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ON A CERTAIN CLASS OF OPERATOR ALGEBRAS AND THEIR DERIVATIONS

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Abstract. Given a von Neumann algebra M with a faithful normal finite trace, we introduce the so-called finite tracial algebra M_f as the intersection of L_p-spaces $L_p(M,\mu)$ over all $p \geq 1$ and over all faithful normal finite traces μ on M. Basic algebraic and topological properties of finite tracial algebras are studied. We prove that all derivations on these algebras are inner.

1 Introduction

In the present paper we introduce a new class of algebras, the so-called *finite tracial algebras,* which are defined as the intersection of non-commutative L_p-spaces $L_p(M,\mu)$ [13] over all $p \in [1,\infty)$ and over all faithful normal finite (f.n.f.) traces μ on a von Neumann algebra M. Equivalently, a finite tracial algebra M_f is the intersection of all non-commutative Arens algebras $L^{\omega}(M,\mu) = \bigcap\limits_{p\geq 1} L_p(M,\mu)$, over all f.n.f. traces μ. It is known that Arens algebras are metrizable locally convex $*$-algebras with respect to the topology generated by the system of L_p-norms for a fixed trace. Algebraic and topological properties of Arens algebras have been investigated in the papers [1, 2, 3, 6, 9].

In the present paper we study basic properties of finite tracial algebras with the topology generated by all L_p-norms $\{\|\cdot\|_p^{\mu}\}$, where $p \in [1,\infty)$ and μ runs over all f.n.f. traces on a given von Neumann algebra M. We prove that a finite tracial algebra M_f is metrizable or reflexive if and only if the center of the von Neumann algebra M is finite-dimensional; in this case M_f coincides with an appropriate Arens algebra. We also give a necessary and sufficient condition for M_f to coincide (as a set) with M. But even in this case one has a new topology on the von Neumann algebra M. We obtain also a description of the dual space for the algebra M_f.

Finally we prove that every derivation on a solid subalgebra of the Arens algebra $L^{\omega}(M,\tau)$ is inner. In particular we obtain that the algebra M_f admits only inner derivations.

Throughout the paper we consider a von Neumann algebra M with a f.n.f trace. Therefore M is a finite von Neumann algebra and thus all closed densely defined operators affiliated with M are measurable with respect to M, i.e. the set of all such operators coincides with the algebra $S(M)$ of all measurable operators and hence also with the algebra $LS(M)$ of all locally measurable operators affiliated with M; moreover the center of $S(M) = LS(M)$ coincides with the set of operators affiliated with the center of M.

2 Preliminaries

Let M be a von Neumann algebra with the positive cone M^+ and let $\mathbf{1}$ denote the identity operator in M.

A positive linear functional μ is called a *finite trace,* if $\mu(uxu^*) = \mu(x)$ for all $x \in M$ and each unitary operator $u \in M$.

A finite trace μ is said to be *faithful* if for $x \in M^+, \mu(x) = 0$ implies that $x = 0$.

A finite trace μ is normal if given any monotone net $\{x_\alpha\}$ increasing to $x \in M$, one has $\mu(x) = \sup \mu(x_\alpha)$.

Let τ be a fixed faithful normal finite (f.n.f.) trace on a von Neumann algebra M. The Radon–Nikodym theorem [11, Theorem 14] implies that given any f.n.f. trace μ on M there exists a positive operator $h \in L_1(M,\tau)$ affiliated with the center of M such that $\mu(x) = \tau(hx)$ for all $x \in M$. This operator h is called the Radon–Nikodym derivative of the trace μ with respect to the trace τ and is denoted by $\frac{d\mu}{d\tau}$.

We recall [11], [13] that given a f.n.f. trace τ on a von Neumann algebra M the space $L_p(M,\tau)$, $p \in [1,\infty)$, is deÏned as

$$L_p(M,\tau) = \{x \in S(M) : |x|^p \in L_1(M,\tau)\}.$$

The space $L_p(M,\tau)$ equipped with the norm $\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}$ is a Banach space and its dual space coincides with $L_q(M,\tau)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and the duality is given by

$$\langle x, a\rangle = f_a(x) = \tau(ax)$$

for all $f_a \in L_p(M,\tau)^*, a \in L_q(M,\tau)$ (see [13, Theorem 4.4]).

Following [9] consider the intersection

$$L^\omega(M,\tau) = \bigcap_{p\in[1,\infty)} L_p(M,\tau).$$

It is known (see also [2], [3], [6]), that $L^\omega(M,\tau)$ is a complete locally convex $*$-algebra with respect to the topology t^τ generated by the system of norms $\{\|\cdot\|\}_{p\in[1,\infty)}$.

Each operator $a \in \bigcup_{q\in[1,\infty)} L_q(M,\tau)$ defines a continuous linear functional f_a on $(L^\omega(M,\tau), t^\tau)$ by the formula $f_a(x) = \tau(ax)$, and conversely given an arbitrary continuous linear functional f on the algebra $(L^\omega(M,\tau), t^\tau)$ there exists an element $a \in \bigcup_{q\in[1,\infty)} L_q(M,\tau)$ such that $f(x) = \tau(ax)$.

3 Finite tracial algebras

Let M be a finite von Neumann algebra. Denote by $\mathcal{F}$ the set of all f.n.f. traces on M and from now on suppose that $\mathcal{F} \neq \emptyset$.

Consider the space

$$M_f = \bigcap_{\mu \in \mathcal{F}} \bigcap_{p \in [1,\infty)} L_p(M,\mu) = \bigcap_{\mu \in \mathcal{F}} L^\omega(M,\mu).$$

On the space M_f one can consider the topology t, generated by the system of norms $\{\|\cdot\|_p^\mu\} : \mu \in \mathcal{F}, p \in [1,\infty)$.

Since each Arens algebra $L^\omega(M,\mu)$, $\mu \in \mathcal{F}$, is a complete locally convex topological $*$-algebra in $S(M)$ from the above definition one easily obtains the following

Theorem 3.1. *(M_f, t) is a complete locally convex topological $*$-algebra.*

Definition 3.1. The topological $*$-algebra M_f is called the *finite tracial algebra* with respect to the von Neumann algebra M.

Remark 3.1. *Finite tracial algebras present examples of so called GW^*-algebras in the sense of* [10].

Recall (see [10]) that a topological $*$-algebra (A, t_A) is called a GW^*-algebra, if A has a W^*-subalgebra B with $(\mathbf{1} + x^*x)^{-1} \in B$ for all $x \in A$ and the unit ball of B is t_A-bounded.

The finite tracial algebra M_f is a GW^*-algebra. Since $M \subset M_f$, it is sufficient to show that the unit ball in M is t-bounded in M_f.

Let $x \in M$, $\|x\|_\infty \leq 1$. For $\mu \in \mathcal{F}$ and $1 \leq p < \infty$, we have

$$\|x\|_p^\mu = \|x\mathbf{1}\|_p^\mu \leq \|x\|_\infty \|\mathbf{1}\|_p^\mu \leq \mu(\mathbf{1})^{\frac{1}{p}},$$

i.e. $\|x\|_p^\mu = \|x\mathbf{1}\|_p^\mu \leq \mu(\mathbf{1})^{\frac{1}{p}}$ for all $x \in M$, $\|x\|_\infty \leq 1$. This means that the unit ball of M is t-bounded in M_f. Therefore M_f is a GW^*-algebra.

Although the algebra M_f contains M, it is a rather small algebra, since it is contained in all $L_p(M,\mu)$ for all $p \geq 1$ and f.n.f. traces μ on M. The following result gives necessary and sufficient conditions for M_f to coincide with M.

Theorem 3.2. *For a finite von Neumann algebra M the following conditions are equivalent*

(i) $M_f = M$;

(ii) *M is a finite sum of homogeneous type $I_n, n \in \mathbb{N}$, von Neumann algebras.*

The proof of this theorem consists of several auxiliary proposition which are interesting on their own. Let us start with the commutative case.

Proposition 3.1. *Let M be a von Neumann algebra with a faithful normal trace and Z be its center. Then the center of the algebra M_f coincides with Z, i.e. $Z(M_f) = Z$. In particular, if M is abelian, then $M_f = M$.*

Proof. Let M be a von Neumann algebra with a faithful normal finite trace τ, and $\tau(\mathbf{1}) = 1$.

Consider $x \in Z(M_f)$, $x \geq 0$ and let $x = \int\limits_0^\infty \lambda\, de_\lambda$ be the spectral resolution of x. Since $x \in Z(M_f)$ and $M \subset M_f$, we have that $e_\lambda \in Z$ for all $\lambda \in \mathbb{R}$. Passing if necessary to the element $\varepsilon\mathbf{1} + x$, we may suppose without loss of generality that $e_1 = 0$.

For $n \in \mathbb{N}$ set

$$p_n = e_{(n+1)^2} - e_{n^2}$$

and

$$y = \sum_{n\in\mathbb{N}} n^2 p_n.$$

Since $xp_n \geq n^2 p_n$ for all $n \in \mathbb{N}$, we have that $0 \leq y \leq x$, and hence $y \in M_f$.

Let

$$F = \{n \in \mathbb{N} : t_n = \tau(p_n) \neq 0\}$$

and

$$h = \sum_{n\in F} \frac{1}{n^2 t_n} p_n \in Z(S(M)).$$

Since

$$\bigvee_{n=1}^{m} p_n = \bigvee_{n=1}^{m} (e_{(n+1)^2} - e_{n^2}) = \sum_{n=1}^{m} (e_{(n+1)^2} - e_{n^2}) = e_{(m+1)^2} - e_1 = e_{(m+1)^2} \uparrow \mathbf{1},$$

one has that

$$\bigvee_{n=1}^{\infty} p_n = \mathbf{1}.$$

Therefore there exists $h^{-1} \in S(M)$. Further we have

$$\tau(h) = \sum_{n\in F} \frac{1}{n^2 t_n} \tau(p_n) = \sum_{n\in F} \frac{1}{n^2 t_n} t_n = \sum_{n\in F} \frac{1}{n^2} \leq \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty,$$

i.e. $h \in L^1(M, \tau)$.

Put $\mu(\cdot) = \tau(h\cdot)$. Since $y \in M_f$, it follows that $y \in L^1(M, \mu)$. Therefore, $\mu(y) < \infty$.

On the other hand,

$$hy = \sum_{n\in F} \frac{1}{n^2 t_n} p_n \sum_{n\in\mathbb{N}} n^2 p_n = \sum_{n\in F} \frac{1}{t_n} p_n,$$

and thus

$$\mu(y) = \tau(hy) = \sum_{n\in F} \frac{1}{t_n} \tau(p_n) = \sum_{n\in F} \frac{1}{t_n} t_n = \sum_{n\in F} 1 = |F|,$$

where F is the cardinality of the set F. Since $\mu(y) < \infty$, this implies that F is a finite set. Let $k = \max\{n : n \in F\}$. Then $\tau(p_n) = 0$ for all $n > k$, and since τ is faithful, we have that $p_n = 0$ for all $n > k$, i.e. $e_{(n+1)^2} = e_{n^2}$. As $e_{n^2} \uparrow \mathbf{1}$, we have that $e_{n^2} = \mathbf{1}$ for all $n > k$. This means that $0 \leq x \leq (k+1)^2\mathbf{1}$, i.e. $x \in Z$. □

Proposition 3.2. *Let M be a type $I_n, n \in \mathbb{N}$, von Neumann algebra. Then $M_f = M$.*

Proof. By [12, Ch. V, Theorem 1.27] the von Neumann algebra M of type I_n ($n \in \mathbb{N}$) can be represented as $M = Z \otimes B(H_n)$, where Z is the center M and H_n is the n-dimensional Hilbert space. Put $\mathcal{F}_Z = \{\tau|_Z : \tau \in \mathcal{F}\}$. Therefore by Proposition 3.1 we obtain

$$\begin{aligned} M_f &= \bigcap_{p\in[1,\infty)} \bigcap_{\tau\in\mathcal{F}} L_p(M,\tau) = \bigcap_{p\in[1,\infty)} \bigcap_{\mu\in\mathcal{F}_Z} L_p(Z,\mu) \otimes B(H_n) = \\ &= \left(\bigcap_{p\in[1,\infty)} \bigcap_{\mu\in\mathcal{F}} L_p(Z,\mu) \right) \otimes B(H_n) = \\ &= Z_f \otimes B(H_n) = Z \otimes B(H_n) = M. \end{aligned}$$

□

Proposition 3.3. *Let M be a finite von Neumann algebra which is isomorphic to the direct sum of an infinite number of homogeneous type I_n ($n \in \mathbb{N}$) von Neunamm algebras. Then $M_f \neq M$.*

Proof. Suppose that $M = \sum_{k\in K}^{\oplus} M_k$, where K is an infinite subset of $\mathbb{N}$, and M_k is a homogeneous type I_k von Neumann algebra.

Since the set K is infinite, there exists a sequence $\{k_n\} \subset K$ such that $k_n \geq 2^n$ for all $n \in \mathbb{N}$. We have that

$$M_{k_n} = Z_{k_n} \otimes B(H_{k_n}),$$

where Z_{k_n} is the center of M_{k_n} and

$$N_n = \mathbf{1}_n \otimes B(H_n) \subset M_{k_n}.$$

Therefore the algebra M contains a subalgebra $*$-isomorphic to the algebra $N = \sum_{n\in\mathbb{N}}^{\oplus} N_n$.

Hence, without loss of generality we may assume that $M = \sum_{n\in\mathbb{N}}^{\oplus} N_n$, where $N_n = B(H_{2^n})$ is the algebra of all $2^n \times 2^n$ matrices over $\mathbb{C}$. On each N_n we consider the unique tracial state (i.e. normalized f.n.f. trace) μ_n and define on M the following f.n.f. trace

$$\tau(x) = \sum_{n\in\mathbb{N}} 2^{-n} \mu_n(x_n),$$

where $x = \sum_{n\in\mathbb{N}}^{\oplus} x_n$. Then every f.n.f. trace μ on M has the form

$$\mu(x) = \tau(hx) = \sum_{n\in\mathbb{N}} 2^{-n} \mu_n(h_n x_n) = \sum_{n\in\mathbb{N}} 2^{-n} \alpha_n \mu_n(x_n),$$

where

$$h = \sum_{n\in\mathbb{N}}^{\oplus} h_n = \sum_{n\in\mathbb{N}}^{\oplus} \alpha_n \mathbf{1}_n \in L_1(M,\tau),$$

i.e. $\alpha_n > 0$, $n \in \mathbb{N}$, $\sum\limits_{n\in\mathbb{N}} 2^{-n}\alpha_n < \infty$.

Take a minimal projection p_n in each $N_n = B(H_{2^n})$. Then $\mu_n(p_n) = 1/2^n$.

Consider the unbounded element $x = \sum_{n\in\mathbb{N}}^{\oplus} np_n$ in $S(M) \setminus M$ and let us prove that $x \in M_f$. For every f.n.f. trace μ on M one has that

$$\mu(x^p) = \sum_{n\in\mathbb{N}} 2^{-n}\alpha_n\mu_n(n^p p_n) = \sum_{n\in\mathbb{N}} 2^{-n}\alpha_n n^p 2^{-n} < \infty,$$

because $n^p 2^{-n} < 1$ for sufficiently large $n \in \mathbb{N}$. Therefore $x \in L_p(M, \mu)$ for all $p \geq 1$ and every f.n.f. trace $\mu \in \mathcal{F}$, i.e. $x \in M_f$. □

Proposition 3.4. *Let M be a type II_1 von Neumann algebra with a f.n.f. trace τ. Then $M_f \neq M$.*

Proof. Suppose that the trace τ is normalized, i.e. $\tau(\mathbf{1}) = 1$, and denote by Φ the canonical center-valued trace on M. Since M is of type II_1, there exists a projection p_1 such that

$$p_1 \sim \mathbf{1} - p_1.$$

Therefore from $\Phi(p_1) + \Phi(p_1^{\perp}) = \Phi(\mathbf{1}) = \mathbf{1}$ and $\Phi(p_1) = \Phi(p_1^{\perp})$ we obtain that

$$\Phi(p_1) = \Phi(p_1^{\perp}) = \frac{1}{2}\mathbf{1}.$$

Suppose that we have constructed mutually orthogonal projections $p_1, p_2, \ldots, p_n$ in M such that

$$\Phi(p_k) = \frac{1}{2^k}\mathbf{1}, k = \overline{1, n}.$$

Set $e_n = \sum\limits_{k=1}^{\infty} p_k$. Then $\Phi(e_n^{\perp}) = \frac{1}{2^n}\mathbf{1}$. Now take a projection $p_{n+1} \leq e_n^{\perp}$ such that

$$p_{n+1} \sim e_n^{\perp} - p_{n+1},$$

i.e.

$$\Phi(p_{n+1}) = \frac{1}{2^{n+1}}\mathbf{1}.$$

In this manner we obtain a sequence $\{p_n\}_{n\in\mathbb{N}}$ of mutually orthogonal projections such that

$$\Phi(p_n) = \frac{1}{2^n}\mathbf{1},\ n \in \mathbb{N}.$$

It is clear that $\tau(p_n) = \tau(\Phi(p_n)) = \frac{1}{2^n}$, $n \in \mathbb{N}$.

From

$$\sum_{n=1}^{\infty} \|np_n\|_1^{\tau} = \sum_{n=1}^{\infty} \tau(np_n) = \sum_{n=1}^{\infty} \frac{n}{2^n} < \infty,$$

it follows that the element $x = \sum\limits_{n=1}^{\infty} np_n$ belongs to $L_1(M, \tau)$, and it is unbounded, i.e. $x \notin M$.

On the other hand, for an arbitrary central element $h \in L_1(M, \tau)$, $h > 0$, and $n \in \mathbb{N}$ we have

$$\tau(hp_n) = \tau(\Phi(hp_n)) = \tau(h\Phi(p_n)) = \tau\left(h\frac{1}{2^n}\right) = \frac{1}{2^n}\tau(h).$$

Therefore for an arbitrary f.n.f. trace μ on M with $\frac{d\mu}{d\tau} = h$ we have

$$\mu(|x|^p) = \tau(x^p) = \tau(hx^p) = \tau\left(h\sum_{n=1}^{\infty} n^p p_n\right) = \sum_{n=1}^{\infty} n^p \tau(h_n p_n) = \tau(h)\sum_{n=1}^{\infty}\frac{n^p}{2^n} < \infty,$$

i.e. $x \in L_p(M,\mu)$ for all $p \geq 1$ and every f.n.f. trace μ. Therefore $x \in M_f \setminus M$. □

Proof of Theorem 3.2. The implication $(i) \Rightarrow (ii)$ follows by Propositions 3.3 and 3.4, while $(ii) \Rightarrow (i)$ follows by Propositions 3.2. □

Now let us describe continuous linear functionals on the space (M_f, t).

Theorem 3.3. *Given any $\mu \in \mathcal{F}$, $1 < q < \infty$, and $a \in L_q(M,\mu)$ the functional $\varphi(x) = \mu(xa)$, $x \in M_f$, is a continuous linear functional on (M_f, t). Conversely for any continuous linear functional φ on (M_f, t) there exist $\mu \in \mathcal{F}$, $1 < q < \infty$ and $a \in L_q(M,\mu)$ such that*

$$\varphi(x) = \mu(xa),\ x \in M_f.$$

Proof. Let $\mu \in \mathcal{F}$, $1 < q < \infty$, $a \in L_q(M,\mu)$. Put

$$\varphi_a(x) = \mu(xa),\ x \in M_f.$$

Take $p \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Since

$$|\varphi_a(x)| = |\mu(xa)| \leq \|a\|_q^\mu \|x\|_p^\mu$$

for all $x \in M_f$, one has that φ_a is a continuous linear functional on (M_f, t).

Conversely, let φ be a continuous linear functional on (M_f, t). By [14, Corollary 1, P. 43] there exist $\mu \in \mathcal{F}$, $1 \leq p < \infty$, $c > 0$, such that

$$|\varphi(x)| \leq c\|x\|_p^\mu$$

for all $x \in M_f$. Since $M \subset M_f$ and M is $\|\cdot\|_p^\mu$-dense in $L_p(M,\tau)$, the functional φ can be uniquely extended onto $L_p(M,\mu)$. By [13, Theorem 4.4] there exists $a \in L_q(M,\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\varphi(x) = \mu(xa)$$

for all $x \in L_p(M,\mu)$. In particular,

$$\varphi(x) = \mu(xa)$$

for all $x \in M_f$, i.e. $\varphi = \varphi_a$. □

If the von Neumann algebra M is a factor then it has a unique (up to a scalar multiple) f.n.f. trace μ. In this case the finite tracial algebra M_f coincides with the Arens algebra $L^\omega(M,\mu)$ and the topology t merges to the topology t^μ generated by the system of norms $\{\|\cdot\|_p^\mu\}_{p\geq 1}$. The following theorem describes the general case where this phenomenon occurs.

Recall some notions of the theory of linear topological spaces. Let E be a locally convex linear topological space. An absolutely convex absorbing set in E is called a barrel. If each barrel in E is a neighborhood of zero, then E is said to be a barreled space.

It is known [14, Theorem 2, P. 200] that every reflexive locally convex space is barreled.

Theorem 3.4. *Let M be a finite von Neumann algebra and suppose that $\mathcal{F} \neq \emptyset$ is the family of all f.n.f. traces on M. The following conditions are equivalent:*

(i) $M_f = L^\omega(M,\mu)$ for some (and hence for all) $\mu \in \mathcal{F}$;

(ii) (M_f, t) is metrizable;

(iii) (M_f, t) is reflexive;

(iv) the center Z of M is finite-dimensional, i.e. $M = \sum\limits_{i=1}^{m} M_i$, where all M_i are I_n-factors or II_1-factors.

Proof. Suppose that Z is finite-dimensional. Then M is a finite direct sum of factors M_i, $i = \overline{1,m}$, for each factor M_i the algebras $(M_i)_f$ and $L^\omega(M_i,\mu_i)$ coincide, and the topology t_i is the same as $t_i^{\mu_i}$. Therefore

$$M_f = \left(\sum_{i=1}^{m} M_i\right)_f = \sum_{i=1}^{m} (M_i)_f = \sum_{i=1}^{m} L^\omega(M_i,\mu_i) = L^\omega(M,\mu),$$

where $\mu = \sum_{i=1}^{m} \mu_i \in \mathcal{F}$, i.e. $M_f = L^\omega(M,\mu)$.

Now since the topology t^μ on the Arens algebra $L^\omega(M,\mu)$ is metrizable [2], it follows that $t = t^\mu$ is also metrizable.

It is known [1] that for finite traces μ the Arens algebra $(L^\omega(M,\mu), t^\mu)$ is reflexive and hence (M_f, t) is also reflexive. Therefore (iv) implies (i), (ii) and (iii).

$(i) \Rightarrow (iv)$. Suppose that $M_f = L^\omega(M,\mu)$ for an appropriate $\mu \in \mathcal{F}$. Then there exists a sequence of mutually orthogonal projections $\{p_n\}$ in Z such that $p_n \neq 0$ for all $n \in \mathbb{N}$. Since the trace μ is finite, one has that $\sum\limits_{k=1}^{\infty} \mu(p_k) < \infty$ and hence there is a subsequence $\{n_k : k \in \mathbb{N}\}$ such that $\mu(p_{n_k}) \leq \frac{1}{2^k}$ for all k.

Set

$$x = \sum_{k=1}^{\infty} k p_k.$$

For $p \geq 1$ we have

$$\mu(|x|^p) = \sum_{k=1}^{\infty} k^p \mu(p_k) = \sum_{k=1}^{\infty} k^p \frac{1}{2^k} < \infty,$$

and hence $x \in L^\omega(M,\mu) = M_f$.

On the other hand, x is a central element in M_f and Proposition 3.1 implies that $x \in Z(M_f) = Z \subseteq M$. But it is clear that the element x is unbounded, i.e. $x \in M$. The contradiction shows that Z is finite-dimensional.

$(ii) \Rightarrow (iv)$. Suppose that (M_f, t) is metrizable. By Theorem 3.1 it is complete and hence it is a Fréchet space. In particular the center of M_f which coincides with Z_f is also a Fréchet space. By Proposition 3.1, $Z_f = Z$ and hence Z is a Fréchet space with respect to the induced topology $t_Z = t|_Z$.

Consider the identity mapping

$$I : (Z, \|\cdot\|_\infty) \to (Z, t_Z),$$

where $\|\cdot\|_\infty$ is the operator norm on Z. From the inequalities

$$\|x\|_p^\mu \leq C_p^\mu \|x\|_\infty$$

(where C_p^μ is an appropriate constant for each $p \geq 1$, $\mu \in \mathcal{F}$) it follows that the mapping I is continuous. Since (Z, t_Z) is a Fréchet space, by the Banach theorem on the inverse operator (see [14, Chapter II, Section 5]) we obtain that the inverse mapping

$$I^{-1} : (Z, t_Z) \to (Z, \|\cdot\|_\infty)$$

is also continuous. This means that for some $p \in [1, \infty)$ and an appropriate $\mu \in \mathcal{F}$ there exists a constant K_p^μ such that

$$\|x\|_\infty \leq K_p^\mu \|x\|_p^\mu \tag{3.1}$$

for all $x \in Z$ (see [14, Theorem 1, P. 42]).

Now suppose that $\dim Z = \infty$. There exists a sequence $\{p_n\}$ of projections in Z such that $p_n \uparrow \mathbf{1}$, $p_n \neq p_{n+1}$. Thus $p_n^\perp \neq 0$, $\mu(p_n^\perp) \to 0$, i.e. $\|p_n^\perp\|_p^\mu \to 0$. From the inequality (3.1) we obtain that $\|p_n^\perp\|_\infty \to 0$.

On the other hand, $\|p_n^\perp\|_\infty = 1$. This contradiction implies that Z is finite-dimensional.

(iii) $\Rightarrow$ *(iv)*. Suppose that M_f is reflexive. Then the center $Z(M_f) = Z$ is also reflexive as a closed subspace of a reflexive space. The set

$$B = \{x \in Z : \|x\|_\infty \leq 1\}$$

is a barrel in (Z, t) and since Z is reflexive, we have that B is a neighborhood of zero in Z. Therefore there exist $p \geq 1$, $\mu \in \mathcal{F}$ and $\varepsilon > 0$ such that

$$\{x \in Z : \|x\|_p^\mu \leq \varepsilon\} \subset B$$

i.e.

$$\|x\|_\infty \leq \varepsilon^{-1} \|x\|_p^\mu$$

for all $x \in Z$. From this as above it follows that Z is finite-dimensional. □

Remark 3.2. *In the von Neumann algebra M the operator topology is stronger than the topology t, t is stronger than t^μ, and t^μ is stronger than each L_p-norm topology for any $p \geq 1$.*

4 Derivations on finite tracial algebras

Derivations on unbounded operator algebras, in particular on various algebras of measurable operators affiliated with von Neumann algebras, appear to be a very attractive special case of general unbounded derivations on operator algebras.

Let A be an algebra over the complex number. A linear operator $D : A \to A$ is called a derivation if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$ defines a derivation D_a on A given as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be inner derivations.

In [4] we have investigated and completely described derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M and on its various subalgebras. Recently the above conjecture was also confirmed for the type I case in the paper [7] by a representation of measurable operators as operator valued functions. Another approach to similar problems in AW^*-algebras of type I was suggested in the recent paper [8].

In the paper [3] we have proved the spatiality of derivations on the non commutative Arens algebra $L^\omega(M,\tau)$ associated with an arbitrary von Neumann algebra M and a faithful normal semi-finite trace τ. Moreover if the trace τ is finite then every derivation on $L^\omega(M,\tau)$ is inner.

In this section we prove that each derivation on a finite tracial algebra is inner.

The following result is an immediate corollary of [5, Proposition 3.6].

Lemma 4.1. *Let M be a von Neumann algebra with a faithful normal trace τ. Given any derivation $D : M \to L^\omega(M,\tau)$ there exists an element $a \in L^\omega(M,\tau)$ such that*

$$D(x) = ax - xa,\ x \in M.$$

Further we need also the following assertion from [7, Proposition 6.17].

Lemma 4.2. *Let A be a $*$-subalgebra of $LS(M)$ such that $M \subseteq A$ and A is solid (that is, if $x \in LS(M)$ and $y \in A$ satisfy $|x| \leq |y|$, then $x \in A$). If $\omega \in LS(M)$ is such that $[\omega, x] \in A$ for all $x \in A$, then there exists $\omega_1 \in A$ such that $[\omega, x] = [\omega_1, x]$ for all $x \in A$.*

The main result of this section is the following theorem.

Theorem 4.1. *Let M be a von Neumann algebra with a faithful normal finite trace τ. If $A \subseteq L^\omega(M,\tau)$ is a solid $*$-subalgebra such that $M \subseteq A$, then every derivation on A is inner.*

Proof. Since $A \subseteq L^\omega(M,\tau)$, by Lemma 4.1 there exits an element $a \in L^\omega(M,\tau)$ such that

$$D(x) = ax - xa,\ x \in M. \tag{4.1}$$

Let us show that in fact

$$D(x) = ax - xa \ \text{ for all } \ x \in A. \tag{4.2}$$

Consider $x \in A$, $x \geq 0$. Then $(\mathbf{1} + x)^{-1} \in M$. As $D(\mathbf{1}) = 0$, by the Leibniz rule it follows that for each invertible $b \in A$ one has

$$D(b) = -bD(b^{-1})b.$$

Therefore

$$D(x) = D(\mathbf{1} + x) = -(\mathbf{1} + x)D((\mathbf{1} + x)^{-1})(\mathbf{1} + x).$$

On the other hand, since $(\mathbf{1} + x)^{-1} \in M$ equality (4.1) implies that

$$D((\mathbf{1} + x)^{-1}) = a(\mathbf{1} + x)^{-1} - (\mathbf{1} + x)^{-1}a.$$

Therefore

$$-(\mathbf{1}+x)D((\mathbf{1}+x)^{-1})(\mathbf{1}+x) = -(\mathbf{1}+x)[a(\mathbf{1}+x)^{-1} - (\mathbf{1}+x)^{-1}a](\mathbf{1}+x) = \\ = -(\mathbf{1}+x)a + a(\mathbf{1}+x) = ax - xa,$$

i.e.

$$D(x) = ax - xa,\ x \in A,\ x \geq 0.$$

Since each element from A is a finite linear combination of positive elements, we obtain equality (4.2) for arbitrary $x \in A$.

Now since A is a solid $*$-subalgebra in $L^\omega(M,\tau)$ containing A, Lemma 4.2 implies that the element a implementing the derivation D may be chosed from the algebra A, i.e.

$$D(x) = ax - xa,\ x \in A$$

for an appropriate $a \in A$. □

Since the algebra M_f is a solid $*$-subalgebra of $L^\omega(M,\tau)$ and contains M, we obtain the following result.

Corollary 4.1. *If M is a von Neumann algebra with a faithful normal trace, then every derivation on M_f is inner.*

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