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SPACES OF GENERALISED SMOOTHNESS IN SUMMABILITY PROBLEMS FOR $\Phi-$ MEANS OF SPECTRAL DECOMPOSITION

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Abstract. We establish conditions for localization of generalised Riesz means of spectral decomposition by system of fundamental functions of Laplace operator in terms of belongingness of the decomposing function to the spaces of generalised smoothness.

1 Introduction

In this paper we establish a condition for localisation of $\Phi-$means of spectral decomposition by the system of fundamental functions of Laplace operator in arbitrary multi-dimensional domain. The result is obtained in terms of belongingness of the decomposing function to the spaces of generalised smoothness, in which the modulus of continuity of a function is estimated by the majorant of non exponential form constructed by the function Φ. The result generalises known theorems in [4] on conditions for localisation of Riesz means given in terms of exponential scale of smoothness and extends our publication in [1].

The structure of the paper is as follows: In Section 2 we give basic definitions, statement of the problem and formulation of the basic theorem on conditions of localisation. Section 3 is devoted to justification of basic results. Namely, Subsection 3.1 contains preliminaries on properties of fundamental functions and Bessel integro-differential operators. Subsection 3.2 being the central part of the work contains the integral representation for $\Phi-$means of spectral decomposition. Further, in Subsection 3.3, quantities defining this representation were estimated. Based on these, in Subsection 3.4 we establish estimates for $\Phi-$means of spectral decomposition and prove a theorem on conditions for localisation of spectral decomposition.

2 Statement of the problem and formulation of the result

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space and for $1 \le p \le \infty$, $L_p(\mathbb{R}^n)$ denote a Lebesgue space with the norm

$$\|f\|_{L_p(\mathbb{R}^n)} = \begin{cases} \left[\int\limits_{\mathbb{R}^n} |f(x)|^p\right]^{1/p}, & \text{if} \quad 1 \le p < \infty, \\ \operatorname*{ess\,sup}\limits_{x \in \mathbb{R}^n} |f(x)|, & \text{if} \quad p = \infty. \end{cases}$$

Denote by $A \hookrightarrow B$ the topological inclusion of space A in space B and observe that $A = B$ if and only if $A \hookrightarrow B \cap B \hookrightarrow A$. The notation $a \asymp b$ means that $c \le \frac{a}{b} \le d$ with $0 < c \le d$ depending on non significant parameters. For $u > 0$ and function φ in $\mathbb{R}^1$, the notations $\varphi(u)$ al. $\downarrow$ and $\varphi(u)$ al. $\uparrow$ are used to mean that $\varphi(u)$ is almost decreasing and $\varphi(u)$ is almost increasing respectively.

Further, let $G \subset \mathbb{R}^n$ be an arbitrary domain whereas $(-\widehat{\Delta})$ an arbitrary self adjoint non-negative extension of Laplace operator in n-dimensional domain G, $u(x,t)$ an ordered spectral representation of the space $L_2(G)$ with respect to $(-\widehat{\Delta})$, $d\rho(t)$ is the corresponding spectral measure and $\{u_i(x,t)\}_{i=1}^m$ is a system of fundamental functions, where $m \le \infty$ is the multiplicity of the representation. Moreover, for any fixed $t \ge 0$, $u_i(x,t) \in C^\infty(G)$ and

$$\Delta u_i(x,t) + t^2 u_i(x,t) = 0, \qquad x \in G. \tag{2.1}$$

For each $f \in L_2(G)$ defined are the Fourier transforms

$$\widehat{f} := \left\{\widehat{f}_i(t)\right\}_{i=1}^m, \qquad \widehat{f}_i(t) = \int\limits_G f(x) u_i(x,t)dx \tag{2.2}$$

and the spectral decomposition by system $u(x,t) = \{u_i(x,t)\}_{i=1}^m$

$$S_\mu(f,x) = \int\limits_0^\mu \widehat{f}(t) u(x,t) d\rho(t), \quad \mu > 0, \tag{2.3}$$

with

$$\widehat{f}.u = \sum_{i=1}^m \widehat{f}_i u_i\,, \quad |\widehat{f}| = \left\{\sum_{i=1}^m \widehat{f}_i^2\right\}^{1/2} \quad \text{and} \quad |u| = \left\{\sum_{i=1}^m u_i^2\right\}^{1/2}.$$

Let $s > 0$ and φ is a function on $\mathbb{R}^1$ with properties: $\varphi(u) \equiv 0$, if $u < 0$, $\varphi(u) > 0$, $\varphi(u)$ almost decreasing for $u \in (0,1]$ and $\varphi(u) \asymp \varphi(v)$ if $u \asymp v$ for $u, v \in (0,1]$.

Moreover for $s > 0$ set $s_0 = s$ if $s < 1$, $s_0 = 1$ if $s \ge 1$ and require that

$$1)\ \varphi_{s_0}(u) = \int\limits_0^u v^{s_0 - 1}\varphi(v)dv < \infty, \quad u \in (0,1], \tag{2.4}$$

$$2)\ \varphi \in C^2(0,1),\ |\varphi'(u)| \le c\varphi(u)u^{-1},\ |\varphi''(u)| \le c\varphi(u)u^{-2}, \tag{2.5}$$

$$3)\ \int\limits_0^u (1-v)^{s-1}\varphi(v)dv = \Gamma(s).$$

Define Φ as s-th integral of Riemann–Liouville

$$\Phi(u) = \frac{1}{\Gamma(s)} \int\limits_0^u (u-v)^{s-1} \varphi(v) dv \qquad u \in (0,1]. \tag{2.6}$$

Let us introduce the $\Phi-$means of spectral decomposition as

$$\sigma_\mu^\Phi(f,x) = \int\limits_0^\mu \widehat{f}(t) u(x,t) \Phi\left(1 - \frac{t^2}{\mu^2}\right) d\rho(t), \quad f \in L_2(G), \tag{2.7}$$

We observe that $\Phi(1) = 1$ and if $\varphi(u) \equiv \Gamma(s+1)$, $u \in (0,1]$, then $\Phi(u) = u^s$ and the $\Phi-$means reduce to the Riesz means $\sigma_\mu^s(f,x)$ of order s.

The problem consists in obtaining conditions for localisation of $\Phi-$means of spectral decomposition. For Riesz means a similar problem has been solved completely by Ilin and Alimov [4] in the framework of spaces of exponential order of smoothness. The result, substantially, was due to the possibility of giving a two-sided estimate of exponential type for Lebesgue function of Riesz means in spaces with exponential order of smoothness. For $\Phi-$means of spectral decomposition such an estimate does not posses exponential character. Thus, for multiple Fourier integral with assumption of convexity on $\varphi(u)$ in [2], the following estimate was established for Lebesgue constant:

$$\int\limits_{R_1 \le |x| \le R_2} |D_\mu^\Phi| dx \asymp \frac{1}{\omega_0(1/\mu)}, \quad \omega_o(u) = \frac{u^{\frac{n-1}{2}-s}}{\varphi(u)} \tag{2.8}$$

Here, $0 < R_1 < R_2 < \infty$, are fixed and for $x, \xi \in \mathbb{R}^n$,

$$D_\mu^\Phi(x) = F^{-1}\left[\Phi(1 - |\xi|^2/\mu^2)\right](x)$$

is the Kernel of $\Phi-$means. These estimates hint that the result on conditions for localisation of $\Phi-$means to be formulated in terms of spaces of generalised smoothness.

Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Omega \subset\subset G$, that is, $\overline{\Omega}$ is compact and $\overline{\Omega} \subset G$. Let $\omega(0) = 0$, $\omega(u)$ is increasing and $\omega(u)u^{-k}$ is almost decreasing for $k \in \mathbb{N}$.

Definition 2.1. *The Nikol'skii type space with generalised smoothness $H_p^{\omega(.)}(\Omega)$ is defined as follows:*

$$H_p^{\omega(.)}(\Omega) = \{f \in L_p(\Omega) : \|f\|_{H_p^{\omega(.)}(\Omega)} < \infty\},$$

where

$$\|f\|_{H_p^{\omega(.)}(\Omega)} = \|f\|_{L_p(\Omega)} + \sup_{0<u\le 1}\left[\frac{\omega_{p,\Omega}^k(f;u)}{\omega(u)}\right]$$

and

$$\omega_{p,\Omega}^k(f;u) = \sup_{|h|\le u} \|\Delta_{h,\Omega}^k f\|_{L_p(\Omega)}, \quad u > 0$$

is the modulus of continuity of order k for function $f \in L_p(\Omega)$ with

$$\Delta_{h,\Omega}^k f(x) = \begin{cases} \Delta_h^k f(x) = \sum\limits_{m=0}^k (-1)^{k-m} C_k^m f(x+mh), & [x, x+kh] \subset \Omega \\ 0, & [x, x+kh] \not\subseteq \Omega. \end{cases}$$

We denote by $\overset{\circ}{H}{}_p^{\omega(.)}(\Omega)$ the closure in $H_p^{\omega(.)}(\Omega)$ of $C_0^\infty(\Omega)$ and note that $\overset{\circ}{H}{}_p^{\omega(.)}(\Omega) \hookrightarrow H_p^{\omega(.)}(\Omega)$, moreover, the inclusion is strict. Then holds true the following:

Theorem 2.1. *Let $s > 0,\ 0 \le \alpha, \beta$ are such that,*

$$\frac{n-2}{2} - s < \alpha \le \beta < \min\left\{\alpha + \frac{3}{2}, \frac{n}{2} + 1\right\} \tag{2.9}$$

and let a function ω satisfy the conditions

$$\omega(u)u^{-\alpha} \text{ al.} \uparrow, \quad \omega(u)u^{-\beta} \text{ al.} \downarrow \quad \text{on} \quad (0,1], \tag{2.10}$$

$$\omega(u) \le c\omega_0(u), \quad u \in (0,1], \quad \omega_0(u) = \frac{u^{((n-1)/2)+s_0-s}}{\varphi_{s_0}(u)}. \tag{2.11}$$

Let $D \subset \Omega \subset\subset G$ and $f \in \overset{\circ}{H}{}_2^{\omega(.)}(\Omega)$ be a function satisfying the condition $f(x) \equiv 0$ for all x in D. Then for each compact $K \subset D$ uniformly in $x \in K$ holds true the relation:

$$\lim_{\mu\to\infty} \sigma_\mu^\Phi(f,x) = 0.$$

The theorem gives conditions for localization of $\Phi-$means of spectral decomposition. In typical situations when

$$\varphi_{s_0}(u) \asymp u^{s_0}\varphi(u), \ \omega_0(u) \asymp \frac{u^{\frac{n-1}{2}-s}}{\varphi(u)}, \ u \in (0,1], \tag{2.12}$$

appears a function $\omega_0(u)$ of the form (2.8). In particular, for Riesz means, that is, for $\varphi(u) = \Gamma(s+1)$ we obtain the requirement

$$\varphi(u) \le cu^{\frac{n-1}{2}-s}. \tag{2.13}$$

For $s < \frac{n-1}{2}$, this results to the sharp condition of localization in terms of exponential order of smoothness for a function $f \in \overset{\circ}{H}{}_2^{\alpha}(\Omega)$ with $\alpha = \frac{n-1}{2} - s$ established by V. A. Ilin and Sh. A. Alimov in [4]. Moreover, as shown by M. L. Goldman in [2] this condition is also sharp in terms of spaces of generalised smoothness. Namely, if

$$\overline{\lim_{u\to+0}} \left[\omega(u)u^{s-\frac{n-1}{2}}\right] = \infty, \qquad 0 \le s < \frac{n-1}{2}, \tag{2.14}$$

then for all $x_0 \in \Omega,\ 1 \le p \le \infty$, there exists a function $f_0 \in \overset{\circ}{H}{}_p^{\omega(.)}(\Omega)$, which is zero in some neighbourhood of x_0 and the Riesz means of spectral decomposition $\sigma_\mu^s(f_0, x_0)$ is unbounded as $\mu \to \infty$.

In obtaining this result the implementation of generalised kernels of fractional order and the corresponding integral operators in the spaces of generalised smoothness, investigated by M.L. Goldman in [3] played an important role.

3 Justification of properties of localisation

3.1 Preliminaries

Properties of fundamental functions

Throughout the paper we use notations and definitions from Section 2. The following properties of fundamental functions are described in [4].

1. $u_i(.,t) \in C^\infty(G), \quad \Delta u_i(x,t) + t^2 u_i(x,t) = 0 \quad$ in G.

2. Holds true the following mean value formula

$$\int_\theta u_i(x_0 + r\theta, t)d\theta = (2\pi)^{\frac{n}{2}} u_i(x_0,t)\frac{J_{(n-2)/2}(rt)}{(rt)^{(n-2)/2}}, \tag{3.1}$$

   where $J_v(.)$ is Bessel function. The integral is evaluated in all angles θ of spherical system of coordinates $(r,\theta),\ 0 < r < \rho(x_0, \partial G)$.

3. For any subdomain $\Omega \subset\subset G, \quad \mu \geq 1, \quad \nu \in [1,\mu]$

$$\sup_{x\in\Omega} \int\limits_{|t-\mu|\leq\nu} |u(x,t)|^2 d\rho(t) \leq c(\Omega)\nu\mu^{n-1} \tag{3.2}$$

4. For $f \in L_2(\Omega)$ holds true Parseval's equality.

$$\int\limits_0^\infty |\widehat{f}(t)|^2 d\rho(t) = \int_G |f(x)|^2 dx$$

5. If $f \in C_0^\infty(G)$, then for any subdomain $\Omega \subset\subset G$ holds true uniformly convergent in Ω decomposition

$$f(x) = \int_0^\infty \widehat{f}(t)u(x,t)d\rho(t) \tag{3.3}$$

Bessel integro-differential operators

Let $g \in C^\infty(\mathbb{R}^1_+)$ be a finite function in the neighbourhood of zero. We define "Bessel's derivative" as follows

$$\mathbb{D}^1[g](r) = \left(r^{-1}\frac{d}{dr}\right)g(r), \quad \mathbb{D}^l = \mathbb{D}(\mathbb{D}^{l-1}), \quad l = 2,3,\dots$$

Let us introduce the corresponding operator of integration

$$\mathbb{I}^\varkappa[g](r) = \left(2^{\varkappa-1}\Gamma(\varkappa)\right)^{-1}\int\limits_0^r g(\eta(r^2-\eta^2)^{\varkappa-1}\eta d\eta, \quad 0 \leq \varkappa \leq 1,$$

$$\mathbb{I}^\gamma = \mathbb{I}^x\mathbb{I}^l \quad \text{for} \quad \gamma = l + \varkappa,\ l \in \mathbb{N},\ 0 \leq \varkappa \leq 1.$$

For $\gamma = l - \varkappa, \quad l \in \mathbb{N}, \quad 0 < \varkappa < 1$, put $\mathbb{D}^\gamma = \mathbb{D}^l \mathbb{I}^\varkappa$. On the functions of these type these operators are inverses of each other, and

$$\mathbb{I}^\gamma \mathbb{D}^\gamma [g] = \mathbb{D}^\gamma \mathbb{I}^\gamma [g] = g.$$

In such a way we define the notation $\mathbb{D}^\gamma = \mathbb{I}^{-\gamma}$ for $\gamma < 0$. The operators can be extended to wider classes of functions for which the introduced formulae make sense. Holds true the following formula (cf. [5, 6]).

$$\mathbb{D}^\gamma \left[r^\sigma J_\sigma(rt) \right] = t^\gamma r^{\sigma - \gamma} J_{\sigma-\gamma}(rt), \quad \sigma > -1, \quad \gamma \in \mathbb{R}^1, \tag{3.4}$$

where the operators are acting in r.

3.2 Representation of $\Phi-$means of spectral decomposition for a finite function

Fourier-Bessel representation for function Φ

Using change of variables in (2.6), we obtain the following formula for later use.

$$\Phi\left(1 - \frac{t^2}{\mu^2}\right) = \frac{1}{\Gamma(s)\mu^{2s}} \int_0^\mu (u^2 - t^2)_+^{s-1} \varphi\left(1 - \frac{u^2}{\mu^2}\right) u du \tag{3.5}$$

We shall also use the following notations

$$I \equiv I_{s,\nu}(u,t,R) := u^{\nu+s+1} t^{-\nu} \int_R^\infty r^{1-s} J_\nu(tr) J_{\nu+s}(ur) dr \tag{3.6}$$

$$\Delta_R(\mu,t) := 2^s \mu^{-2s} \int_0^\mu \varphi\left(1 - \frac{u^2}{\mu^2}\right) I_{s,\nu}(u,t,R) du \tag{3.7}$$

Proposition 3.1. *For $s > 0, \quad \nu > -1$ holds true the following formula*

$$\Phi\left(1 - \frac{t^2}{\mu^2}\right) = t^{-\nu} \int_0^\infty B_\mu^\Phi(r) J_\nu(rt) dr, \qquad t, \mu > 0 \tag{3.8}$$

where

$$B_\mu^\Phi(r) = \frac{r^{1-s} 2^s}{\mu^{2s}} \int_0^\mu u^{\nu+s+1} \varphi\left(1 - \frac{u^2}{t^2}\right) J_{\nu+s}(ur) du. \tag{3.9}$$

Proof. For $0 < A \leq \infty$ we denote

$$T_A(\mu,t) := t^{-\nu} \int_0^A B_\mu^\Phi(r) J_\nu(rt) dr$$

$$\psi_A(u,t) := \frac{2^s u^{\nu+s+1}}{t^\nu \mu^{2s}} \varphi\left(1 - \frac{u^2}{\mu^2}\right) \int_0^A r^{1-s} J_{\nu+s}(ur) J_\nu(tr) dr. \tag{3.10}$$

Then substituting (3.9) in T_A and changing the order of integration we obtain

$$T_A(\mu,t) = \int_0^\mu \phi_A(u,t)du, \quad 0 < A < \infty.$$

From the formula (Bateman-Erdelyi)

$$\int_0^\infty r^{1-s} J_{\nu+s}(ur) J_\nu(tr) dr = \frac{2^{1-s}t^\nu}{\Gamma(s)u^{\nu+s}}(u^2-t^2)_+^{s-1}$$

follows that

$$\psi_\infty(u,t) = \frac{2u}{\Gamma(s)\mu^{2s}}\varphi\left(1-\frac{u^2}{\mu^2}\right)\left(u^2-t^2\right)_+^{s-1}.$$

Taking into account equations (3.5)–(3.7), we obtain

$$T_A(\mu,t) = \int_0^\mu \psi_\infty(u,t)du - \int_0^\mu \left[\psi_\infty(u,t)-\psi_A(u,t)\right]du = \Phi\left(1-\frac{t^2}{\mu^2}\right) - \Delta_A(\mu,t) \quad (3.11)$$

and to show relations (3.8)-(3.9), we only need to check that

$$\Delta_A(\mu,t) \to 0 \quad \text{as} \quad A\to\infty, \quad \mu,t>0.$$

The estimate for $\Delta_A(\mu,t)$ is given in Subsection 3.3. □

Representation of $\Phi-$means

Let $x_0 \in G,\ f \in C_0^\infty(K_{x_0,M})$, where $K_{x_0,M} = \{x : |x-x_0| \le M\}, \quad 0 < M < \rho(x_0,\partial G)$ and

$$F(r) = (2\pi)^{-\frac{n}{2}} r^{n-2} \int_\theta f(x_0+r\theta)dr \quad (3.12)$$

Put $x = x_0 + r\theta$ in (3.3) and integrate with respect to θ. By virtue of uniform convergence, the integral with respect to θ can be taken inside and the mean value formula (3.1) be applied to get

$$F(r) = \int_0^\infty \widehat{f}(t)u(x_0,t)t^{(2-n)/2}\left[r^{(n-2)/2}J_{(n-2)/2}(rt)\right]d\rho(t).$$

For $f \in C_0^\infty(K_{x_0,M})$ such decomposition is fast convergent. Applying operator $\mathbb{D}^\gamma$, $\gamma \in \mathbb{R}^1$ under the integral sign with respect to t and taking into account equation (3.4), we obtain

$$\mathbb{D}^\gamma F(r) = \int_0^\infty \widehat{f}(t)u(x_0,t)t^{-\nu}r^\nu J_\nu(rt)d\rho(t), \quad \nu = \frac{n-2}{2} - \gamma. \quad (3.13)$$

Proposition 3.2. *Let* $x_0 \in G,\ f \in C_0^\infty(K_{x_0,M}),\ M < \rho(x_0, \partial G),\ s > 0, \gamma < n/2\ \nu = \frac{n-2}{2} - \gamma,\ \delta = \nu + 1/2 - s \le 0$. *Then*

$$\sigma_\mu^\Phi(f, x_0) = \int\limits_0^\infty r^{-\nu} \mathbb{D}^\gamma F(r) B_\mu^\Phi(r) dr \tag{3.14}$$

where $B_\mu^\Phi(r)$ *is of the form* (3.9).

Proof. For $A > 0$ denote

$$S_A(\mu) = \int\limits_0^A r^{-\nu} \mathbb{D}^\gamma F(r) B_\mu^\Phi(r) dr$$

Substitute here expansion (3.13) which is fast convergent and interchange the order of integration with respect to t and r. Then

$$S_A(\mu) = \int\limits_0^\infty \hat{f}(t) u(x_0, t) \left(\int_0^\infty t^{-\nu} B_\mu^\Phi(r) J_\nu(rt) dr \right) d\rho(t).$$

The internal integral is $T_A(\mu, t)$, and equations (3.11) and (2.7) result in

$$S_A(\mu) = \sigma_\mu^\Phi(f, x_0) - \int\limits_0^\infty \hat{f}(t) u(x_0, t) \Delta_A(\mu, t) d\rho(t).$$

To prove relation (3.14) we need to show that

$$\sigma_A(\mu) := \sigma_\mu^\Phi(f, x_0) - \int\limits_0^\infty \hat{f}(t) u(x_0, t) \Delta_A(\mu, t) d\rho(t) \to 0 \quad \text{as } A \to \infty. \tag{3.15}$$

The proof of relation (3.15) depends on proposition 3.4 of Subsection 3.3. □

Corollary 3.1. *Let in Proposition 3.2* $f(x) \equiv 0$ *if* $|x - x_0| \le R$ *for* $0 < R < M$. *Then*

$$\sigma_\mu^\Phi(f, x_0) = \int\limits_R^\infty r^{-\nu} \mathbb{D}^\gamma F(r) B_\mu^\Phi(r) dr \tag{3.16}$$

Actually in (3.14), $\mathbb{D}^\gamma F(r) \equiv 0$ for $0 < r < R$.

Proposition 3.3. *Let in Proposition 3.2* $f(x) \equiv 0$ *if* $|x - x_0| \le R$ *for* $0 < R < M$. *Then holds true the representation*

$$\sigma_\mu^\Phi(f, x_0) = \int\limits_0^\infty \hat{f}(t) u(x_0, t) \Delta_A(\mu, t) d\rho(t). \tag{3.17}$$

Proof. Put decomposition (3.13) in (3.16). As in the proof of Proposition 3.2 interchange the order of integration with respect to t and r to obtain

$$\sigma_\mu^\Phi(f,x_0)=\int\limits_0^\infty \hat f(t)u(x_0,t)t^{-\nu}\left(\int\limits_R^\infty B_\mu^\Phi(r)J_\nu(rt)dr\right)d\rho(t).$$

Let us substitute representation (3.9) for $B_\mu^\Phi(r)$ and interchange the order of integration with respect to u and r (correctness of such operation can be verified as in the proof of Proposition 3.1). Finally using notations (3.6)–(3.7), we arrive at the representation (3.17). □

Corollary 3.2. *The value of* ν *in representation* (3.17) *remains free within the intervals* $-1<\nu\le s-1/2$*. We shall make use of this, choosing* ν *depending on the property of smoothness of decomposable function.*

3.3 Estimate for $\Delta_R(\mu,t)$

Below we give estimates for the value of $\Delta_R(\mu,t)$ in all domains of the arguments. The estimates are necessary for the study of properties of $\sigma_\mu^\Phi(f,x_0)$ (cf. (3.17)) as well as for verification of representations in Subsection 3.2.

Proposition 3.4. *Let* $\delta=\nu+\frac{1}{2}-s$

1. *For* $\mu\ge\frac{3}{R}$, $\quad 0<t<1/R$, $\quad\delta\le 0$ *holds true the estimate*

$$|\Delta_R(\mu,t)|\le c_1(\mu R)^\delta\varphi_1\left(\frac{1}{\mu R}\right) \tag{3.18}$$

2. *For* $\mu\ge\frac{1}{R}$, $\quad t>1/R$ *holds true the estimates*

$$|\Delta_R(\mu,t)|\le c_2R^{-s}\mu^{\delta+1}\varphi_1\left(\frac{1}{\mu R}\right)t^{-(\nu+1/2)}|\mu-t|^{-1},\quad |\mu-t|>\frac{1}{R} \tag{3.19}$$

$$|\Delta_R(\mu,t)|\le c_3R^{s_0-s}\mu^{\delta+s_0}\varphi_{s_0}\left(\frac{1}{\mu R}\right)t^{-(\nu+1/2)},\quad |\mu-t|\le\frac{1}{R} \tag{3.20}$$

where c_1,c_2,c_3 *do not depend on* t,μ,R*.*

Proposition 3.5. *Let* $0<\mu<3R$, $\quad\delta=\nu+\frac{1}{2}-s$*. Then*

$$|\Delta_R(\mu,t)|\le c_1\quad \textit{if}\quad 0<t\le 2\mu,\ \delta\le 0 \tag{3.21}$$

$$|\Delta_R(\mu,t)|\le c_2\mu^{\nu+\frac{3}{2}}t^{-(\nu+\frac{3}{2})},\quad \textit{if}\quad t>2\mu,\ \nu\ge-\frac{1}{2}. \tag{3.22}$$

where c_1,c_2 *do not depend on* t,μ,R*.*

The proofs of Propositions 3.4 and 3.5 depend on the following lemmas.

Lemma 3.1. *Let* $s>0$, $\quad\nu>-1$, $\quad I=I_{s,\nu}(\mu,t,R)$*, see equation* (3.6)*.*

1. For $u > \frac{2}{R}, \quad 0 < t < \frac{1}{R}$ *holds true the estimate*

$$|I| \leq c_1 u^{\nu+s-1/2} R^{\nu+\frac{1}{2}-s}, \quad \text{if} \quad \nu \leq s - 1/2. \tag{3.23}$$

2. For $u > \frac{1}{2R}, \quad t > \frac{1}{2R}$ *holds true the estimate*

$$|I| \leq c_2 R^{1-s} u^{\nu+s+1/2} t^{-(\nu+\frac{1}{2})} \left[R|t-u| + (R|t-u|)^{1-s_0} \right]^{-1}. \tag{3.24}$$

3. For $0 < u < \frac{3}{R}$ *holds true the estimate*

$$I \leq c_3 \begin{cases} u^{2s-1}, & \text{if} \quad 0 < t \leq u/2, \ \nu \leq s - 1/2 \\ u^{2s-s_0}|u-t|^{-(1-s_0)}, & \text{if} \quad u/2 < t < 2u \\ u^{\nu+2s+1/2} t^{-(\nu+3/2)}, & \text{if} \quad t > 2u, \quad \nu > -1/2. \end{cases} \tag{3.25}$$

Here c_1, c_2, c_3 *do not depend on* t, μ, R.

4. Holds true the following formula

$$\int_0^u I_{s,\nu}(v,t,R)dv = u^{-1} I_{s+1,\nu}(u,t,R) \tag{3.26}$$

and

$$B := \int_0^u (u^2 - v^2) I_{s,\nu}(v,t,R)dv = 2u^{-1} I_{s+1,\nu}(u,t,R) \tag{3.27}$$

The estimates (3.23)–(3.25) are obtained in standard way with the help of asymptotic formula for Bessel functions. Some of these estimates are well known in [8] and [4] and the remaining can be proved in similar way. Formula (3.26) is based on the well known relation from [5]. To get relation (3.27) integrate B by parts. Then taking into account (3.26), we obtain

$$B = (u^2 - v^2) v^{-1} I_{s+1,\nu}(v,t,R)|_{u=0}^{v} + 2 \int_0^u I_{s+1,\nu}(v,t,R)dv.$$

Both substitutions are zero (the lower because of (3.25)) and using relation (3.26) we arrive at the required value of B.

Remark 3.1. *For the Riesz means put* $\varphi(u) = \Gamma(s+1)$ *in* (3.7)*, and then use equation* (3.26) *to get*

$$\Delta_R(\mu, t) = c_0 \mu^{-(2s+1)} I_{s+1,\nu}(\mu, t, R),$$

and all the required estimates follow from Lemma 3.1. However, in general cases, the proofs of Propositions 3.4 and 3.5 require additional efforts.

Lemma 3.2. *Let* $\mu > 1/R, \ \ s > 0, \ \nu > -1.$ *Then*

$$B_R(\mu,t) := 2^s \mu^{-2s} \int\limits_0^{\mu-\frac{1}{2R}} \varphi\left(1-\frac{u^2}{\mu^2}\right) I_{s,\nu}(u,t,R)du$$

hold the following estimates: for $\mu > 1/R \ \ t > 1/R$

$$|B_R(\mu,t)| \le c_1 R^{-s} \varphi\left(\frac{1}{\mu R}\right) \mu^{\nu+\frac{1}{2}-s} t^{-(\nu+1/2)} \left(1+R|\mu-t|\right)^{-1}$$

and for $\mu > 3/R, \ \ 0 < t < 1/R$

$$|B_R(\mu,t)| \le c_2 \varphi\left(\frac{1}{\mu R}\right) (\mu R)^{\nu-s-1/2}, \quad \textit{for} \ \ \nu \le s+1/2.$$

Here c_1, c_2 *do not depend on* $\mu, t, R.$

Proof. 1. Let $\varkappa = 1-(\mu-\frac{1}{2R})^2/\mu^2$. Holds true a type of Taylor's formula

$$\varphi\left(1-\frac{u^2}{\mu^2}\right) = \varphi(\varkappa) + \frac{\varphi'(\varkappa)}{\mu^2}\left[\left(\mu-\frac{1}{2R}\right)^2 - u^2\right]$$

$$+\frac{2}{\mu^4}\int\limits_u^{\mu-\frac{1}{2R}} (v^2-u^2)\varphi''\left(1-\frac{v^2}{\mu^2}\right) vdv,$$

which can be checked using integration by parts for the integral on the right-hand side. Put this in $B_R(\mu,t)$ to get three summands. The first is calculated using equation (3.26) and the second using (3.27). In the third interchange the order of integration with respect to u and v and then apply equation (3.27) w.r.t u to the resulting integral. Finally, we arrive at the representation

$$B_R(\mu,t) = 2^{s+2}\mu^{-2s-4}\Bigg\{\frac{\mu^4}{4}\varphi(\varkappa)\left(\mu-\frac{1}{2R}\right)^{-1} I_{s+1,\nu}\left(\mu-\frac{1}{2R},t,R\right) +$$

$$+\frac{\varphi'(\varkappa)\mu^2}{2\left(\mu-\frac{1}{2R}\right)} I_{s+2,\nu}\left(\mu-\frac{1}{2R},t,R\right) + \int\limits_0^{\mu-\frac{1}{2R}} \varphi''\left(1-\frac{v^2}{\mu^2}\right) I_{s+2,\nu}(v,t,R)dv\Bigg\}.$$

2. Note that $\mu-\frac{1}{2R} \asymp \mu, \quad \varkappa \asymp \frac{1}{\mu R}$ if $\mu > 1/R$ and

$$|\varphi'(\varkappa)| \le c\varkappa^{-1}\varphi(\varkappa) \asymp \mu R \varphi\left(\frac{1}{\mu R}\right).$$

Therefore taking into account Lemma 3.1, the first two summands in $B_R(\mu,t)$ give the required estimates. For the third summand we notice that

$$\left|\varphi''\left(1-\frac{v^2}{\mu^2}\right)\right| \le c\frac{\varphi(1-v^2/\mu^2)\mu^4}{(\mu^2-v^2)^2} \le \frac{c'\varphi(\varkappa)\mu^4}{(\mu^2-v^2)^2} \asymp \frac{\varphi(1/\mu R)\mu^2}{(\mu-v)^2}$$

and it gives an increment on $B_R(\mu,t)$ which is at most

$$B'_R := \mu^{-2s-2}\varphi\left(\frac{1}{\mu R}\right)\int\limits_0^{\mu-\frac{1}{R}} |I_{s+2,\nu}(v,t,R)|\,(\mu-v)^{-2}dv.$$

3. Let $\mu > 1/R,\quad t > 1/R$. Comparing estimates (3.24)–(3.25) evaluated for $I_{s+2,\nu}(v,t,R)$ we observe that

$$|I_{s+2,\nu}(v,t,R)| \le \frac{cR^{-s-1}v^{s+\nu+5/2}t^{-(\nu+1/2)}}{1+R|t-v|},\quad 0<v<\mu-1/2R.$$

Therefore

$$B'_R \le cR^{-s-1}\varphi\left(\frac{1}{\mu R}\right)\mu^{\nu+1/2}t^{-(\nu+1/2)}A_R(\mu,t)$$

where

$$A_R(\mu,t) = \int\limits_0^{\mu-\frac{1}{2R}} (1+R|t-v|)^{-1}(\mu-v)^{-2}dv$$

So, for $\mu>1/R,\quad t>1/R$, it remains to check that

$$A_R(\mu,t) \le cR\,[1+R|\mu-t|]^{-1}.$$

1) Let $t \ge \mu - 1/2R$, then

$$\big[1+R|t-v|\big]^{-1} \le [1+R\,(t-(\mu-1/2R))]^{-1} \asymp \big[1+R|\mu-t|\big]^{-1}.$$

And indeed,

$$A_R(\mu,t) \le c\big[1+R|t-\mu|\big]^{-1}\int\limits_0^{\mu-\frac{1}{2R}} (\mu-v)^{-2}dv = c_1\,[1+R|t-\mu|]^{-1}.$$

2) Let $t<\mu-1/2R$, then

$$A_R(\mu,t) = \int\limits_0^{t}\dots + \int\limits_t^{\mu-\frac{1}{2R}}\dots = A_R^{(1)}+A_R^{(2)}.$$

Let us estimate $A_R^{(1)}$. If $t\le\mu/2$, then $\mu-t\ge t,\quad \mu-t\ge\mu/2>1/2R$ and

$$A_R^{(1)} \asymp (\mu-t)^{-2}\int\limits_0^{t}[1+R(t-v)]^{-1}\,dv \asymp \frac{\ln(Rt+1)}{(\mu-t)\mu R} \le \frac{c}{\mu-t} \le \frac{c'R}{1+(\mu-t)R}.$$

If $t>\mu/2$, then $\mu-t<t$ and

$$A_R^{(1)} = \frac{1}{R}\int\limits_0^{Rt}(1+u)^{-1}\left(\mu-t+\frac{u}{R}\right)^{-2}du = \frac{1}{R}\int\limits_0^{R(\mu-t)}\dots + \frac{1}{R}\int\limits_{R(\mu-t)}^{Rt}\dots$$

The first integral is at most

$$R^{-1}(\mu-t)^{-2}\int_0^{R(\mu-t)} du=(\mu-t)^{-1}\asymp R\left[1+R|\mu-t|\right]^{-1},$$

and the second integral is at most

$$R\left[1+R|\mu-t|\right]^{-1}\int_{R(\mu-t)}^{Rt} u^{-2}du\le cR\left[1+R|\mu-t|\right]^{-2},$$

which gives the required estimate for $A_R^{(1)}$. The value of $A_R^{(2)}$ can be estimated in similar way.

4. Let now $\mu>3/R,\quad t<1/R$. Then comparing estimates (3.23) and (3.25) for $I_{s+2,\nu}(v,t,R)$ shows that

$$|I_{s+2,\nu}(v,t,R)|\le c\begin{cases} v^{2s+3} & \text{if}\quad 0<v<2/R,\quad \nu\le s+1/2,\\ v^{\nu+s+3/2}R^{\nu-s-3/2} & \text{if}\quad 2/R<v<\mu-1/2R,\quad \nu\le s+3/2.\end{cases}$$

Partition the integral B'_R into two, using the estimates provided and taking into account that $v<2/R$ implies that $\mu-v\asymp\mu$. Finally we obtain the required inequality for B'_R which means for $B_R(\mu,t)$.

□

Lemma 3.3. *Let* $s>0,\ 0<\mu,t,R<\infty,\ 0\le\sigma<\mu$. *Then*

$$A_\sigma:=\int_0^\mu \varphi\left(1-\frac{u^2}{\mu^2}\right)|t-u|^{s_0-1}udu\le c\mu^{1+s_0}\varphi_{s_0}\left(1-\frac{\sigma^2}{\mu^2}\right),$$

where $c>0$ *does not depend on* μ,t,R,σ.

Proof. 1. In the case $s\ge1,\ s_0=1$ the lemma is obvious.

$$A_\sigma:=\mu^2\int_0^{1-\frac{\sigma^2}{\mu^2}}\varphi(v)dv=\mu^2\varphi_1\left(1-\frac{\sigma^2}{\mu^2}\right) \tag{3.28}$$

2. Now let $0<s<1, s_0=s$. Denote

$$\psi(u)=\varphi\left(1-\frac{u^2}{\mu^2}\right)u\chi_{(\sigma,\mu)}(u),\quad \xi_t(u)=|t-u|^{s-1}.$$

Without loss of generality, it is possible to assume that $\varphi(u)$ is decreasing. Then non-increasing rearrangements are equal:

$$\psi^*(v)=\psi(\mu-v)\asymp\varphi\left(\frac{v}{\mu}\right)(\mu-v)\chi_{(0,\,\mu-\sigma)}(v),\quad \xi_t^*(v)=2^{1-s}v^{s-1}.$$

Therefore using a well known theorem on rearrangements (cf. [7, p. 94]), we obtain

$$A_\sigma = \int_0^\infty \psi(u)\xi_t(u)du \le \int_0^\infty \psi^*(v)\xi_t^*(v)dv \asymp \int_0^{\mu-\sigma} \varphi\left(\frac{v}{\mu}\right)(\mu-v)^{vs-1}dv$$

$$\le \mu^{s+1}\int_0^{1-\frac{\sigma}{\mu}} \varphi(\xi)\xi^{s-1}d\xi = \mu^{s+1}\varphi_s\left(1-\frac{\sigma}{\mu}\right) \asymp \mu^{s+1}\varphi_s\left(1-\frac{\sigma^2}{\mu^2}\right).$$

□

Lemma 3.4. *Let* $\mu > 1/R, s > 0, \nu > -1, \delta = \nu + 1/2 - s$,

$$C_R(\mu,t) = 2^s\mu^{-2s}\int_{\mu-\frac{1}{2R}}^{\mu} \varphi\left(1-\frac{u^2}{\mu^2}\right) I_{s,\nu}(u,t,R)du.$$

Then hold true the following estimates: For $\mu > 1/R,\ \ t > 1/R$

$$|C_R(\mu,t)| \le C_1R^{-s}\mu^{\delta+1}\varphi_1\left(\frac{1}{\mu R}\right)t^{-(\nu+1/2)}|\mu-t|^{-1}, \quad |\mu-t| \ge \frac{1}{R}, \tag{3.29}$$

$$|C_R(\mu,t)| \le C_2R^{s_0-s}\mu^{\delta+s_0}\varphi_{s_0}\left(\frac{1}{\mu R}\right)t^{-(\nu+1/2)}, \quad |\mu-t| \le \frac{1}{R}, \tag{3.30}$$

for $\mu > 3/R,\ 0 < t < 1/R$

$$|C_R(\mu,t)| \le C_3(\mu R)^\delta\varphi_{s_1}\left(\frac{1}{\mu R}\right), \quad \text{if} \quad \nu \le s - 1/2. \tag{3.31}$$

Proof. 1. If $|\mu - t| \ge \frac{1}{R}$, then for $u \in (\mu - \frac{1}{2R}, \mu)$ we have $|u-t| \ge \frac{1}{2R}$ and $|u-t| \ge \frac{|\mu-t|}{2}$. Therefore employing estimate (3.11) with the second summand in $[\dots]$ neglected leads to the estimate

$$|C_R(\mu,t)| \le CR^{-s}\mu^{\delta-1}t^{-(\nu+1/2)}|\mu-t|^{-1}\int_{\mu-\frac{1}{2R}}^{\mu} \varphi\left(1-\frac{u^2}{\mu^2}\right)udu.$$

Recalling the notation A_σ for $s_0 = 1 \quad \sigma = \mu - \frac{1}{2R}$ in equation (3.28) and we arrive at (3.29).

2. If $|\mu - t| \le \frac{1}{R}$, then for $u \in (\mu - \frac{1}{2R}, \mu)$ we have $|u-t| \le \frac{3}{R}$. Estimate (3.24) with the first summand in $[\dots]$ neglected leads to the estimate

$$|C_R(\mu,t)| \le CR^{s_0-s}\mu^{\delta-1}t^{-(\nu+1/2)}A_\sigma, \quad \sigma = \mu - 1/2R$$

and employing Lemma 3.3 we obtain the estimate (3.30).

3. Estimate (3.31) follows from the relations (3.23) and (3.28).

□

Proof of Proposition 3.4 Lemmas 3.2 and 3.4 result in Proposition 3.4. Indeed, $\Delta_R(\mu,t) = B_R(\mu,t) + C_R(\mu,t)$, estimates for $C_R(\mu,t)$ are of the required form and estimates for $B_R(\mu,t)$ in any case are not worst, because

$$\varphi_{s_0}(u) \geq \int\limits_{u/2}^{u} \varphi(v)v^{s_0-1}dv \asymp \varphi(u)u^{s_0}, \quad u \in (0,1].$$

□

Proof of Proposition 3.5. We have $0 < \mu < 3/R$.

1. Let $t > 2\mu$. Substituting in (3.7) the last estimate from (3.25) and for $u/\mu = v$, we obtain

$$|\Delta_R(\mu,t)| \leq c\,(\mu/t)^{\nu+3/2} \int\limits_0^1 v^{\varkappa}\varphi(1-v^2)dv, \quad \varkappa = \nu + 2s + 1/2.$$

   Since $\varkappa > -1$ and the integral is convergent this gives estimate (3.22).

2. Let $t \leq 2\mu$. For $s \geq 1$ comparison of estimates (3.25) shows that $|I| \leq cu^{2s-1}$, $0 < u < \mu$ for $-1/2 \leq \nu \leq s - 1/2$ from which in similar way to item 1, we obtain (3.21) for $0 < s < 1$. These considerations are true everywhere except the interval $\Delta = (t/2, \min\{2t, \mu\})$ on which we need to employ the second estimate from (3.25) and require to estimate the integral

$$\mathcal{N} := c_0\mu^{-2s} \int\limits_{\Delta} u^s \varphi\left(1 - \frac{u^2}{\mu^2}\right) |u-t|^{s-1}du.$$

   If $t < \mu/4$, then $\varphi\left(1 - \frac{u^2}{\mu^2}\right) \asymp \varphi(1)$ for $u \in \Delta$ and

$$\mathcal{N} \asymp \mu^{-2s}t^s\varphi(1) \int\limits_{t/2}^{2t} |u-t|^{s-1}du \asymp \mu^{-2s}t^{2s}\varphi(1) = O(1).$$

   And if $t > \mu/4$, then $u^{s-1} \asymp \mu^{s-1}$ for $u \in \Delta$

$$\mathcal{N} \asymp \mu^{-s-1} \int\limits_{\Delta} \varphi\left(1 - \frac{u^2}{\mu^2}\right) |u-t|^{s-1}udu \leq \mu^{-s-1}A_0 \leq c_1\varphi_s(1)$$

   also cf. Lemma 3.3 for $\sigma = 0$ and finally we again obtain (3.21).

□

Proof of relation (3.15). $\sigma_R(\mu) \to 0 \quad \text{as} \quad R \to \infty,\ \mu > 0.$

1. Assume that $R > 3/\mu$ and $R > 1$. Then

$$\sigma_R(\mu) = \int_0^{1/R} \dots + \int_{1/R}^{\mu/2} \dots + \int_{\mu/2}^{\infty} \dots = \sigma_0 + \sigma_1 + \sigma_2.$$

To estimate σ_0 we use inequality (3.18), then

$$|\sigma_0| \leq c(\mu R)^\delta \varphi_1\left(\frac{1}{\mu R}\right) \int_0^{1/R} |\hat{f}(t)||u(x_0,t)|d\rho(t). \tag{3.32}$$

But from properties (3.3) and (3.4) follows that

$$\int_0^1 |\hat{f}(t)||u(x_0,t)|d\rho(t) \leq c\|f\|_{L_2}\left\{\int_0^1 |u(x_0,t)|^2 d\rho(t)\right\}^{1/2} \tag{3.33}$$

and therefore $\sigma_0 \to 0$ as $R \to \infty$.

2. To estimate σ_1 we use inequality (3.19), and

$$|\sigma_1| \leq c(\mu) R^{-s} \varphi_1\left(\frac{1}{\mu R}\right) \int_{1/R}^{\mu} |\hat{f}(t)||u(x_0,t)|t^{(-\nu+1/2)} d\rho(t).$$

If $\nu \leq -1/2$, then the integral is $O(1)$. And if $\nu \geq -1/2$, then

$$|\sigma_1| \leq c(\mu) R^{\nu+1/2-s} \varphi_1\left(\frac{1}{\mu R}\right) \int_0^{\infty} |\hat{f}(t)||u(x_0,t)|d\rho(t)$$

and since $\delta = \nu + 1/2 - s \leq 0$, $\sigma_1 \to 0$ as $R \to \infty$.

3. Let us estimate σ_2. Taking into account that, if $0 < s < 1$, then

$$\varphi_1(u) = \int_0^u \varphi(u) v^{s-1} u^{1-s} dv \leq u^{1-s} \int_0^u \varphi(v) v^{s-1} dv = u^{1-s}\varphi_s(u), \tag{3.34}$$

we employ estimate (3.20) for σ_2 to get

$$|\sigma_2| \leq c\mu^{\delta+s_0} R^{s_0-s} \varphi_{s_0}\left(\frac{1}{\mu R}\right) \int_{\mu/2}^{\infty} |\hat{f}(t)||u(x_0,t)|t^{-(\nu+1/2)} d\rho(t),$$

and $\sigma_2 \to 0$ as $R \to \infty$.

□

3.4 Estimate of Φ−means of spectral decomposition

Proposition 3.6. *Let $s>0$, α, $\beta,\omega(u)$ be as in Theorem 2.1, $\Omega \subset\subset G$, $x_0\in\Omega$, $0<R<\rho(x_0,\partial\Omega)$, $R\le 1$, $f\in C_0^\infty(\Omega)$, $f(x)\equiv 0$ for $|x-x_0|\le R$. Then for all $\mu\ge 3/R$ holds true the following inequality:*

$$|\sigma_\mu^\Phi(f,x_0)|\le c\|f\|_{H_2^\omega}\left\{(\mu R)^{s_0-s-1/2}\varphi_{s_0}\left(\frac{1}{\mu R}\right)\omega\left(\frac{1}{\mu}\right)\mu^{n/2}+(\mu R)^\delta\varphi_1\left(\frac{1}{\mu R}\right)\omega(R)V_\beta(R)\right\} \tag{3.35}$$

Here c does not depend on μ, R, f, x_0 and

$$V_\beta(R)=\begin{cases} R^{-\frac{n}{2}}, & \beta<\frac{n}{2}\\ R^{-\frac{n}{2}}\log_2\left(\frac{1}{R}\right) & \beta=\frac{n}{2}\\ R^{-\beta} & \beta>\frac{n}{2}\end{cases}$$

$\delta\le\min\{0,(n+1)/2-s-\beta\}$.

To proof Proposition 3.6, in addition to Proposition 3.4 we require the following:

Lemma 3.5. *Let condition (2.11) be satisfied for $\alpha\le\beta<\alpha+3/2$. Then for any domain $\Omega\subset\subset G$ and function $f\in C_0^\infty(\Omega)$*

$$\sup_{\mu\ge1}\left\{\int_\mu^{3\mu}|\hat f(t)|^2 d\rho(t)\right\}^{1/2}\le c(\Omega)\omega\left(\mu^{-1}\right)\|f\|_{H_2^\omega} \tag{3.36}$$

The proof of the lemma is similar to that of Lemma 3.1 in [4], giving the estimate for H_2^r.

Proof of Proposition 3.6

1. In virtue of conditions on α and β we have

$$\varepsilon_0:=\min\{s+\alpha-(n-2)/2,\ 3/2+\alpha-\beta\}>\max\{0,\ \alpha-(n-1)/2\}=:\varepsilon_1$$

Put $\nu=(n-3)/2-\alpha+\varepsilon$, $\delta=\nu+1/2-s$, where $\varepsilon_1<\varepsilon\le\varepsilon_0$. Then $\nu>-1$ and the conditions

$$(n-3)/2<\alpha+\nu\le\beta+\nu\le n/2,\quad \delta\le\min\{0,(n+1)/2-s-\beta\} \tag{3.37}$$

are satisfied. We use this value of ν in representation (3.17) and estimate the contribution of each summand.

$$\sigma_\mu^\Phi(f,x_0)=\int_0^{1/R}\dots+\int_{1/R}^{\mu/2}\dots+\int_{\mu/2}^{3\mu/2}\dots+\int_{3\mu/2}^{\infty}\dots=\sigma_0+\sigma_1+\sigma_2+\sigma_3.$$

2. Let us estimate σ_0. Inequalities (3.32) and (3.33) show that the integral over the interval $(0,1)$ gives the value which is not greater than the value of the second

summand on the right-hand side of (3.35). Further, based on (3.36) and (2.2), we obtain

$$\left|\int_1^{1/R}\dots\right| \le \sum_{k=0}^{[\log_2 1/R]}\Big\{\int_{2^k}^{2^{k+1}}|\hat f(t)|^2 d\rho(t)\Big\}^{1/2}\Big\{\int_{2^k}^{2^{k+1}}|u(x_0,t)|^2 d\rho(t)\Big\}^{1/2}$$
$$\le c\|f\|_{H_2^\omega}\sum_{k=0}^{[\log_2 1/R]}\omega\left(2^{-k}\right)2^{kn/2}\equiv c\|f\|_{H_2^\omega}\Sigma_R.$$

But $\omega(u)u^{-\beta}$ is almost decreasing on $(0,1]$ and therefore

$$\Sigma_R=\sum_{k=0}^{[\log_2 1/R]}\omega\left(2^{-k}\right)2^{k\beta}2^{k(n/2-\beta)}\le c\omega(R)R^{-\beta}\sum_{k=0}^{[\log_2 1/R]}2^{k(n/2-\beta)}.$$

Finally $|\sigma_0|$ gives the second term on the right-hand side of (3.35).

3. To estimate σ_1 we use inequality (3.19) with $\mu-t\asymp\mu$ and obtain

$$|\sigma_1|\le cR^{-s}\mu^\delta\varphi_1\left(\frac{1}{\mu R}\right)\int_{1/R}^{\mu/2}t^{-(\nu+1/2)}|\hat f(t)||u(x_0,t)|d\rho(t). \tag{3.38}$$

Similar to step 2 for $\lambda=[\log_2(\mu R)]$

$$\int_{1/R}^{\mu/2}\dots\le c\sum_{k=1}^{\lambda}\left(\frac{\mu}{2^k}\right)^{-(\nu+1/2)}\Big(\int_{\mu/2^{k+1}}^{\mu/2^k}|\hat f(t)|^2 d\rho(t)\Big)^{1/2}\Big(\int_{\mu/2^{k+1}}^{\mu/2^k}|u(x_0,t)|^2 d\rho(t)\Big)^{1/2}$$
$$\le c_1\|f\|_{H_2^\omega}\sum_{k=0}^{\lambda}\left(\frac{\mu}{2^k}\right)^{\frac{n-1}{2}-\nu}\omega\left(\frac{2^k}{\mu}\right)\equiv c_1\|f\|_{H_2^\omega}\Sigma_{\mu,R}. \tag{3.39}$$

And we have

$$\Sigma_{\mu,R}\le c\omega\left(\frac{1}{\mu}\right)\mu^\beta\begin{cases}\mu^{\frac{n-1}{2}-\nu-\beta}, & \text{if } \nu+\beta<(n-1)/2,\\ \log_2(\mu R), & \text{if } \nu+\beta=(n-1)/2,\\ R^{\nu+\beta-\frac{n-1}{2}}, & \text{if } \nu+\beta>(n-1)/2.\end{cases}$$

Since $\nu+\beta\le n/2$,

$$\Sigma_{\mu,R}\le c\mu^{\frac{n-1}{2}-\nu}\omega\big(1/\mu\big)(\mu R)^{1/2}. \tag{3.40}$$

The estimates (3.38)–(3.40) show that σ_1 contributes to the first summand on the right hand-side of (3.35) and for $0<s<1$ inequality (3.34) should be considered in addition.

4. Let us estimate σ_2. Inequalities (3.19)–(3.20) and (3.34) result in

$$|\Delta_R(\mu,t)|\le cR^{s_0-s}\mu^{\delta+s_0}\varphi_{s_0}\left(\frac{1}{\mu R}\right)(1+R|\mu-t|)^{-1}t^{-(\nu+1/2)}.$$

Using inequality (3.34)

$$|\sigma_2| \le c\,(\mu R)^{s_0-1}\varphi_{s_0}\left(\frac{1}{\mu R}\right)\omega\left(\frac{1}{\mu}\right)\|f\|_{H_2^\omega}\left(\int\limits_{\mu/2}^{3\mu/2}\frac{|u(x_0,t)|^2d\rho(t)}{(1+R|\mu-t|)^2}\right)^{\frac{1}{2}}.$$

$$\int\limits_{\mu/2}^{3\mu/2}\dots \quad \le c\sum_{k=0}^{\infty}2^{-2k}\int\limits_{2^k-1\le R|\mu-t|\le 2^{k+1}}|u(x_0,t)|^2d\rho(t) \le c_1\mu^{n-1}R^{-1}\sum_{k=0}^{\infty}2^{-k}.$$

Finally σ_2 gives the required contribution to the right-hand side of (3.35).

5. To estimate σ_3 we use (3.8) and take into account that $t-\mu\asymp t$

$$|\sigma_3| \le cR^{-s}\mu^{\delta+1}\varphi_1\left(\frac{1}{\mu R}\right)\int\limits_{3\mu/2}^{\infty}t^{-(\nu+3/2)}|\hat{f}(t)||u(x_0,t)|d\rho(t).$$

Partition to the integrals over the intervals $[\mu 2^k,\mu 2^{k+1}]$, in similar way to steps 2 and 3, we obtain the estimate

$$\int\limits_{3\mu/2}^{\infty}\dots \quad \le c\|f\|_{H_2^\omega}\sum_{k=0}^{\infty}\left(\mu 2^k\right)^{\frac{n-2}{2}-\nu}\omega\left(\frac{1}{\mu 2^k}\right) \equiv c\|f\|_{H_2^\omega}\Sigma_\mu.$$

Taking into account results of step 1 and that $\omega(u)u^{-\alpha}$ is almost increasing,

$$\Sigma_\mu = \sum_{k=0}^{\infty}\left(\mu 2^k\right)^{\alpha-\varepsilon}\omega\left(\frac{1}{\mu 2^k}\right) \le c\mu^{\alpha-\varepsilon}\omega\left(\frac{1}{\mu}\right)\sum_{k=0}^{\infty}2^{-k\varepsilon}.$$

Substitute these estimates and take into account (3.34) if $s<1$. Finally we get the contribution of σ_3 which is even of smaller order than the first summand on the right hand-side of (3.35).

□

Proof of Theorem 2.1 Let $R=\rho(K,\partial D)$. Then for any function $f\in C_0^\infty(\Omega)$ such that $f\equiv 0$ in D and any point $x_0\in K$ the estimate (3.35) applicable with $R=R(K)>0$ fixed. Then

$$\left|\sigma_\mu^\Phi(f,x_0)\right| \le c(K)\|f\|_{H_2^\omega}\left[\omega\left(\frac{1}{\mu}\right)\Big/\omega_0\left(\frac{1}{\mu}\right)+\mu\delta\varphi_1\left(\frac{1}{\mu}\right)\right]$$

Since $\delta\le 0$, $\quad \mu^\delta\varphi_1\left(\frac{1}{\mu}\right)\to 0$ as $\mu\to\infty$. And finally apply condition (2.11) to get

$$\left|\sigma_\mu^\Phi(f,x_0)\right| \le c_1(K)\|f\|_{H_2^\omega},\quad \forall x_0\in K,\ \ \forall\mu\ge 1.$$

Then using the densness of $C_0^\infty(\Omega)$ in $\overset{\circ}{H}{}_p^{\omega(.)}(\Omega)$ and uniform convergence of spectral decomposition and hence $\Phi-$means for $C_0^\infty(\Omega)$, in standard scheme we obtain that $\sigma_\mu^\Phi(f,x)\to f(x)$ as $\mu\to\infty$ uniformly in $x\in K$ for any function $f\in\overset{\circ}{H}{}_p^{\omega(.)}(\Omega)$, $\ \ f\equiv 0$ in D. □

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