, (8.19)]:

$$\mathcal{M}_{w_0} f = \int_G f(g)\, w(g)\, dg, \qquad \text{where } w(g) = \rho(g)w_0. \tag{5.5}$$

It is possible to use different admissible vectors v_0 and w_0 for wavelet transform (4.2) and reconstruction formula (5.5), respectively, cf. Example 15.

Let either

- ρ be not a square integrable representation (even modulo a subgroup);

or

- w_0 be an inadmissible vector of a square integrable representation ρ.

A suitable invariant pairing in this case is not associated with integration over the Haar measure on G. In this case we speak about a *Hardy pairing*. The following example explains the name.

Example 8 (Hardy pairing). Let G be the $ax + b$ group and its representation ρ (3.4) in Example 3. An invariant pairing on G, which is not generated by the Haar measure $a^{-2} da\, db$, is:

$$\langle f_1, f_2 \rangle_H = \lim_{a \to 0} \int_{-\infty}^{\infty} f_1(a, b)\, f_2(a, b)\, \frac{db}{a}. \tag{5.6}$$

For this pairing, we can consider functions $\frac{1}{\pi \mathrm{i}} \frac{1}{x+\mathrm{i}}$ or e^{-x^2}, which are not admissible vectors in the sense of square integrable representations. For example, for $v_0 = \frac{1}{\pi \mathrm{i}} \frac{1}{x+\mathrm{i}}$ we obtain:

$$[\mathcal{M} f](x) = \lim_{a \to 0} \int_{-\infty}^{\infty} f(a, b)\, \frac{a^{-\frac{1}{p}}}{\pi \mathrm{i}(x + \mathrm{i}a - b)}\, db = -\lim_{a \to 0} \frac{a^{-\frac{1}{p}}}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{f(a, b)\, db}{b - (x + \mathrm{i}a)}.$$

In other words, it expresses the boundary values at $a = 0$ of the Cauchy integral $[-\mathcal{C} f](x + \mathrm{i}a)$.

Here is an important example of non-linear pairing.

Example 9. Let $G = \mathrm{Aff}$ and an invariant homogeneous functional on G be given by the L_∞-version of Haar functional (5.4):

$$\langle f_1, f_2\rangle_\infty = \sup_{g\in G} |f_1(g) f_2(g)| \,. \tag{5.7}$$

Define the following two functions on $\mathbb{R}$:

$$v_0^+(t) = \begin{cases} 1, & \text{if } t = 0; \\ 0, & \text{if } t \neq 0, \end{cases} \qquad \text{and} \qquad v_0^*(t) = \begin{cases} 1, & \text{if } |t| \le 1; \\ 0, & \text{if } |t| > 1. \end{cases} \tag{5.8}$$

The respective contravariant transforms are generated by representation ρ_∞ (3.4) are:

$$\begin{aligned} [\mathcal{M}_{v_0^+} f](t) &= f^+(t) = \left\langle f(a,b), \rho_\infty(a,b) v_0^+(t)\right\rangle_\infty = \sup_a |f(a,t)| \,, & (5.9)\\ [\mathcal{M}_{v_0^*} f](t) &= f^*(t) = \langle f(a,b), \rho_\infty(a,b) v_0^*(t)\rangle_\infty = \sup_{a>|b-t|} |f(a,b)| \,. & (5.10) \end{aligned}$$

Transforms (5.9) and (5.10) are the *vertical* and *non-tangential maximal functions* [31, § VIII.C.2], respectively.

Example 10. Consider again $G = \mathrm{Aff}$ equipped now with an invariant linear functional, which is a Hardy-type modification (cf. (5.6)) of L_∞-functional (5.7):

$$\langle f_1, f_2\rangle_{\overset{H}{\infty}} = \varlimsup_{a\to 0} \sup_{b\in\mathbb{R}} (f_1(a,b) f_2(a,b)), \tag{5.11}$$

where $\varlimsup$ is the upper limit. Then, the covariant transform $\mathcal{M}^H$ for this pairing for functions v^+ and v^* (5.8) becomes:

$$\begin{aligned} [\mathcal{M}^H_{v_0^+} f](t) &= \left\langle f(a,b), \rho_\infty(a,b) v_0^+(t)\right\rangle_{\overset{H}{\infty}} = \varlimsup_{a\to 0} f(a,t), & (5.12)\\ [\mathcal{M}^H_{v_0^*} f](t) &= \langle f(a,b), \rho_\infty(a,b) v_0^*(t)\rangle_{\overset{H}{\infty}} = \varlimsup_{\substack{a\to 0\\ |b-t|<a}} f(a,b). & (5.13) \end{aligned}$$

They are the *normal* and *non-tangential* upper limits from the upper-half plane to the real line, respectively.

Note the obvious inequality $\langle f_1, f_2\rangle_\infty \ge \langle f_1, f_2\rangle_{\overset{H}{\infty}}$ between pairings (5.7) and (5.11), which produces the corresponding relation between respective contravariant transforms.

There is an explicit duality between the covariant transform and the contravariant transform. Discussion of the grand maximal function in the Remark 4.3 shows usefulness of the covariant transform over a family of fiducial functionals. Thus, we shall not be surprised by the contravariant transform over a family of reconstructing vectors as well.

Definition 5.2. *Let $w : \mathrm{Aff} \to L_1(\mathbb{R})$ be a function. We define a new function $\rho_1 w$ on* Aff *with values in $L_1(\mathbb{R})$ via the point-wise action $[\rho_1 w](g) = \rho_1(g) w(g)$ of ρ_∞* (3.4). *If $\sup_g \|w(g)\|_1 < \infty$, then, for $f \in L_1(\mathrm{Aff})$, we define the* extended contravariant transform *by:*

$$[\mathcal{M}_w f](x) = \int_{\mathrm{Aff}} f(g)\, [\rho_1 w](g)\, dg. \tag{5.14}$$

Note, that (5.14) reduces to the contravariant transform (5.5) if we start from the constant function $w(g) = w_0$.

Definition 5.3. *We call a function r on $\mathbb{R}$ a* nucleus *if:*

i. r is supported in $[-1,1]$,

ii. $|r| < \frac{1}{2}$ almost everywhere, and

iii. $\int_{\mathbb{R}} r(x)\,dx = 0$, cf. (4.5).

Clearly, for a nucleus r, the function $s = \rho_1(a,b)r$ has the following properties:

i. s is supported in a ball centred at b and radius a,

ii. $|s| < \frac{1}{2a}$ almost everywhere, and

iii. $\int_{\mathbb{R}} s(x)\,dx = 0$.

In other words, $s = \rho_1(a,b)r$ is an *atom*, cf. [40, § III.2.2] and any atom may be obtained in this way from some nucleus and certain $(a,b) \in \mathrm{Aff}$.

Example 11. Let $f(g) = \sum_j \lambda_j \delta_{g_j}(g)$ with $\sum_j |\lambda_j| < \infty$ be a countable sum of point masses on Aff. If all values of $w(g_j)$ are nucleuses, then (5.14) becomes:

$$[\mathcal{M}_w f](x) = \int_{\mathrm{Aff}} f(g)\,[\rho_1 w](g)\,dg = \sum_j \lambda_j s_j, \tag{5.15}$$

where $s_j = \rho_1(g_j)w(g_j)$ are atoms. The right-hand side of (5.15) is known as an *atomic decomposition* of a function $h(x) = [\mathcal{M}_w f](x)$, see [40, § III.2.2].

6 Intertwining properties of covariant transforms

The covariant transform has obtained its name because of the following property.

Theorem 6.1. [23, 25]. *Covariant transform* (4.1) *intertwines ρ and the left regular representation Λ* (5.1) *on $L(G,U)$:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}. \tag{6.1}$$

Corollary 6.1. *The image space $\mathcal{W}(V)$ is invariant under the left shifts on G.*

The covariant transform is also a natural source of *relative convolutions* [19, 29], which are operators $A_k = \int_G k(g)\rho(g)\,dg$ obtained by integration a representation ρ of a group G with a suitable kernel k on G. In particular, inverse wavelet transform $\mathcal{M}_{w_0} f$ (5.5) can be defined from the relative convolution A_f as well: $\mathcal{M}_{w_0} f = A_f w_0$.

Corollary 6.2. *Covariant transform* (4.1) *intertwines the operator of convolution K (with kernel k) and the operator of relative convolution A_k, i.e. $K\mathcal{W} = \mathcal{W}A_k$.*

If the invariant pairing is defined by integration with respect to the Haar measure, cf. Example 7, then we can show an intertwining property for the contravariant transform as well.

Proposition 6.1. [20, Proposition 2.9]. *Inverse wavelet transform $\mathcal{M}_{w_0}$ (5.5) intertwines left regular representation Λ (5.1) on $L_2(G)$ and ρ:*

$$\mathcal{M}_{w_0}\Lambda(g) = \rho(g)\mathcal{M}_{w_0}. \tag{6.2}$$

Corollary 6.3. *The image $\mathcal{M}_{w_0}(L(G)) \subset V$ of a left invariant space $L(G)$ under the inverse wavelet transform $\mathcal{M}_{w_0}$ is invariant under the representation ρ.*

Remark 6.1. It is an important observation, that the above intertwining property is also true for some contravariant transforms which are not based on pairing (5.4). For example, in the case of the affine group all pairings (5.6), (5.11) and (non-linear!) (5.7) satisfy (6.2) for the respective representation ρ_p (3.4).

There is also a simple connection between a covariant transform and right shifts.

Proposition 6.2. [24, 25]. *Let G be a Lie group and ρ be a representation of G in a space V. Let $[\mathcal{W}f](g) = F(\rho(g^{-1})f)$ be a covariant transform defined by a fiducial operator $F : V \to U$. Then the right shift $[\mathcal{W}f](gg')$ by g' is the covariant transform $[\mathcal{W}'f](g) = F'(\rho(g^{-1})f)]$ defined by the fiducial operator $F' = F \circ \rho(g^{-1})$.*

In other words the covariant transform intertwines right shifts $R(g) : f(h) \mapsto f(hg)$ on the group G with the associated action

$$\rho_B(g) : F \mapsto F \circ \rho(g^{-1}) \tag{6.3}$$

on fiducial operators:

$$R(g) \circ \mathcal{W}_F = \mathcal{W}_{\rho_B(g)F}, \qquad g \in G. \tag{6.4}$$

Although the above result is obvious, its infinitesimal version has interesting consequences. Let G be a Lie group with a Lie algebra $\mathfrak{g}$ and ρ be a smooth representation of G. We denote by $d\rho_B$ the derived representation of the associated representation ρ_B (6.3) on fiducial operators.

Corollary 6.4. [24, 25]. *Let a fiducial operator F be a null-solution, i.e. $AF = 0$, for the operator $A = \sum_j a_j d\rho_B^{X_j}$, where $X_j \in \mathfrak{g}$ and a_j are constants. Then the covariant transform $[\mathcal{W}_F f](g) = F(\rho(g^{-1})f)$ for any f satisfies*

$$D(\mathcal{W}_F f) = 0, \qquad \text{where} \quad D = \sum_j \bar{a}_j \mathfrak{L}^{X_j}.$$

Here, $\mathfrak{L}^{X_j}$ are the left invariant fields (Lie derivatives) on G corresponding to X_j.

Example 12. Consider representation ρ (3.4) of the $ax + b$ group with the $p = 1$. Let A and N be the basis of $\mathfrak{g}$ generating one-parameter subgroups A and N (3.3), respectively. Then, the derived representations are:

$$[d\rho^{\mathsf{A}} f](x) = -f(x) - xf'(x), \qquad [d\rho^{\mathsf{N}} f](x) = -f'(x).$$

The corresponding left invariant vector fields on $ax+b$ group are:

$$\mathfrak{L}^{\mathsf{A}} = a\partial_a, \qquad \mathfrak{L}^{\mathsf{N}} = a\partial_b.$$

The mother wavelet $\frac{1}{x+\mathrm{i}}$ in (4.6) is a null solution of the operator

$$-d\rho^{\mathsf{A}} - \mathrm{i}d\rho^{\mathsf{N}} = I + (x+\mathrm{i})\frac{d}{dx}. \tag{6.5}$$

Therefore, the image of the covariant transform with fiducial operator F_+ (4.6) consists of the null solutions to the operator $-\mathfrak{L}^{\mathsf{A}} + \mathrm{i}\mathfrak{L}^{\mathsf{N}} = \mathrm{i}a(\partial_b + \mathrm{i}\partial_a)$, that is in the essence Cauchy–Riemann operator $\partial_{\bar{z}}$ (2.3) in the upper half-plane.

Example 13. In the above setting, the function $p(x) = \frac{1}{\pi}\frac{1}{x^2+1}$ (4.7) is a null solution of the operator:

$$(d\rho^{\mathsf{A}})^2 - d\rho^{\mathsf{A}} + (d\rho^{\mathsf{N}})^2 = 2I + 4x\frac{d}{dx} + (1+x^2)\frac{d^2}{dx^2}.$$

The covariant transform with the mother wavelet $p(x)$ is the Poisson integral, its values are null solutions to the operator $(\mathfrak{L}^{\mathsf{A}})^2 - \mathfrak{L}^{\mathsf{A}} + (\mathfrak{L}^{\mathsf{N}})^2 = a^2(\partial_b^2 + \partial_a^2)$, which is Laplace operator Δ (2.3).

Example 14. Fiducial functional F_m (4.9) is a null solution of the following functional equation:

$$F_m - F_m \circ \rho_\infty(\tfrac{1}{2}, \tfrac{1}{2}) - F_m \circ \rho_\infty(\tfrac{1}{2}, -\tfrac{1}{2}) = 0.$$

Consequently, the image of wavelet transform $\mathcal{W}_p^m$ (4.10) consists of functions which solve the equation:

$$(I - R(\tfrac{1}{2}, \tfrac{1}{2}) - R(\tfrac{1}{2}, -\tfrac{1}{2}))f = 0 \quad \text{or} \quad f(a,b) = f(\tfrac{1}{2}a, b + \tfrac{1}{2}a) + f(\tfrac{1}{2}a, b - \tfrac{1}{2}a).$$

The last relation is the key to the stopping time argument [14, Chapter VI, Lemma 2.2] and the dyadic squares technique, see for example [40, § IV.3], [14, Chapter VII, Theorem 1.1] or the picture on the front cover of the latter book.

The moral of the above Examples 12–14 is: there is a significant freedom in choice of covariant transforms. However, some fiducial functionals have special properties, which suggest the suitable technique (e.g., analytic, harmonic, dyadic, etc.) following from this choice.

7 Composing the covariant and the contravariant transforms

From Propositions 6.1, 6.2 and Remark 6.1 we deduce the following

Corollary 7.1. *The composition $\mathcal{M}_w \circ \mathcal{W}_F$ of a covariant $\mathcal{M}_w$ and contravariant $\mathcal{W}_F$ transforms is a map $V \to V$, which commutes with ρ, i.e., intertwines ρ with itself.*

In particular for the affine group and representation (3.4)*, $\mathcal{M}_w \circ \mathcal{W}_F$ commutes with shifts and dilations of the real line.*

Since the image space of $\mathcal{M}_w \circ \mathcal{W}_F$ is an Aff-invariant space, we shall be interested in the smallest building blocks with the same property. For the Hilbert spaces, any group invariant subspace V can be decomposed into a direct integral $V = \oplus \int V_\mu \, d\mu$ of *irreducible* subspaces V_μ, i.e. V_μ does not have any non-trivial invariant subspace [16, § 8.4]. For representations in Banach spaces complete reducibility may not occur and we shall look for *primary* subspace, i.e. space which is not a direct sum of two invariant subspaces [16, § 8.3]. We already identified such subspaces as generalized Hardy spaces in Definition 3.1. They are also related to covariant functional calculus [21, § 6] [25].

For irreducible Hardy spaces, we can use the following general principle, which has several different formulations, cf. [16, Theorem 8.2.1]:

Lemma 7.1 (Schur). [2, Lemma 4.3.1]. *Let ρ be a continuous unitary irreducible representation of G on the Hilbert space H. If a bounded operator $T : H \to T$ commutes with $\rho(g)$, for all $g \in G$, then $T = kI$, for some $\lambda \in \mathbb{C}$.*

Remark 7.1. A revision of proofs of the Schur's Lemma, even in different formulations, show that the result is related to the existence of joint invariant subspaces for all operators $\rho(g)$, $g \in G$.

In the case of classical wavelets, the relation between wavelet transform (4.2) and inverse wavelet transform (5.5) is suggested by their names.

Example 15. For an irreducible square integrable representation and admissible vectors v_0 and w_0, there is the relation [2, (8.52)]:

$$\mathcal{M}_{w_0} \mathcal{W}_{v_0} = kI, \tag{7.1}$$

as an immediate consequence from the Schur's lemma. Furthermore, square integrability condition (4.3) ensures that $k \neq 0$. The exact value of the constant k depends on v_0, w_0 and the Duflo–Moore operator [7, § 8.2] [2].

It is of interest here, that two different vectors can be used as analysing vector in (4.2) and for the reconstructing formula (5.5). Even a greater variety can be achieved if we use additional fiducial operators and invariant pairings.

For the affine group, recall the decomposition from Proposition 3.1 into invariant subspaces $L_2(\mathbb{R}) = H_2 \oplus H_2^\perp$ and the fact, that the restrictions ρ_2^+ and ρ_2^- of ρ_2 (3.4) on H_2 and $H_2^\perp$ are not unitary equivalent. Then, Schur's lemma implies:

Corollary 7.2. *Any bounded linear operator $T : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ commuting with ρ_2 has the form $k_1 I_{H_2} \oplus k_2 I_{H_2^\perp}$ for some constants k_1, $k_2 \in \mathbb{C}$. Consequently, the Fourier transform maps T to the operator of multiplication by $k_1 \chi_{(0,+\infty)} + k_2 \chi_{(-\infty,0)}$.*

Of course, Corollary 7.2 is applicable to the composition of covariant and contravariant transforms. In particular, the constants k_1 and k_2 may have zero values: for example, the zero value occurs for $\mathcal{W}$ (4.2) with an admissible vector v_0 and non-tangential limit $\mathcal{M}^H_{v_0^*}$ (5.13)—because a square integrable function $f(a, b)$ on Aff vanishes for $a \to 0$.

Example 16. The composition of contravariant transform $\mathcal{M}_{v_0^*}$ (5.10) with covariant transform $\mathcal{W}_\infty$ (4.10) is:

$$\begin{aligned}
[\mathcal{M}_{v_0^*}\mathcal{W}_\infty f](t) &= \sup_{a>|b-t|}\left\{\frac{1}{2a}\int_{b-a}^{b+a}|f(x)|\,dx\right\} \\
&= \sup_{b_1<t<b_2}\left\{\frac{1}{b_2-b_1}\int_{b_1}^{b_2}|f(x)|\,dx\right\}.
\end{aligned} \tag{7.2}$$

Thus, $\mathcal{M}_{v_0^*}\mathcal{W}_\infty f$ coincides with *Hardy–Littlewood maximal function* f^M (2.6), which contains important information on the original function f [31, § VIII.B.1]. Combining Propositions 6.1 and 6.2 (via Remark 6.1), we deduce that the operator $M : f \mapsto f^M$ commutes with ρ_p: $\rho_p M = M\rho_p$. Yet, M is non-linear and Corollary 7.2 is not applicable in this case.

Example 17. Let the mother wavelet $v_0(x) = \delta(x)$ be the Dirac delta function, then the wavelet transform $\mathcal{W}_\delta$ generated by ρ_∞ (3.4) on $C(\mathbb{R})$ is $[\mathcal{W}_\delta f](a,b) = f(b)$. Take the reconstruction vector $w_0(t) = (1-\chi_{[-1,1]}(t))/t/\pi$ and consider the respective inverse wavelet transform $\mathcal{M}_{w_0}$ produced by Hardy pairing (5.6). Then, the composition of both maps is:

$$\begin{aligned}
[\mathcal{M}_{w_0}\circ\mathcal{W}_\delta f](t) &= \lim_{a\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} f(b)\,\rho_\infty(a,b)w_0(t)\,\frac{db}{a} \\
&= \lim_{a\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} f(b)\,\frac{1-\chi_{[-a,a]}(t-b)}{t-b}\,db \\
&= \lim_{a\to 0}\frac{1}{\pi}\int_{|b|>a}\frac{f(b)}{t-b}\,db.
\end{aligned} \tag{7.3}$$

The last expression is the *Hilbert transform* $\mathcal{H} = \mathcal{M}_{w_0}\circ\mathcal{W}_\delta$, which is an example of a *singular integral operator* (SIO) [40, § I.5], [35, § 2.6] defined through the principal value (2.5) (in the sense of Cauchy). By Corollary 7.2 we know that $\mathcal{H} = k_1 I_{H_2}\oplus k_2 I_{H_2^\perp}$ for some constants k_1, $k_2\in\mathbb{C}$. Furthermore, we can directly check that $\mathcal{H}J = -J\mathcal{H}$, for the reflection J from (3.8), thus $k_1 = -k_2$. An evaluation of $\mathcal{H}$ on a simple function from H_2 (say, the Cauchy kernel $\frac{1}{x+\mathrm{i}}$) gives the value of the constant $k_1 = -\mathrm{i}$. Thus, $\mathcal{H} = (-\mathrm{i}I_{H_2})\oplus(\mathrm{i}I_{H_2^\perp})$.

In fact, the previous reasons imply the following

Proposition 7.1. [39, § III.1.1]. *Any bounded linear operator on $L_2(\mathbb{R})$ commuting with quasi-regular representation ρ_2 (3.4) and anticommuting with reflection J (3.8) is a constant multiple of Hilbert transform (7.3).*

Example 18. Consider the covariant transform $\mathcal{W}_q$ defined by the inadmissible wavelet $q(t)$ (4.8), the conjugated Poisson kernel. Its composition with the contravariant transform $\mathcal{M}^H_{v_0^+}$ (5.12) is

$$[\mathcal{M}^H_{v_0^+} \circ \mathcal{W}_q f](t) = \overline{\lim_{a\to 0}} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x)\,(t-x)}{(t-x)^2+a^2}\, dx \tag{7.4}$$

We can see that this composition satisfies to Proposition 7.1, the constant factor can again be evaluated from the Cauchy kernel $f(x) = \frac{1}{x+\mathrm{i}}$ and is equal to 1. Of course, this is a classical result [15, Theorem 4.1.5] in harmonic analysis that (7.4) provides an alternative expression for Hilbert transform (7.3).

Example 19. Let $\mathcal{W}$ be a covariant transfrom generated either by the functional $F_\pm$ (4.6) (i.e. the Cauchy integral) or $\frac{1}{2}(F_+ - F_-)$ (i.e. the Poisson integral) from the Example 4. Then, for contravariant transform $\mathcal{M}^H_{v_0^+}$ (5.9) the composition $\mathcal{M}^H_{v_0^+}\mathcal{W}$ becomes the normal boundary value of the Cauchy/Poisson integral, respectively. The similar composition $\mathcal{M}^H_{v_0^*}\mathcal{W}$ for reconstructing vector v_0^* (5.8) turns to be the non-tangential limit of the Cauchy/Poisson integrals.

The maximal function and SIO are often treated as elementary building blocks of harmonic analysis. In particular, it is common to define the Hardy space as a closed subspace of $L_p(\mathbb{R})$ which is mapped to $L_p(\mathbb{R})$ by either the maximal operator (7.2) or by the SIO (7.3) [40]*§ III.1.2 and § III.4.3 [8]. From this perspective, the coincidence of both characterizations seems to be non-trivial. On the contrast, we presented both the maximal operator and SIO as compositions of certain co- and contravariant transforms. Thus, these operators act between certain Aff-invariant subspaces, which we associated with generalized Hardy spaces in Definition 3.1. For the right choice of fiducial functionals, the coincidence of the respective invariant subspaces is quite natural.

The potential of the group-theoretical approach is not limited to the Hilbert space $L_2(\mathbb{R})$. One of possibilities is to look for a suitable modification of Schur's Lemma 7.1, say, to Banach spaces. However, we can proceed with the affine group without such a generalisation. Here is an illustration to a classical question of harmonic analysis: to identify the class of functions on the real line such that $\mathcal{M}^H_{v_0^*}\mathcal{W}$ becomes the identity operator on it.

Proposition 7.2. *Let B be the space of bounded uniformly continuous functions on the real line. Let $F : B \to \mathbb{R}$ be a fiducial functional such that:*

$$\lim_{a\to 0} F(\rho_\infty(1/a,0)f) = 0, \qquad \text{for all } f \in B \text{ such that } f(0) = 0 \tag{7.5}$$

and $F(\rho_\infty(1,b)f)$ is a continuous function of $b \in \mathbb{R}$ for a given $f \in B$.

Then, $\mathcal{M}^H_{v_0^*} \circ \mathcal{W}_F$ is a constant multiple of the identity operator on B.

Proof. First of all we note that $\mathcal{M}^H_{v_0^+}\mathcal{W}_F$ is a bounded operator on B. Let $v^*_{(a,b)} = \rho_\infty(a,b)v^*$. Obviously, $v^*_{(a,b)}(0) = v^*(-\frac{b}{a})$ is an eigenfunction for operators $\Lambda(a',0)$, $a' \in \mathbb{R}_+$ of the left regular representation of Aff:

$$\Lambda(a',0)v^*_{(a,b)}(0) = v^*_{(a,b)}(0). \tag{7.6}$$

This and the left invariance of pairing (5.2) imply that $\mathcal{M}^H_{v_0^*} \circ \Lambda(1/a, 0) = \mathcal{M}^H_{v_0^*}$ for any $(a, 0) \in \mathrm{Aff}$. Then, applying intertwining properties (6.1) we obtain that

$$\begin{aligned}
[\mathcal{M}^H_{v_0^*} \circ \mathcal{W}_F f](0) &= [\mathcal{M}^H_{v_0^*} \circ \Lambda(1/a, 0) \circ \mathcal{W}_F f](0) \\
&= [\mathcal{M}^H_{v_0^*} \circ \mathcal{W}_F \circ \rho_\infty(1/a, 0) f](0).
\end{aligned}$$

Using the limit $a \to 0$ (7.5) and the continuity of $F \circ \rho_\infty(1, b)$ we conclude that the linear functional $l : f \mapsto [\mathcal{M}^H_{v_0^*} \circ \mathcal{W}_F f](0)$ vanishes for any $f \in \mathbb{B}$ such that $f(0) = 0$. Take a function $f_1 \in B$ such that $f_1(0) = 1$ and define $c = l(f_1)$. From linearity of l, for any $f \in B$ we have:

$$l(f) = l(f - f(0)f_1 + f(0)f_1) = l(f - f(0)f_1) + f(0)l(f_1) = cf(0).$$

Furthermore, using intertwining properties (6.1) and (6.2):

$$\begin{aligned}
[\mathcal{M}^H_{v_0^*} \circ \mathcal{W}_F f](t) &= [\rho_\infty(1, -t) \circ \mathcal{M}^H_{v_0^*} \circ \mathcal{W}_F f](0) \\
&= [\mathcal{M}^H_{v_0^*} \circ \mathcal{W}_F \circ \rho_\infty(1, -t) f](0) \\
&= l(\rho_\infty(1, -t) f) \\
&= c[\rho_\infty(1, -t) f](0) \\
&= cf(t).
\end{aligned}$$

This completes the proof. □

To get the classical statement we need the following lemma.

Lemma 7.2. *For $w(t) \in L_1(\mathbb{R})$, define the fiducial functional on B:*

$$F(f) = \int_{\mathbb{R}} f(t)\, w(t)\, dt. \tag{7.7}$$

Then F satisfies the conditions (and thus the conclusions) of Proposition 7.2.

Proof. Let f be a continuous bounded function such that $f(0) = 0$. For $\varepsilon > 0$ chose

- $\delta > 0$ such that $|f(t)| < \varepsilon$ for all $|t| < \delta$;
- $M > 0$ such that $\int_{|t|>M} |w(t)|\, dt < \varepsilon$.

Then, for $a < \delta/M$, we have the estimation:

$$\begin{aligned}
|F(\rho_\infty(1/a, 0) f)| &= \left| \int_{\mathbb{R}} f\,(at)\ w(t)\, dt \right| \\
&\leq \left| \int_{|t|<M} f\,(at)\ w(t)\, dt \right| + \left| \int_{|t|>M} f\,(at)\ w(t)\, dt \right| \\
&\leq \varepsilon(\|w\|_1 + \|f\|_\infty).
\end{aligned}$$

Finally, for a uniformly continuous function g for $\varepsilon > 0$ there is $\delta > 0$ such that $|g(t + b) - g(t)| < \varepsilon$ for all $b < \delta$ and $t \in \mathbb{R}$. Then

$$|F(\rho_\infty(1, b)g) - F(g)| = \left| \int_{\mathbb{R}} (g(t + b) - g(t))\, w(t)\, dt \right| \leq \varepsilon \|w\|_1.$$

This proves the continuity of $F(\rho_\infty(1, b)g)$ at $b = 0$ and, by the group property, at any other point as well. □

Remark 7.2. A direct evaluation shows, that the constant $c = l(f_1)$ from the proof of Proposition 7.2 for fiducial functional (7.7) is equal to $c = \int_{\mathbb{R}} w(t)\, dt$. Of course, for non-trivial boundary values we need $c \neq 0$. On the other hand, admissibility condition (4.5) requires $c = 0$. Moreover, the classical harmonic analysis and the traditional wavelet construction are two "orthogonal" parts of the same covariant transform theory in the following sense. We can present a rather general bounded function $w = w_a + w_p$ as a sum of an admissible mother wavelet w_a and a suitable multiple w_p of the Poisson kernel. An extension of this technique to unbounded functions leads to *Calderón–Zygmund decomposition* [40, § I.4].

The table integral $\int_{\mathbb{R}} \frac{dx}{x^2+1} = \pi$ tells that the "wavelet" $p(t) = \frac{1}{\pi}\frac{1}{1+t^2}$ (4.7) is in $L_1(\mathbb{R})$ with $c = 1$, the corresponding wavelet transform is the Poisson integral. Its boundary behaviour from Proposition 7.2 is the classical result, cf. [14, Chapter I, Corollary 3.2]. The comparison of our arguments with the traditional proofs, e.g. in [14], does not reveal any significant distinctions. We simply made an explicit usage of the relevant group structure, which is implicitly employed in traditional texts anyway, cf. [4]. Further demonstrations of this type can be found in [1, 9].

8 Transported norms

If the functional F and the representation ρ in (4.1) are both linear, then the resulting covariant transform is a linear map. If $\mathcal{W}_F$ is injective, e.g. due to irreducibility of ρ, then $\mathcal{W}_F$ transports a norm $\|\cdot\|$ defined on V to a norm $\|\cdot\|_F$ defined on the image space $\mathcal{W}_F V$ by the simple rule:

$$\|u\|_F := \|v\|\,, \qquad \text{where the unique } v \in V \text{ is defined by } u = \mathcal{W}_F v. \tag{8.1}$$

By the very definition, we have the following

Proposition 8.1. *i. $\mathcal{W}_F$ is an isometry $(V, \|\cdot\|) \to (\mathcal{W}_F V, \|\cdot\|_F)$.*

ii. If the representation ρ acts on $(V, \|\cdot\|)$ by isometries then $\|\cdot\|_F$ is left invariant.

A touch of non-triviality occurs if the transported norm can be naturally expressed in the original terms of G.

Example 20. It is common to consider a unitary square integrable representation ρ and an admissible mother wavelet $f \in V$. In this case, wavelet transform (4.2) becomes an isometry to square integrable functions on G with respect to a Haar measure [2, Theorem 8.1.3]. In particular, for the affine group and setup of Example 3, the wavelet transform with an admissible vector is a multiple of an isometry map from $L_2(\mathbb{R})$ to the functions on the upper half-plane, i.e., the $ax+b$ group, which are square integrable with respect to the Haar measure $a^{-2}\, da\, db$.

A reader expects that there are other interesting examples of the transported norms, which are not connected to the Haar integration.

Example 21. In the setup of Example 4, consider the space $L_p(\mathbb{R})$ with representation (3.4) of Aff and Poisson kernel $p(t)$ (4.7) as an inadmissible mother wavelet. The norm transported by $\mathcal{W}_P$ to the image space on Aff is [36, § A.6.3]:

$$\|u\|_p = \sup_{a>0} \left(\int_{-\infty}^{\infty} |u(a,b)|^p \, \frac{db}{a} \right)^{\frac{1}{p}} . \tag{8.2}$$

In the theory of Hardy spaces, the L_p-norm on the real line and transported norm (8.2) are naturally intertwined, cf. [36, Theorem A.3.4.1(iii)], and are used interchangeably.

The second possibility to transport a norm from V to a function space on G uses an contravariant transform $\mathcal{M}_v$:

$$\|u\|_v := \|\mathcal{M}_v u\| . \tag{8.3}$$

Proposition 8.2. *i. The contravariant transform $\mathcal{M}_v$ is an isometry $(L, \|\cdot\|_v) \to (V, \|\cdot\|)$.*

ii. If the composition $\mathcal{M}_v \circ \mathcal{W}_F = cI$ is a multiple of the identity on V then transported norms $\|\cdot\|_v$ (8.3) and $\|\cdot\|_F$ (8.1) differ only by a constant multiplier.

The above result is well-known for traditional wavelets.

Example 22. In the setup of Example 15, for a square integrable representation and two admissible mother wavelets v_0 and w_0 we know that $\mathcal{M}_{w_0}\mathcal{W}_{v_0} = kI$ (7.1), thus transported norms (8.1) and (8.3) differ by a constant multiplier. Thus, norm (8.3) is also provided by the integration with respect to the Haar measure on G.

In the theory of Hardy spaces the result is also classical.

Example 23. For the fiducial functional F with property (7.5) and the contravariant transform $\mathcal{M}^H_{v_0^*}$ (5.13), Proposition 7.2 implies $\mathcal{M}^H_{v_0^*} \circ \mathcal{W}_F = cI$. Thus, the norm transported to Aff by $\mathcal{M}^H_{v_0^*}$ from $L_p(\mathbb{R})$ up to factor coincides with (8.2). In other words, the transition to the boundary limit on the Hardy space is an isometric operator. This is again a classical result of the harmonic analysis, cf. [36, Theorem A.3.4.1(ii)].

The co- and contravariant transforms can be used to transport norms in the opposite direction: from a classical space of functions on G to a representation space V.

Example 24. Let V be the space of σ-finite signed measures of a bounded variation on the upper half-plane. Let the $ax+b$ group acts on V by the representation adjoint to $[\rho_1(a,b)f](x,y) = a^{-1}f(\frac{x-b}{a}, \frac{y}{a})$ on $L_2(\mathbb{R}^2_+)$, cf. (3.2). If the mother wavelet v_0 is the indicator function of the square $\{0 < x < 1, 0 < y < 1\}$, then the covariant transform of a measure μ is $\tilde{\mu}(a,b) = a^{-1}\mu(Q_{a,b})$, where $Q_{a,b}$ is the square $\{b < x < b+a, 0 < y < a\}$. If we request that $\tilde{\mu}(a,b)$ is a bounded function on the affine group, then μ is a Carleson measure [14, § I.5]. A norm transported from $L_\infty(\text{Aff})$ to the appropriate subset of V becomes the Carleson norm of measures. The indicator function of a tent taken as a mother wavelet will lead to an equivalent definition.

It was already mentioned in Remark 4.3 and Example 11 that we may be interested to mix several different covariant and contravariant transforms. This motivate the following statement.

Proposition 8.3. *Let $(V, \|\cdot\|)$ be a normed space and ρ be a continuous representation of a topological locally compact group G on V. Let two fiducial operators F_1 and F_2 define the respective covariant transforms $\mathcal{W}_1$ and $\mathcal{W}_2$ to the same image space $W = \mathcal{W}_1 V = \mathcal{W}_2 V$. Assume, there exists an contravariant transform $\mathcal{M} : W \to V$ such that $\mathcal{M} \circ \mathcal{W}_1 = c_1 I$ and $\mathcal{M} \circ \mathcal{W}_2 = c_2 I$. Define by $\|\cdot\|_{\mathcal{M}}$ the norm on U transpordef from V by $\mathcal{M}$. Then*

$$\|\mathcal{W}_1 v_1 + \mathcal{W}_2 v_2\|_{\mathcal{M}} = \|c_1 v_1 + c_2 v_2\|, \quad \text{for any } v_1, v_2 \in V. \tag{8.4}$$

Proof. Indeed:

$$\begin{aligned}\|\mathcal{W}_1 v_1 + \mathcal{W}_2 v_2\|_{\mathcal{M}} &= \|\mathcal{M} \circ \mathcal{W}_1 v_1 + \mathcal{M} \circ \mathcal{W}_2 v_2\| \\ &= \|c_1 v_1 + c_2 v_2\|,\end{aligned}$$

by the definition of transported norm (8.3) and the assumptions $\mathcal{M} \circ \mathcal{W}_i = c_i I$. □

Although the above result is simple, it does have important consequences.

Corollary 8.1 (Orthogonality Relation). *Let ρ be a square integrable representation of a group G in a Hilbert space V. Then, for any two admissible mother wavelets f and f' there exists a constant c such that:*

$$\int_G \langle v, \rho(g) f\rangle \overline{\langle v', \rho(g) f'\rangle}\, dg = c\, \langle v, v'\rangle \qquad \text{for any } v_1, v_2 \in V. \tag{8.5}$$

Moreover, the constant $c = c(f', f)$ is a sesquilinear form of vectors f' and f.

Proof. We can derive (8.5) from (8.4) as follows. Let $\mathcal{M}_f$ be the inverse wavelet transform (5.5) defined by the admissible vector f, then $\mathcal{M}_f \circ \mathcal{W}_f = I$ on V providing the right scaling of f. Furthermore, $\mathcal{M}_f \circ \mathcal{W}_{f'} = \bar{c} I$ by (7.1) for some complex constant c. Thus, by (8.4):

$$\|\mathcal{W}_f v + \mathcal{W}_{f'} v'\|_{\mathcal{M}} = \|v + \bar{c} v'\|.$$

Now, through the polarisation identity [18, Problem 476] we get the equality (8.5) of inner products. □

The above result is known as the *orthogonality relation* in the theory of wavelets, for some further properties of the constant c see [2, Theorem 8.2.1].

Here is an application of Proposition 8.3 to harmonic analysis, cf. [15, Theorem 4.1.7]:

Corollary 8.2. *The covariant transform $\mathcal{W}_q$ with conjugate Poisson kernel q (4.8) is a bounded map from $(L_2(\mathbb{R}), \|\cdot\|)$ to $(L(\mathrm{Aff}), \|\cdot\|_2)$ with norm $\|\cdot\|_2$ (8.2). Moreover:*

$$\|\mathcal{W}_q f\|_2 = \|f\|, \qquad \text{for all } f \in L_2(\mathbb{R}).$$

Proof. As we establish in Example 18 for contravariant transform $\mathcal{M}^{H}_{v_0^+}$ (5.12), $\mathcal{M}^{H}_{v_0^+} \circ \mathcal{W}_q = -\mathrm{i}I$ and $\mathrm{i}I$ on H_2 and $H_2^{\perp}$, respectively. Take the unique presentation $f = u+u^{\perp}$, for $u \in H_2$ and $u^{\perp} \in H_2^{\perp}$. Then, by (8.4)

$$\|\mathcal{W}_q f\|_2 = \left\|-\mathrm{i}u + \mathrm{i}u^{\perp}\right\| = \left\|u + u^{\perp}\right\| = \|f\| .$$

This completes the proof. □

9 Conclusion

We demonstrated that both, real and complex, techniques in harmonic analysis have the same group-theoretical origin. Moreover, they are complemented by the wavelet construction. Therefore, there is no any confrontation between these approaches and they can be lined up as in Table 1. In other words, the binary opposition of the real and complex methods resolves via Kant's triad thesis-antithesis-synthesis: complex-real-covariant.

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Covariant scheme	**Complex variable**	**Real variable**
Covariant transform is $\mathcal{W}_F^\rho : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v)$. In particular, the wavelet transform for the mother wavelet v_0 is $\tilde{v}(g) = \langle v, \rho(g)v_0\rangle$.	The Cauchy integral is generated by the mother wavelet $\frac{1}{2\pi\mathrm{i}}\frac{1}{x+\mathrm{i}}$. The Poisson integral is generated by the mother wavelet $\frac{1}{\pi}\frac{1}{x^2+1}$	The averaging operator $\tilde{f}(b) = \frac{1}{2a}\int_{b-a}^{b+a} f(t)\,dt$ is defined by the mother wavelet $\chi_{[-1,1]}(t)$, to average the modulus of $f(t)$ we use all elements of the unit ball in $L_\infty[-1,1]$.
The covariant transform maps vectors to functions on G or, in the induced case, to functions on the homogeneous space G/H.	Functions are mapped from the real line to the upper half-plane parametrised by either the $ax+b$-group or the homogeneous space $\mathrm{SL}_2(\mathbb{R})/K$.	Functions are mapped from the real line to the upper half-plane parametrised by either the $ax+b$-group or the homogeneous space $\mathrm{SL}_2(\mathbb{R})/A$.
Annihilating action on the mother wavelet produces functional relation on the image of the covariant transform	The operator $-d\rho^{\mathsf{A}} - \mathrm{i}d\rho^{\mathsf{N}} = I + (x+\mathrm{i})\frac{d}{dx}$ annihilates the mother wavelet $\frac{1}{2\pi\mathrm{i}}\frac{1}{x+\mathrm{i}}$, thus the image of wavelet transform is in the kernel of the Cauchy–Riemann operator $-\mathfrak{L}^{\mathsf{A}} + \mathrm{i}\mathfrak{L}^{\mathsf{N}} = \mathrm{i}a(\partial_b + \mathrm{i}\partial_a)$. Similarly, for the Laplace operator.	The mother wavelet $v_0 = \chi_{[-1,1]}$ satisfies the equality $\chi_{[-1,1]} = \chi_{[-1,0]} + \chi_{[0,1]}$, where both terms are again scaled and shifted v_0. The image of the wavelet transform is suitable for the stopping time argument and the dyadic squares technique.
An invariant pairing $\langle\cdot,\cdot\rangle$ generates the contravariant transform $[\mathcal{M}_{w_0}^\rho f]\,\langle f(g), \rho(g)w_0\rangle$ for	The contravariant transform with the invariant Hardy pairing on the $ax+b$ group produces boundary values of functions on the real line.	The covariant transform with the invariant sup pairing produces the vertical and non-tangential maximal functions.
The composition $\mathcal{M}_v \circ \mathcal{W}_F$ of the covariant and contravariant transforms is a multiple of the identity on irreducible components.	SIO is a composition of the Cauchy integral and its boundary value.	The Hardy–Littlewood maximal function is the composition of the averaging operator and the contravariant transform from the invariant sup pairing.
The Hardy space is an invariant subspace of the group representation.	The Hardy space consists of the limiting values of the Cauchy integral. SIO is bounded on this space.	The Hardy–Littlewood maximal operator is bounded on the Hardy space H_p .

Table 1: The correspondence between different elements of harmonic analysis.

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