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To the authors, reviewers, and readers of the Eurasian Mathematical Journal

The L.N. Gumilyov Eurasian National University was the main organizer of the Eurasian Mathematical Journal in 2010.

The University provided essential financial support which allowed paying honorariums to the authors and reviewers. This support, together with the efforts of the international highly qualified Editorial Board consisting of more than 60 prominent mathematicians from more than 25 countries, allowed establishing a top level journal, which became a really international one.

In 2010 - 2013 there were published 133 scientific papers written by 182 mathematicians representing 37 countries. Among the authors there are full members and corresponding members of National Academies of Sciences of several countries.

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The journal has impact-factor of MathNet-Ru (0,328 for year 2012) and recently the Content Selection & Advisory Board of Scopus has advised that the journal will be accepted for inclusion in Scopus. The following comments of the reviewer were attached: “This is a first rate journal with excellent contribution of many leading figures (in partial differential equations especially though the standard is quite high of all articles that I have seen).”

The main goal of financial support by the L.N. Gumilyov Eurasian National University being achieved, the University has informed us that it will stop financing honorariums to the authors and reviewers starting with the issue EMJ, volume 5 (2014), number 3.

On behalf of the authors, reviewers, and readers of the EMJ, the Editorial Board expresses deep gratitude to the L.N. Gumilyov Eurasian National University for organizing the journal and generous financial support, and hopes that the journal will preserve its high level and in future will become one of the leading mathematical journals.

Editorial Board

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HÖLDER ANALYSIS AND GEOMETRY ON BANACH SPACES: HOMOGENEOUS HOMEOMORPHISMS AND COMMUTATIVE GROUP STRUCTURES, APPROXIMATION AND TZAR'KOV'S PHENOMENON PART I

S.S. Ajiev

Communicated by O.V. Besov

Dedicated to my father

Key words: Hölder classification of spheres, Hölder-Lipschitz mappings, approximation of uniformly continuous mappings, Tsar'kov's phenomenon, Mazur mappings, Lozanovskii factorisation, homogeneous right inverses, metric projection, asymmetric uniform convexity and smoothness, dimension-free estimates, λ-horn condition, local unconditional structure, Nikol'skii-Besov, Lizorkin-Triebel, Sobolev, non-commutative L_p and IG-spaces, Banach lattices, wavelets, three-space problem, Kalton-Pełczyńki decomposition, bounded extension of Hölder mappings, Markov type and cotype, complemented subspaces, UMD

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Abstract. In an explicit quantitative and often precise manner, we construct the homogeneous Hölder homeomorphisms and study the approximation of uniformly continuous mappings by the Hölder-Lipschitz ones between the pairs of abstract and concrete metric and (quasi) Banach spaces including, in particular, Banach lattices, general non-commutative L_p-spaces, the classes IG and IG_+ of independently generated spaces (for example, non-commutative-valued Bochner-Lebesgue spaces) and anisotropic Sobolev, Nikol'skii-Besov and Lizorkin-Triebel spaces of functions on an open subset or a class of domains of an Euclidean space defined with underlying mixed L_p-norms in terms of differences, local approximations by polynomials, wavelet decompositions and systems of closed operators, such as holomorphic functional calculus and Fourier multipliers of smooth Littlewood-Paley decompositions. Our approach also allows to treat both the finite (as in the initial and/or boundary value problems in PDE) and infinite l_p-sums of these spaces, their duals and "Bochnerizations". Many results are automatically extended to the setting of the function spaces with variable smoothness, including the weighted ones. The sharpness of the approximation results, shown for the majority of the pairs under some mild conditions and underpinning the corresponding sharpness of the Hölder continuity exponents of the homogeneous homeomorphisms, indicates that the range of the exponents is often a proper subset of $(0, 1]$, that is the presence of Tsar'kov's phenomenon. We also consider the approximation by the mappings taking

the values in the convex envelope of the range of the original approximated mapping. The negative results on the absence of uniform embeddings of the balls of some function spaces, particularly including BMO, VMO, Nikol'skii-Besov and Lizorkin-Triebel spaces with $q = \infty$ and their VMO-like separable subspaces, into any Hilbert space are established. Relying on the solution to the problem of the global Hölder continuity of metric projections and the existence of the Hölder continuous homogeneous right inverses of closed surjective operators and retractions onto closed convex subsets, as well as our results on the bounded extendability of the Hölder-Lipschitz mappings and re-homogenisation technique, we develop and employ our key explicit quantitative tools, such as the global (on arbitrary bounded subsets) Hölder continuity of the duality mapping and Lozanovskii factorisation, the answer to the three-space problem for the Hölder classification of infinite-dimensional spheres, the Hölder continuous counterpart of the Kalton-Pełczyński decomposition method, the Hölder continuity of the homogeneous homeomorphism induced by the complex interpolation method and such counterparts of the classical Mazur mappings as the abstract and simple Mazur ascent and complex Mazur descent. Important role is also played by the study of the local unconditional structure and other complementability results, as well as the existence of equivalent geometrically friendly norms.

1 Introduction

This is the first part of the article. The content of the second part is briefly described below.

While even uniform homeomorphisms of spheres play important role in nonlinear functional analysis [15], the existence of a homogeneous Hölder homeomorphism between two spaces permits, for example, to transfer counterexamples [49], group action, homogeneous Hölder group structures (see Theorems 5.14 − 5.16 below), entropy and non-compactness estimates, or topological results [40] used in PDE from one space into another. It can be especially convenient when one of the spaces is as well studied as the Hilbert space. In turn, the knowledge of the Hölder regularity exponents allows to "measure the Hölder closedness" of pairs of non-isomorphic spaces to each other and to approximate uniformly continuous mappings by Hölder-Lipschitz ones. In the setting of infinite-dimensional spaces, the latter research direction goes back to A.F. Timan [59], while the decisive contribution (including even the approximation of set-valued mapping) was made by S.V. Konyagin and I.G. Tsar'kov [39, 62, 63, 64, 65] (see Sections 10 and 11 for more details).

Namely, I.G. Tsar'kov [39, 62, 63, 64, 15] had considered the pairs (X, Y), where $X \in \{L_p, l_p\}_{p\in[1,\infty]}$ or a metric space, and $Y \in \{L_q, l_q\}_{q\in[1,\infty]}$ or a superreflexive space. He discovered an interesting phenomenon: f can be arbitrary well-approximated in $C(B_X, Y)$ (B_X is the unit ball in X) by an α-Hölder function if, and only if, $\alpha \in (0, \min(p, 2)/\max(q, 2)]$. In the case of a metric X and a superreflexive Y, he used the Frechet extension theorem and embedding into $l_\infty(B_X)$, along with the existence of $1/p$-Hölder retractions from l_∞ onto the bounded closed convex subsets of a p-uniformly convex space. Moreover, he reduced the case of the pairs of the Lebesgue and l_p-spaces to the case of the pair (l_2, l_2) by means of studying and employing the

Hölder-Lipschitz regularity of the classical Mazur mapping and, eventually, utilising Kirszbraun's extension theorem.

Hölder classification and approximation problems are very strongly related: the sharpness of the latter can lead to the sharpness of the former (see Sections 6.2 and 11).

There are seven essentially different classifications (equivalence relations) of Banach spaces by:

1) isomorphisms (preserving the linear structure and topology);

2) continuous homeomorphisms (preserving the topology);

3) isometries (preserving the metric);

4) local isomorphisms (preserving the linear structure and topology of finite-dimensional subspaces);

5) uniform homeomorphisms (preserving the uniform structure);

6) uniform homeomorphisms of spheres (preserving the uniform structure and homogeneity);

7) Hölder homeomorphisms of spheres (preserving the uniform structure in a strong quantitative sense and homogeneity).

Using the **notation** $X \overset{i}{\sim} Y$ for the ith equivalence relation between Banach spaces, one can conclude that the relations $(1, 3, 4, 5)$ are too discrete, while the second is too synthetic. Indeed, it was established by Banach himself that l_p and L_p are not isomorphic for $p \neq 2$, while M.I. Kadets [38] proved that all separable Banach spaces are (continuously) homeomorphic. In turn, S. Mazur and S. Ulam [47] discovered that every isometry of two Banach spaces is a composition of a translation and a linear isometry. The classification by local isomorphisms generated by the partial order relation of λ-finite representability (see Definition 2.3) attracted the most attention thanks to its close ties with the Rademacher type and cotype and the super-properties of Banach spaces. While $X \overset{4}{\sim} X^{**}$ (principle of local reflexivity) and $l_p \overset{4}{\sim} L_p([0,1])$, different Rademacher types or cotypes yield $l_p \overset{4}{\not\sim} l_q$ for $p \neq q$ $(p, q \in [1, \infty))$. The fifth classification appeared to be even more discrete than the forth: M. Ribe [55] showed that $X \overset{5}{\sim} Y$ implies $X \overset{4}{\sim} Y$, while W.B. Johnson, J. Lindenstrauss and G. Schechtman [36] have observed that, if a Banach space X is uniformly homeomorphic to l_p for $p \in (1, \infty)$, then it is isomorphic to l_p. Moreover, $l_p \overset{5}{\not\sim} L_p$ for $p \in (0, \infty) \setminus \{2\}$ according to J. Bourgain [21] (case $p \in [1, 2)$), E. Gorelik [34] ($p \in (2, \infty)$) and A. Weston [69] ($p \in (0, 1]$). (See also [15].)

The most balanced sixth classification was established by E. Odell, Th. Schlumprecht in their celebrated work [49]. Substituting $\overset{6}{\sim}$ with $\overset{7}{\sim}$ in their result on the classification of the Banach lattices, we obtain Theorem 6.2 below, meaning that $\overset{6}{\sim}$ and $\overset{7}{\sim}$ coincide on the class of Banach lattices. The investigation of the properties of the more quantitative seventh classification has begun with the introduction of the Mazur transform by S. Mazur [46], and is also the first main task of this article.

It is well-known since the 19th century that a (uniformly) continuous function $f \in C = C([a, b])$ for $[a, b] \subset (-\infty, \infty)$ can be arbitrary well-approximated in $C([a, b])$ by smooth functions, such as S.N. Bernstein polynomials (Weierstrass theorem), Fejer trigonometric polynomials and Steklov averages

$$f_\varepsilon(x) = \frac{1}{\varepsilon}\int\limits_x^{x+\varepsilon} \tilde{f}(\tau)d\tau,$$

where $\tilde{f}$ is the extension of f to $\mathbb{R}$ by the constants $f(a)$ and $f(b)$. Steklov's approach is suitable for both finite and infinite interval $[a,b]$ and provides very convenient explicit estimates in terms of the modulus of continuity (of the first order in C)

$$\omega(\varepsilon, f) = \sup_{|x-y|\le\varepsilon} |f(x) - f(y)| : \ \|f - f_\varepsilon\|_C \le \omega(\varepsilon, f), \|f_\varepsilon\|_{Lip} \le \omega(\varepsilon, f)/\varepsilon$$

and

$$f_\varepsilon([a,b]) \subset f([a,b])$$

for $\varepsilon > 0$.

As the second main task, we study the counterparts of this result in the setting of uniformly continuous functions from a bounded subset A of a metric or a (quasi) Banach space X into a (quasi) Banach space Y for abstract and particular pairs (X, Y), where X and Y satisfy certain natural additional conditions (like being isomorphic or Lipschitz-homeomorphic to "better" abstract spaces). In particular, both X and Y is allowed to be either contained or isomorphic to an arbitrary space from the union of our six Γ-groups of the classes of the specific parameterised spaces described below.

In fact, our approaches also permit to treat the pairs with X and/or Y taken to be l_p-sums of function spaces (that can correspond to the spaces naturally appearing in some boundary, or initial value problem), function spaces with variable smoothness (including weighted spaces) investigated by O.V. Besov [19], spaces with dominating derivatives and both the duals and l_p-sums of the spaces mentioned above. The basic groups of the function spaces we are dealing with include anisotropic Nikol'skii-Besov and Lizorkin-Triebel spaces of functions on an open subset of $\mathbb{R}^n$ that are defined in terms of either averaged differences, or local polynomial approximation, or wavelet decompositions, or systems of closed operators with underlying mixed L_p-norms (see subsection 2.1). Particular examples of the classes defined in terms of the systems of closed operators are Nikol'skii-Besov and Lizorkin-Triebel spaces defined in terms of the Fourier multipliers of the smooth Littlewood-Paley decompositions considered by S.M. Nikol'skii, P.I. Lizorkin and H. Triebel [48, 61] (including Lizorkin – Triebel spaces; see Remarks 2.4 and 10.3). The group Γ_1 includes also the duals of these function spaces. The classical information about function spaces is in [58, 22, 48, 20, 61, 29, 37, 52, 35, 60, 19]. In the setting of function spaces, it appears quantitatively relevant to the problem that the domain of the definition of functions may satisfy the C-flexible λ-horn condition due to O.V. Besov [18].

The first step towards the application of the quantitative methods based on the quasi-Euclidean approach developed in [5, 7, 9, 12] is the choice, if necessary, of a geometrically-friendly equivalent norm in a space under consideration. Thus, in Section 2 we define and divide into six Γ-groups all the parameterised spaces under consideration, describe subfamilies of equivalent norms on some of them and relations between different classes of space and provide a quantitative description of their asymmetric uniform convexity and uniform smoothness. A large class of auxiliary IG-spaces, including, in particular, l_p-sums of L_p-spaces with mixed norm (and other IG-spaces), was introduced, studied and employed in [5, 6, 7, 9, 12]. The class IG_+ extends IG including also the l_p-sums and "Bochnerizations" of the Lebesgue and sequence spaces of functions (possibly, on a discrete set) with values in noncommutative L_p-spaces [53].

Section 3 contains the elementary properties of the Hölder-Lipschitz mappings and the auxiliary results (including some involving the matter of sharpness) on the existence of either ordinary or Hölder continuous (globally on arbitrary bounded subsets) homogeneous inverses for closed linear surjections between Banach spaces. We also introduce the notions related to the Hölder equivalence of spheres of abstract spaces. Moreover, Lemma 3.2 constitutes the answer to the three-space problem for our classification $\overset{7}{\sim}$, while Theorem 3.4 is the Hölder continuous counterpart of the N. Kalton's nonlinear version of A. Pełczyński's decomposition method.

Section 4 contains the definitions and properties of our relatively abstract but occasionally sharp key explicit quantitative tools: the global (on arbitrary bounded subsets) Hölder continuity of the duality mapping and Lozanovskii factorisation and the Hölder continuity of the homogeneous homeomorphism induced by the complex interpolation method. The latter mapping and its uniform continuity are due to M. Daher [28] and N.J. Kalton [15].

In Section 5, we employ the latter key abstract tool and develop a re-homogenisation technique to construct and study our counterparts of the Mazur mapping that we call abstract and simple Mazur ascents and complex Mazur descent. Their compositions appear to be the Hölder homeomorphisms between the spheres of the pairs of compatible IG_{0+}-spaces that are sharp in the setting of the IG_0-spaces and occasionally sharp in the setting of the IG_{0+}-spaces.

Section 6 contains the main results of the paper on the homogeneous Hölder homeomorphisms in a form that permits to trace the constants. We start with complete description of the Banach lattices that are in one equivalence class with the Hilbert spaces and proceed by employing our abstract and constructive tools from Sections 4 and 5 to provide quantitative Hölder classification of the spheres of all the spaces under consideration with respect to the spheres of the Hilbert spaces, including also some spaces that are not equivalent to a Hilbert space. Indeed, relying on the solution to Smirnov's problem due to P. Enflo [32] and our results [3, 4, 11] on the finite representability of c_0 in (anisotropic) $BMO(G)$, $VMO(G)$, $BMO(G) \cap L_\infty(G)$, $VMO \cap L_\infty(G)$, Nikol'skii-Besov and Lizorkin-Triebel spaces, as well as their VMO-like subspaces, we show that the unit balls of these spaces cannot be uniformly embedded into any separable or nonseparable Hilbert space.

In Section 7, we introduce commutative homogeneous Hölder group structures (compatible with the norm and the existing linear structure) on all our spaces under consideration, even on those that do not admit any C^*-algebra structure.

Section 8 contains various results related to complementability of subspaces of abstract and specific Banach spaces, including the existence of certain complemented subspaces, that are employed either directly in the second group of the main results in Section 8, or via some key auxiliary results that are either our counterpart of the

Kalton-Pełczyńki decomposition method in Section 3, or the presence of Tsar'kov's phenomenon (our main sharpness tool) in Section 11.

Section 9 comprises, in an explicit and quantitative form relying on the asymmetric uniform convexity and smoothness and Markov type and cotype, the basic auxiliary properties of abstract and specific Hölder-Lipschitz mappings employed in our approaches to the second main task of the article: globally Hölder-continuous retractions and metric projections onto closed convex subsets of Banach spaces and the bounded extendability of the Hölder-Lipschitz mappings between Banach spaces.

The second group of the main results that are on the approximation of uniformly continuous mappings is situated in Section 10, where we utilise all our key tools developed in the previous sections, as well as the sharpness tools from Section 11. We first establish the approximation results in abstract and semi-abstract settings of mappings from metric, quasi-Banach and IG-spaces into quasi-Banach and IG-spaces, and, then, apply some of these results, as well as our other tools, to treat the approximation of the uniformly continuous mappings between the pairs of either abstract Banach lattices, or our Γ-groups of the specific spaces under consideration.

In Section 11, we benefit from some uniform complementability results given in Section 8 (see also [11]) by detecting the presence of Tsar'kov's phenomenon for the majority of the pairs of the specific spaces under consideration.

The numbering of the equations is used sparingly. Since the majority of references inside every logical unit are to the formulas inside the unit, equations are **numbered independently** inside every proof of a corollary, lemma and theorem, or a definition (if there are any numbered formulas). The name of the corresponding logical unit does not accompany the number of the formula in the references from inside this unit.

2 Definitions, designations and basic properties

Let $\mathbb{N}$ be the set of the natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $I_n = [1, n] \cap \mathbb{N}$ for $n \in \mathbb{N}$ and $I_m = \prod_{i=1}^n I_{m_i}$ for $m \in \mathbb{N}^n$; for $\alpha, \beta \in \mathbb{N}_0^n$, assume that $\alpha \leq \beta$ means the partial order relation generated by the coordinate order relations; $\max(\alpha, \beta) = \min\{\gamma : \gamma \geq \alpha, \gamma \geq \beta\}$; $\mathbb{R}^n$ is the n-dimensional Euclidean space with the standard basis $\langle e^1, \dots, e^n \rangle$, $x = (x_1, \dots, x_n) = \sum_1^n x_i e^i = (x_i)$; $x_{\min} := \min_i x_i$ and $x_{\max} := \max_i x_i$.

Let p' be the conjugate to $p \in [1, \infty]^n$, i.e. $1/p'_i + 1/p_i = 1$ $(1 \leq i \leq n)$.

For $A \subset \mathbb{N}_0^n$, let $|A|$ designate the number of the elements of A; for $\alpha \in \mathbb{N}_0^n$,

$$\hat{A}_\alpha = \{\beta : \beta \in \mathbb{N}_0^n, \beta \leq \alpha\};\ \hat{A} = \bigcup_{\alpha \in A} \hat{A}_\alpha.$$

In what follows, one can assume $\gamma_a = ((\gamma_a)_1, \dots, (\gamma_a)_n) \in (0, \infty)^n$ and $|\gamma_a| = \sum_1^n (\gamma_a)_i = n$ to be fixed.

For $x, y \in \mathbb{R}^n$, $t > 0$, let $[x, y]$ be the segment in $\mathbb{R}^n$ with the ends x and y; $xy = (x_i y_i)$, $t^y = (t^{y_i})$; $x/y = \left(\frac{x_i}{y_i}\right)$ for $y_i \neq 0$, and $t/\gamma_a = \left(\frac{t}{(\gamma_a)_i}\right)$. Assuming $|x|_{\gamma_a} = \max_{1 \leq i \leq n} |x_i|^{1/(\gamma_a)_i}$, we have

$$|x + y|_{\gamma_a} \leq c_{\gamma_a} \left(|x|_{\gamma_a} + |y|_{\gamma_a}\right).$$

For $E \subset \mathbb{R}^n$ and $b \in \mathbb{R}^n$, let

$$|E|_{\gamma_a} = \inf_{x \in E} |x|_{\gamma_a}, \ x \pm bE = \{y : y = x \pm bz, z \in E\}$$

and

$$G_t = \{x : x \in G, |x - \partial G|_{\gamma_a} > t\}$$

for an open $G \subset \mathbb{R}^n$.

Definition 2.1. *For $m \in \mathbb{N}$, $h > 0$, $\gamma \in \mathbb{R}$, $x \in \mathbb{R}^n$, assume that $\Delta_i^m(h)f(x)$ is the difference of the function f of the mth order with the step h in the direction of e^i at the point x, and*

$$\Delta_i^m(h,E)f(x) = \begin{cases} \Delta_i^m(h)f(x) & \text{for } [x, x+mhe^i] \in E, \\ 0 & \text{for } [x, x+mhe^i] \notin E, \end{cases}$$

$$\delta_{i,a}^m(t,x,f,E)_\gamma = \left(\int_{-1}^{1} \left| \Delta_i^m(t^{(\gamma_a)_i}u, E) f(x + \gamma t^{(\gamma_a)_i} u) \right|^a du \right)^{1/a}.$$

Sometimes, in the absence of ambiguity, we use a shorter form $\delta_{i,a}^m(t,x,f)$ instead of $\delta_{i,a}^m(t,x,f,E)_\gamma$.

If ϕ is an integrable function on a Lebesgue-measurable set $E \subset \mathbb{R}^n$, and $|E|$ is the Lebesgue measure, then $\phi_E = |E|^{-1} \int_E \phi \, dx$. Assume $Q_0 = [-1,1]^n$. For $v \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, we say that $Q_v(x) = x + v^{\gamma_a} Q_0$ is the parallelepiped of the γ_a-radius v with the centre x; χ_E is the characteristic function of E, and $\Theta = \chi_{(0,\infty)} : \mathbb{R} \to \{0,1\}$ is Heaviside's Θ–function.

For $p \in (0,\infty)$, $q \in (0,\infty]$, a (countable) index set I, and a quasi-Banach space A, let $l_q(I,A)$ and $l_\infty(I,A)$ be, correspondingly, the (quasi) Banach spaces of all sequences $\alpha = \{\alpha_k\}_{k \in I} \subset A$ with the finite quasi-norms

$$\|\alpha | l_q(I,A)\| := \left(\sum_{k \in I} \|\alpha_k\|_A^q \right)^{1/q} < \infty \text{ and } \|\alpha | l_\infty(I,A)\| := \sup_{k \in I} \|\alpha_k\|_A.$$

Assume also that $c_0(I,A)$ is the closure in $l_\infty(I,A)$ of all $\alpha \in A^I$ with the finite support set $\{k \in I : \alpha_k \neq 0\}$. More generally, for $n \in \mathbb{N}$ and $r \in (0,\infty]^n$, let $l_r(I^n, A)$ be the (quasi) Banach space of all sequences $\alpha = \{\alpha_k\}_{k \in I^n} \subset A$ with the finite quasi-norm

$$l_r(I^n, A) := l_{r_n}\left(I, \left(l_{r_{n-1}} \ldots (l_{r_1}(I, A \underbrace{) \ldots)\right)}_{n \text{ brackets}}\right).$$

For the sake of brevity, we also use the notation

$$l_r(I^n) = l_r(I^n, \mathbb{R}^n), \ c_0(I) = c_0(I, \mathbb{R}^n) \text{ and } l_r = l_r(\mathbb{N}^n).$$

For $p \in (0,\infty]$, let $L_{*p} = L_{*p}(\mathbb{R}_+)$ be the (quasi) normed space of all functions f measurable on $\mathbb{R}_+$ with the finite (quasi) norm

$$\|f|L_{*p}(\mathbb{R}_+)\| := \left(\int_{\mathbb{R}_+} |f(t)|^p dt/t \right)^{1/p} \text{ for } p < \infty \text{ or } \|f|L_{*\infty}(\mathbb{R}_+)\| := \|f|L_\infty(\mathbb{R}_+)\|.$$

For $G \subset \mathbb{R}^n$ and $f : G \to \mathbb{R}$ by means of $\overline{f} : \mathbb{R}^n \to \mathbb{R}$, we designate the function

$$\overline{f(x)} := \begin{cases} f(x), & \text{for } x \in G, \\ 0, & \text{for } x \in \mathbb{R}^n \setminus G. \end{cases}$$

For $p \in (0, \infty]^n$, let $L_p(G)$ be the space of all measurable functions $f : G \to \mathbb{R}^n$ with the finite mixed quasi-norm

$$\|f|L_p(G)\| = \left(\int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}} \dots \left(\int\limits_{\mathbb{R}} |\overline{f}|^{p_1} \, dx_1 \right)^{p_2/p_1} \dots \right)^{p_n/p_{n-1}} dx_n \right)^{1/p_n},$$

where, for $p_i = \infty$, one understands $\left(\int_{\mathbb{R}} |g(x_i)|^{p_i} \, dx_i\right)^{1/p_i}$ as $\operatorname{ess\,sup}_{x_i \in \mathbb{R}} |g(x_i)|$. The classical quantitative geometry of these spaces had been studied for a long time (see, for example, [45]).

For a measurable space (Ω, μ), an ideal (symmetric) space $Y = Y(\Omega)$ and a Banach space X, let $Y(\Omega, X)$ be the space of the Bochner-measurable functions $f : \Omega \mapsto X$ with the finite (quasi)norm

$$\|f|Y(\Omega, X)\| := \| \, \|f(\cdot)\|_X \, | \, Y(\Omega)\| \, .$$

If another measure ν, absolutely continuous with respect to μ with the density $\frac{d\nu}{d\mu} = \omega$, is used instead of μ, the corresponding space is denoted by $Y(G, \omega, X)$.

For example, $L_p(\mathbb{R}^n, l_q)$ with $p, q \in [1, \infty]$ is the Banach space of the measurable function sequences $f = \{f_k\}_{k=1}^{\infty}$ with the finite norm $\| \, \|\{f_k(\cdot)\}_{k \in \mathbb{N}}| \, l_q\| |L_p(\mathbb{R}^n)\|$.

For $s \in \mathbb{N}^n$, $\varsigma \in [1, \infty)$, $p \in [1, \infty]^n$ and an open $G \subset \mathbb{R}^n$, let the Sobolev space $W_p^s(G) = W_p^s(G)_\varsigma$ be the Banach space of the measurable functions f defined on G and possessing the Sobolev generalised derivatives $D_i^{s_i} f$ and the finite norm

$$\|f|W_p^s(G)\|^\varsigma := \|f|L_p(G)\|^\varsigma + \|f|w_p^s(G)\|^\varsigma = \|f|L_p(G)\|^\varsigma + \sum_{i=1}^{n} \|D_i^{s_i} f|L_p(G)\|^\varsigma.$$

Definition 2.2. *For $p \in (0, \infty)^n$, $q \in (0, \infty)$ and $n \in \mathbb{N}$, let $lt_{p,q} = lt_{p,q}(\mathbb{Z}^n \times J)$ be the quasi-Banach space of the sequences $\{t_{i,j}\}_{i \in \mathbb{Z}^n}^{j \in J}$ with $J \in \{\mathbb{N}_0, \mathbb{Z}\}$ endowed with the (quasi) norm*

$$\|\{t_{i,j}\}|lt_{p,q}\| := \left\| \left\{ \sum_{i \in \mathbb{Z}^n} t_{i,j} \chi_{F_{i,j}} \right\}_{j \in J} \right| L_p(\mathbb{R}^n, l_q(J)) \Bigg\|,$$

where $\{F_{i,j}\}_{i \in \mathbb{Z}^n}^{j \in J}$ is a fixed nested family of the decompositions $\{F_{i,j}\}_{i \in \mathbb{Z}^n}$ of $\mathbb{R}^n$ into unions of congruent parallelepipeds $\{F_{i,j}\}_{i \in \mathbb{Z}^n}$ satisfying

$$\cup_{i \in \mathbb{Z}^n} F_{i,j} = \mathbb{R}^n, \; |F_{i,j} \cap F_{k,j}| = 0 \text{ for every } j \in J, \; i \neq k,$$

and either $|F_{i_0,j_0} \cap F_{i_1,j_1}| = 0$, or $F_{i_0,j_0} \cap F_{i_1,j_1} = F_{i_0,j_0}$ for every i_0, i_1 and $j_0 > j_1$. We shall assume that this system regular in the sense that the length of the kth side $l_{k,j}$ of the parallelepipeds $\{F_{i,j}\}_{i \in \mathbb{Z}^n}$ of the jth decomposition (level) satisfies

$$c_l b^{-j(\lambda_a)_k} \leq l_{k,j} \leq c_u b^{-j(\lambda_a)_k} \text{ for } k \in I_n$$

for some positive constants c_l, c_u *and* $b > 1$. *Let also the* 0*-level parallelepipeds* $\{F_{i,0}\}_{i\in\mathbb{Z}^n}$ *be of the form* $Q_t(z)$.

For a parallelepiped $F \in \{F_{i,j}\}_{i\in\mathbb{Z}^n}^{j\in J}$ *or* $F = \mathbb{R}^n$, *by means of* $lt_{p,q}(F)$ *we designate the subspace of* $lt_{p,q}(\mathbb{Z}^n \times J)$ *defined by the condition:* $t_{i,j} = 0$ *if* $F_{i,j} \not\subset F$.

The symbol $lt_{p,q}$ *denotes either* $lt_{p,q}(\mathbb{Z}^n \times J)$ *or* $lt_{p,q}(F)$.

The space $lt_{p,q}$ is isometric to a complemented subspace of $L_p(\mathbb{R}^n, l_q)$, while its dual $lt^*_{p,q}$ is isomorphic to $lt_{p',q'}$ for $p, q \in (1,\infty)$ (see section 8).

Remark 2.1. To motivate a definition of anisotropic Nikol'skii-Besov and Lizorkin-Triebel spaces in terms of the coordinates of wavelet decompositions, we need to introduce the notion of the non-stationary multiresolution analysis of multivariable wavelets and provide the characterisations (equivalent norms) of some anisotropic Nikol'skii-Besov and Lizorkin-Triebel spaces of functions defined on $\mathbb{R}^n$ in terms of these decompositions as has been done in [8] in exactly the sam setting. Since our approach does not need all this information, we simply mention the properties of these spaces that we shall employ.

Let us assume that anisotropic Nikol'skii-Besov and Lizorkin-Triebel spaces $B^s_{p,q}(\mathbb{R}^n)_w$ and $L^s_{p,q}(\mathbb{R}^n)_w$ for $s_* \in \mathbb{R}$, $q \in (1,\infty)$ and $p \in (1,\infty)^n$ endowed with the wavelet norms are such sequence spaces that there are isometries $I_{B,s}$ and $I_{L,s}$ depending only on s that make them isometric, correspondingly, to the space $l_q(\mathbb{N}, l_p)$ and a 1-complemented subspace of $lt_{p,q}\,(\mathbb{Z}^n \times J, l_q(M_{\max}))$ described by the zero values of the coordinates with the indexes from a subset $R \subset \mathbb{Z}^n \times J \times I_{M_{\max}}$ with $\mathbb{Z}^n \times J \times \{1\} \cap R = \emptyset$, where $M_{\max} + 1 = \sup_{j\in J} \prod_{k=1}^n \frac{l_{k,j}}{l_{k,j+1}}$ (see Definition 2.2). Here the intersection condition assures that the subspace isomorphic to $L^s_{p,q}(\mathbb{R}^n)_w$ contains a 1-complemented (isometric) copy of $lt_{p,q}$.

For (quasi) Banach spaces X and Y, $\mathcal{C}(X,Y)$ and $\mathcal{L}(X,Y)$ are the classes of the closed and the bounded linear operators correspondingly. For $C \geq 1$, we say that a subspace Y of a (quasi) Banach space X is C-complemented (in X) if there exists a projection P onto Y satisfying $\|P|\mathcal{L}(X)\| \leq C$.

For $B \subset X$, let $\mathrm{co}(B)$ and $\overline{\mathrm{co}}(B)$ be the convex envelope and the closed convex envelope of B in X correspondingly. Let also

$$S_X = \{x \in X : \|x\|_X = 1\} \text{ and } B_X = \{x \in X : \|x\|_X \leq 1\}$$

be the unit sphere and the unit ball of a Banach space X.

The bi-linear form representing the duality between a (quasi) normed space A and $A^* = \mathcal{L}(A, \mathbb{R})$ is written as

$$(A^*, A) \ni (f, x) \mapsto \langle f, x\rangle = f(x).$$

For an operator T from X into Y, let $D(T)$, $\mathrm{Ker}\, T$ and $\mathrm{Im}\, T$ be its domain, kernel and image correspondingly.

Definition 2.3. *Let* X, Y *be (quasi) Banach spaces and* $\lambda \geq 1$. *Then the Banach-Mazur distance* $d_{BM}(X,Y)$ *between them is equal to* ∞ *if they are not isomorphic and is, otherwise, defined by*

$$d_{BM}(X,Y) := \inf\{\|T\| \cdot \|T^{-1}\| : T : X \xrightarrow{onto} Y,\ \mathrm{Ker}\, T = 0\}.$$

The space X *is* λ*-finitely represented in* Y *if for every finite-dimensional subspace* $X_1 \subset X$*,*

$$\inf\{d_{BM}(X_1, Y_1) : Y_1 \textit{ is a subspace of } Y\} = \lambda.$$

If λ *is equal to 1, then* X *is simply said to be finitely represented in* Y*.*

It is said that Y *contains almost isometric copies of* X *(or contains* X *almost isometrically) if*

$$\inf\{d_{BM}(X, Y_1) : Y_1 \textit{ is a subspace of } Y\} = 1.$$

We also say that X *is* λ*-isomorphic to* Y *if there exists an isomorphism* $T : X \leftrightarrow Y$ *with* $\|T|\mathcal{L}(X,Y)\|\|T^{-1}|\mathcal{L}(Y,X)\| \le \lambda$*.*

Metric spaces X *and* Y *are* C-Lipschitz homeomorphic *if there exists an invertible homeomorphism* $\phi \in H^1(X,Y)$ *satisfying*

$$\|\phi|H^1(X,Y)\| \cdot \|\phi^{-1}|H^1(Y,X)\| \le C.$$

For $r \in [1,\infty]$, a finite or countable set I and a set of quasi-Banach spaces $\{X_i\}_{i\in I}$, let its l_r-sum $l_r(I, \{A_i\}_{i\in I})$ be the space of the sequences $x = \{x_i\}_{i\in I} \in \prod_{i\in I} X_i$ with the finite norm

$$\|x|l_r(I, \{A_i\}_{i\in I})\| := \left(\sum_{i\in I} \|x_i\|^r_{A_i}\right)^{1/r}.$$

For $\gamma_a \in (0,\infty)^n$ and $s \in [0,\infty)$, let $A^*_s = \{\alpha : \alpha \in \mathbb{N}^n_0, (\alpha, \gamma_a) \le s\}$.

For $z \in \mathbb{R}^n$ and $v > 0$, assume that $\tau_z f(x) = f(x - z)$ and $\sigma_v f(x) = f(v^{-\gamma_a} x)$. For a finite $A \subset \mathbb{N}^n_0$, let $\mathcal{P}_A$ be the space of the polynomials of the form $\sum_{\alpha\in A} c_\alpha x^\alpha$ with $\{c_\alpha\}_{\alpha\in A} \subset \mathbb{R}$. For $a \in [1,\infty]^n$, let $p_A \in \mathcal{L}(L_a(Q_0), \mathcal{P}_A)$ be a certain projector onto $\mathcal{P}_A$. To insure the translation invariance of $\mathcal{P}_A$, we shall **always assume** that $A = \hat{A}$. Recall that, for a Banach space X and its subspace $Y \subset A$, X/Y denotes their quotient or factor space.

Definition 2.4. *For* $a \in [1,\infty]^n$*, let* $p_{A,v,z} = \tau_z \circ \sigma_v \circ p_A \circ \sigma_v^{-1} \circ \tau_z^{-1}$*.*

For $\varepsilon > 0$*,* $a \in (0,\infty]^n$*, let* $\pi_{A,v,z} : L_a(Q_v(z)) \to \mathcal{P}_A$ *be an operator of the best* L_a*-approximation satisfying*

$$\|f - \pi_{A,v,z} f|L_a(Q_v(z))\| = \min_{g\in\mathcal{P}_A} \|f - g|L_a(Q_v(z))\|, \ f \in L_a(Q_v(z)).$$

For $G \subset \mathbb{R}^n$*,* $f \in L_{a,loc}(G)$*,* $v > 0$ *and* $a \in (0,\infty]^n$*, we define the* $\mathcal{D}$*-functionals*

$$\mathcal{D}_a(v,x,f,G,A) := \begin{cases} \|f|L_a(Q_v(x))/\mathcal{P}_A\| v^{-(\gamma_a, 1/a)}, & \text{if} \quad Q_v(x) \subset G, \\ 0 & \quad \text{otherwise} \end{cases}$$

and

$$\mathcal{D}_a(v,x,f,G,p_A) := \begin{cases} \|f - p_{A,v,x}|L_a(Q_v(x))\| v^{-(\gamma_a, 1/a)}, & \text{if} \quad Q_v(x) \subset G, \\ 0 & \quad \text{otherwise.} \end{cases}$$

Remark 2.2. ([1, 2]) Let us note that

$$\|f - \pi_{A,v,x} f|L_a(Q_v(x))\| \asymp \|f - p_{A,v,x}|L_a(Q_v(x))\| \asymp \|f|L_a(Q_v(x))/\mathcal{P}_A\|$$

uniformly in v and x.

It is interesting to note that, while switching from one $\mathcal{D}$-functional to another provides only an equivalent norm, the geometric properties under consideration will depend on the parameters only, remaining identical. The same is true regarding switching to another projector in the definition of the second $\mathcal{D}$-functional.

2.1 Classes of spaces of Nikol'skii-Besov and Lizorkin-Triebel types

Above we have already considered Sobolev spaces and Nikol'skii-Besov and Lizorkin-Triebel spaces endowed with the wavelet norms. Let us define several more classes of spaces of Nikol'skii-Besov and Lizorkin-Triebel types. In these definitions, we use a parameter $\varsigma \in (0, \infty]$, which is essential in the study of the geometric properties of function spaces but not the topological (isomorphic) ones (we have equivalent (quasi) norms for different $\varsigma \in (0, \infty]$). It will normally be omitted for the sake of simplicity. If its presence and value should be emphasized, we say that the space under consideration is endowed with **the ς-product norm, or, just, the ς-norm**, and/or add ς as a subindex. For a seminormed (homogeneous) space $x(G)$ of functions defined on $G \subset \mathbb{R}^n$ and an ideal space $Y(G)$, we assume that its intersection $x(G) \cap Y(G)$ is endowed with the ς-norm too:

$$\|f|x(G) \cap Y(G)\|^{\varsigma} := \|f|Y(G)\|^{\varsigma} + \|f|x(G)\|^{\varsigma}.$$

Moreover, **we shall always assume** the parameter ς to be equal to one of the other parameters or its components of $x(G)$ and $Y(G)$ except for the smoothness or anisotropy components.

We start with the spaces defined in terms of the averaged (shifted) axis-directional differences. While the study of these norms and their equivalence with the other norms was one of the primary tasks of, for example, [2] (including the setting of arbitrary open subsets G), we shall refrain from the usage of the results of this type here in order to cover the sets of the parameters not covered by the equivalence results, and because the geometric constants depend on the specific equivalent norm chosen.

Let Pr_i be the orthogonal projector on the ith axis in $\mathbb{R}^n$, and, for any $y \in (I - Pr_i)(G)$,

$$In_i(y) := (I - Pr_i)^{-1}(y) \cap G = \{x \in G : x = y + te_i, t \in \mathbb{R}\}.$$

Definition 2.5. *For an ideal space $Y = Y(G)$, $p \in (0, \infty]^n$, $q \in (0, \infty]$, $r > 0$, $s \in (0, \infty)$, $s/\gamma_a < m \in \mathbb{N}_0^n$, $a \in (0, \infty]^n$, $\gamma \geq 0$, and an open set $G \subset \mathbb{R}^n$, by means of $b^s_{Y,q,a}(G)$, we designate the (quasi) semi-normed space of the measurable functions $f \in L_{a_i,loc}(In_i(y), dx_i)$ for a.e. $y \in (I - Pr_i)(G)$ with the finite (quasi) semi-norm*

$$\|f|b^s_{Y,q,a}(G)\|^{\varsigma} := \sum_{i=1}^{n} \left(\int_0^{\infty} \left\| \delta^{m_i}_{i,a_i}(t, \cdot, f, G_{rt})_{\gamma} | Y(G) \right\|^q t^{-qs} \, \frac{dt}{t} \right)^{\varsigma/q} . \tag{1}$$

By means of $\overset{\circ}{b}{}^s_{Y,\infty,a}(G)$*, we designate the subspace of* $b^s_{Y,\infty,a}(G)$ *of the functions satisfying the condition* $\lim_{t\to 0}\left\|\delta^{m_i}_{i,a_i}(t,f,G_{rt})_\gamma|Y(G)\right\| t^{-s}=0$*;*

$$B^s_{Y,q,a}(G) := b^s_{Y,q,a}(G)\cap Y(G) \text{ and } \overset{\circ}{B}{}^s_{Y,\infty,a}(G) = \overset{\circ}{b}{}^s_{Y,\infty,a}(G)\cap Y(G).$$

We designate the completions of these spaces by means of the same symbols whenever they appear not to be complete.

Note that $b^s_{p,q,a}(G) := b^s_{L_p,q,a}(G)$, and, for $\gamma=0$, one has $B^s_{p,q,1}(G)=B^s_{p,q}(G)$ [20].

Definition 2.6. *For an ideal space* $Y=Y(G)$*,* $p\in(0,\infty]^n$*,* $q\in(0,\infty]$*,* $r>0$*,* $s\in(0,\infty)$*,* $s/\gamma_a<m\in\mathbb{N}^n_0$*,* $a\in(0,\infty]^n$*,* $\gamma\in\mathbb{R}$*, and an open set* $G\subset\mathbb{R}^n$*, by means of* $l^s_{p,q,a}(G)$*, we designate the (quasi) semi-normed space of the measurable functions* $f\in L_{a_i,loc}(In_i(y),dx_i)$ *for a.e.* $y\in(I-Pr_i)(G)$ *with the finite (quasi) semi-norm*

$$\|f|l^s_{Y,q,a}(G)\|^\varsigma := \sum_{i=1}^n\left\|\left(\int_0^\infty\left(\delta^{m_i}_{i,a_i}(t,\cdot,G_{rt},f)_\gamma\right)^q t^{-qs}\,\frac{dt}{t}\right)^{1/q}\middle|Y(G)\right\|^\varsigma. \tag{2}$$

By means of $\overset{\circ}{l}{}^s_{Y,\infty,a}(G)$*, we designate the subspace of* $l^s_{Y,\infty,a}(G)$ *of the functions satisfying the condition*

$$\lim_{\tau\to 0}\left\|\sup_{t\in(0,\tau)}\delta^{m_i}_{i,a_i}(t,\cdot,f,G_{rt})_\gamma t^{-s}\middle|Y(G)\right\|=0,$$

or, that is equivalent for $Y=L_p$ *with* $p\in(0,\infty)^n$ *due to the Lebesgue and Levi theorems, satisfying* $\lim\limits_{t\to 0}\delta^{m_i}_{i,a_i}(t,x,f,G_{rt})_\gamma t^{-s}=0$ *for a.e.* x*;*

$$L^s_{Y,q,a}(G) := l^s_{Y,q,a}(G)\cap Y(G) \text{ and } \overset{\circ}{L}{}^s_{Y,\infty,a}(G) = \overset{\circ}{l}{}^s_{Y,\infty,a}(G)\cap Y(G).$$

We designate the completions of these spaces by means of the same symbols whenever they appear not to be complete.

Note that $l^s_{p,q,a}(G) := l^s_{L_p,q,a}(G)$, and, for $\gamma=0$, one has $L^s_{p,q}(G)=L^s_{p,q,1}(G)$ [20].

Let us define the anisotropic local approximation spaces of Nikol'skii-Besov and Lizorkin-Triebel type in terms of the $\mathcal{D}$-functional as follows.

Definition 2.7. *For* $p\in(0,\infty]^n, q\in(0,\infty]$*,* $a\in(0,\infty]^n$*,* $s\in(0,\infty)$*,* $D=\hat{D}\subset\mathbb{N}^n_0$*,* $|D|<\infty$ *and an ideal space* $Y=Y(G)$ *by means of* $\widetilde{b}^{s,D}_{Y,q,a}(G)$ *and* $\widetilde{l}^{s,D}_{Y,q,a}(G)$*, we designate, correspondingly, the anisotropic (quasi) semi-normed space of the functions* $f\in L_{a,loc}(G)$ *with the finite (quasi) semi-norm*

$$\|f|\widetilde{b}^{s,D}_{Y,q,a}(G)\| := \left(\int_0^\infty\|t^{-s}\mathcal{D}_a(t,\cdot,f,G,D)|Y(G)\|^q\frac{dt}{t}\right)^{\frac{1}{q}},\ \widetilde{b}^{s,D}_{p,q,u}(G) := \widetilde{b}^{s,D}_{L_p,q,u}(G),\ \textit{and}$$

$$\|f|\widetilde{l}^{s,D}_{Y,q,a}(G)\| := \left\|\left(\int_0^\infty(t^{-s}\mathcal{D}_a(t,\cdot,f,G,D))^q\frac{dt}{t}\right)^{\frac{1}{q}}\middle|Y(G)\right\|,\ \widetilde{l}^{s,D}_{p,q,u}(G) := \widetilde{l}^{s,D}_{L_p,q,u}(G).$$

Assume also that

$$\widetilde{B}^{s,D}_{Y,q,a}(G) = \widetilde{b}^{s,D}_{Y,q,a}(G) \cap Y(G) \text{ and } \widetilde{L}^{s,D}_{Y,q,a}(G) = \widetilde{l}^{s,D}_{Y,q,a}(G) \cap Y(G).$$

By means of $\overset{\circ}{\widetilde{b}}{}^{s,D}_{Y,\infty,a}(G)$, *we designate the subspace of* $\widetilde{b}^{s,D}_{Y,\infty,a}(G)$ *of the functions* f *satisfying the condition*

$$\lim_{t\to 0} \|\mathcal{D}_a(t,\cdot,f,G,D)|Y(G)\|\, t^{-s} = 0.$$

By means of $\overset{\circ}{\widetilde{l}}{}^{s,D}_{Y,\infty,a}(G)$, *we designate the subspace of* $\widetilde{l}^{s,D}_{Y,\infty,a}(G)$ *of the functions* f *satisfying the condition*

$$\lim_{\tau\to 0} \left\| \sup_{t\in(0,\tau)} \mathcal{D}_a(t,\cdot,f,G,D)t^{-s} \middle| Y(G) \right\| = 0,$$

or, that is equivalent for $Y = L_p$ *with* $p \in (0,\infty)^n$ *due to the Lebesgue and Levi theorems, satisfying*

$$\lim_{t\to 0} \mathcal{D}_a(t,\cdot,f,G,D)t^{-s} = 0 \text{ for a.e. } x.$$

We designate the completions of these spaces by means of the same symbols whenever they appear not to be complete.

Let us note that $BMO^{\gamma_a}(G) = \widetilde{b}^{0,A_0^*}_{\infty,\infty,1}(G)$ and $VMO^{\gamma_a}(G) = \overset{\circ}{\widetilde{b}}{}^{0,A_0^*}_{\infty,\infty,1}(G)$.

Definition 2.8. *Under the conditions of Definitions* 2.5 − 2.7, *we say that* $b^s_{Y,q,a}(G)$ $(B^s_{Y,q,a}(G))$, $l^s_{Y,q,a}(G)$ $(L^s_{Y,q,a}(G))$, $\widetilde{b}^{s,D}_{Y,q,a}(G)$ $(\widetilde{B}^{s,D}_{Y,q,a}(G))$, *or* $\widetilde{l}^{s,D}_{Y,q,a}(G)$ $(\widetilde{L}^{s,D}_{Y,q,a}(G))$ *is compatible with its underlying space* $Y(G)$ *if, for some* $C > 0$ *and every* $t > 0$, *one has, respectively,*

$$\left(\int_t^\infty \|\tau^{-s}\delta^{m_i}_{i,a_i}(\tau,\cdot,G_{r\tau},f)_\gamma|Y(G)\|^q d\tau/\tau\right)^{1/q} \le Ct^{-s}\|f|Y(G)\|,$$

$$\left\|\left(\int_t^\infty (\tau^{-s}\delta^{m_i}_{i,a_i}(\tau,\cdot,G_{r\tau},f)_\gamma)^q d\tau/\tau\right)^{1/q}\middle| Y(G)\right\| \le Ct^{-s}\|f|Y(G)\|,$$

$$\left(\int_t^\infty \|\tau^{-s}\mathcal{D}_a(\tau,\cdot,f,G,D)|Y(G)\|^q d\tau/\tau\right)^{1/q} \le Ct^{-s}\|f|Y(G)\|, \text{ or}$$

$$\left\|\left(\int_t^\infty (\tau^{-s}\mathcal{D}_a(\tau,\cdot,f,G,D))^q d\tau/\tau\right)^{1/q}\middle| Y(G)\right\| \le Ct^{-s}\|f|Y(G)\|.$$

Generalized Minkowski inequality implies that we, particularly, have the compatibility for $s > 0$ and $Y = L_p$ with $a_{\max} \le p_{\min}$ in the case of any Nikol'skii-Besov, or Lizorkin-Triebel space under the consideration.

We **assume** all the non-homogeneous spaces under consideration to be **compatible** with their underlying ideal spaces.

Remark 2.3. *a*) It was shown by O.V. Besov that, when an open $G \subset \mathbb{R}^n$ is a domain satisfying the flexible λ-horn condition (a class that is strictly larger than the class of extension domains) and $s > 0$, $B^s_{p,q,1}(G) = B^s_{p,q}(G)$ [20], where the latter space is the original Nikol'skii-Besov space endowed with the norm where the ordinary differences (or L_p-moduli of continuity) are replacing the averaged differences. The counterpart of this result for $L^s_{p,q}(G)$ was established in [2] for $s/\lambda_a > \overline{1} = (1, \dots, 1)$.

b) Let us note that in the presence of the compatibility we obtain equivalent norms in the nonhomogeneous Nikol'skii-Besov and Lizorkin-Triebel spaces defined above by substituting the integration $\int_0^\infty$ in their seminorms with the integration $\int_0^h$ for any fixed $h > 0$.

To incorporate the definitions of function spaces in terms of the entire functions of exponential type due to S.M. Nikol'skii [48] and in terms of more general smooth Littlewood-Paley decompositions studied by P.I. Lizorkin and H. Triebel (including Lizorkin-Triebel spaces $F^s_{p,q}$) [61], as well as spaces defined in terms of a holomorphic functional calculus, we define a very large class of abstract Nikol'skii-Besov and Lizorkin-Triebel spaces that possesses quite a few interesting geometric properties even in its full generality, not mentioning the coincidence of the former space with the corresponding spaces in Definitions 2.2, 2.5 − 2.7 above with admissible a (see Definition 2.8 below) and, for example, $G = \mathbb{R}^n$.

Definition 2.9. *Let* $G \subset \mathbb{R}^n$, $p \in [1, \infty]^n$, $q, \varsigma \in [1, \infty]$, $s \in \mathbb{R}$, $b > 1$ *and* $\mathcal{F} = \{F_k\}_{k \in \mathbb{N}_0} \subset \mathcal{C}(L_p(G))$ *be a system of closed operators satisfying*

$$f \in L_p(G) \text{ and } F_k f = 0 \text{ for } k \in \mathbb{N}_0 \Rightarrow f = 0.$$

By means of $B^s_{p,q,\mathcal{F}}(G)$ *and* $L^s_{p,q,\mathcal{F}}(G)$, *we designate, respectively, the Banach spaces of functions defined on* G *with the finite norms*

$$\|f|B^s_{p,q,\mathcal{F}}(G)\|^\varsigma = \|f|L_p(G)\|^\varsigma + \left(\sum_{k\in\mathbb{N}_0} b^{ksq} \|F_k f|L_p(G)\|^q\right)^{\varsigma/q},$$

$$\|f|L^s_{p,q,\mathcal{F}}(G)\|^\varsigma = \|f|L_p(G)\|^\varsigma + \left\| \left(\sum_{k\in\mathbb{N}_0} b^{ksq} |F_k f(\cdot)|^q\right)^{1/q} \right| L_p(G) \right\|^\varsigma.$$

Remark 2.4. *a*) S.M. Nikol'skii [48] has defined functions spaces employing the anisotropic Fourier multipliers $F_k : f \longmapsto \phi(b^{k\lambda}\cdot) * f$, where ϕ and, thus, $F_k f$ are functions of exponential type. These spaces appeared to be global approximation spaces by the entire functions of exponential type, or the meeting place of the theorems of S. Bernstein and Jackson type.

b) The Nikol'skii-Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$ (see [61]) are defined in terms of various more general smooth Littlewood-Paley decompositions (Fourier multipliers), than those with compact Fourier transforms, making them (together with their restrictions $B^s_{p,q}(G)$ and $F^s_{p,q}(G)$ on G) some of the most popular particular cases of $B^s_{p,q,\mathcal{F}}(G)$ and $L^s_{p,q,\mathcal{F}}(G)$ and their subspaces. This setting includes the case of the spiral/parabolic anisotropy due to Caldéron and Torchinski [25]

Unfortunately, this remarkably well-developed framework does not cover, for example, Nikol'skii-Besov and Lizorkin-Triebel spaces defined on an non-extension domain, requiring work with classical spaces endowed with intrinsic norms [20, 17].

c) Under the conditions of the last definition, let also $\Omega \subset \mathbb{C}$ be open with $b^{-k}\Omega \subset \Omega$ for $k \in \mathbb{N}_0$, $g \in H_\infty(\Omega)$ (bounded holomorphic function on Ω), and let $A \in \mathcal{C}(L_p(G))$ admit the bounded $H_\infty(\Omega)$ functional calculus

$$H_\infty(\Omega) \ni h \mapsto h(A) \in \mathcal{L}(L_p(G)) \text{ with } \|f(A)|\mathcal{L}(L_p(G))\| \leq \|f|H_\infty(\Omega)\|.$$

Assuming that $F_k = g(b^{-k}A)$, we obtain the Nikol'skii-Besov and Lizorkin-Triebel spaces $B^s_{p,q,\mathcal{F}}(G)$ and $L^s_{p,q,\mathcal{F}}(G)$ defined in terms of the bounded H_∞-calculus.

Definition 2.10. *We say that the parameter a is* **admissible** *or* **in the admissible range** *for $Y \in \Gamma_1$ if either*

$$Y \in \left\{B^s_{p,q,a}(G),\ L^s_{p,q,a}(G),\ b^s_{p,q,a}(G),\ l^s_{p,q,a}(G)\right\}$$

and $s > (\gamma_a(1/p - 1/a))_{\max}$, or

$$Y \in \left\{\tilde{B}^{s,A}_{p,q,a}(G),\ \tilde{L}^{s,A}_{p,q,a}(G),\ \tilde{b}^{s,A}_{p,q,a}(G),\ \tilde{l}^{s,A}_{p,q,a}(G)\right\}$$

and $s > (\gamma_a, 1/p - 1/a)$, or

$$Y \in \left\{B^s_{p',q',a'}(G)^*,\ L^s_{p',q',a'}(G)^*,\ b^s_{p',q',a'}(G)^*,\ l^s_{p',q',a'}(G)^*\right\}$$

and $s > -(\gamma_a(1/p - 1/a))_{\max}$, or

$$Y \in \left\{\tilde{B}^{s,A}_{p',q',a'}(G)^*,\ \tilde{L}^{s,A}_{p',q',a'}(G)^*,\ \tilde{b}^{s,A}_{p',q',a'}(G)^*,\ \tilde{l}^{s,A}_{p',q',a'}(G)^*\right\}$$

and $s > -(\gamma_a, 1/p - 1/a)$.

The C-flexible λ-horn condition was introduced by O.V. Besov [18] to solve the problem of complex and real interpolation of Nikol'skii-Besov and Lizorkin-Triebel spaces defined on an irregular domain.

Definition 2.11. ([18]) *Let $\lambda = \gamma_a$ (see the beginning of section 2). A domain $G \subset \mathbb{R}^n$ satisfies the C-flexible λ–horn condition if, for some $\delta_0 \in (0,1]$ and $T \in (0,\infty)$, and for every $x \in G$, there exists a path*

$$\rho(t^\lambda) = \big(\rho_1(t^{\lambda_1}, x), \ldots, \rho_n(t^{\lambda_n}, x)\big) = \rho(t^\lambda, x),\ t \in [0,T]$$

with the properties:

a) *$\rho_i(u,x)$ is continuous in x on G, absolutely continuous in t on $[0, T^{\lambda_i}]$ for every $i \in I_n$, $|\rho'_{it}(x,t)| \leq 1$ for $x \in G$ and a.e. $t \in [0, T^{\lambda_i}]$, and $\rho(0,x) = 0$.*

b) *For $V(\lambda, x, \delta_0) = \bigcup_{t\in(0,T]} [\rho(t^\lambda) + t^\lambda \delta_0^\lambda Q_0]$, one has $x + V(\lambda, x, \delta_0) \subset G$.*

Remark 2.5. *a*) The admissibility of a for Y, when it is not a dual space, is closely related to the cases when we do not need to take the completion in Definitions 2.5 − 2.7.

b) The results presented in this article, excluding some related to sharpness, hold for the classes of function spaces of Nikol'skii-Besov and Lizorkin-Triebel type (and there duals, subspaces, factor-spaces and finitely represented spaces) with variable smoothness (including the weighted spaces) which are defined by substituting the power t^s in the definitions of the corresponding spaces in with a more general function $\omega(t,x)$ (see [19]).

c) The same is true for the spaces with dominating derivatives and l_p-sums of the spaces under consideration.

d) The classical Nikol'skii-Besov and Lizorkin-Triebel spaces defined in terms of the differences (used instead of the a-averaged differences in the definition of $B^s_{p,q,a}(G)$ and $L^s_{p,q,a}(G)$) have the same properties (dealt with in this paper) as $B^s_{p,q}(\mathbb{R}^n)_w$ and $L^s_{p,q}(\mathbb{R}^n)_w$ on almost all occasions when G satisfies the C-flexible λ-horn condition (see Definition 2.11).

e) The Besov $B^s_{p,q}(\mathbb{R}^n)$ (note that the original Nikol'skii-Besov spaces $B^s_{p,q}(G)$ defined in terms of differences are denoted by the same symbol) and Lizorkin – Triebel $F^s_{p,q}(\mathbb{R}^n)$ spaces (see [48, 61]) defined in terms of Littlewood-Paley decomposition are particular cases of $B^s_{p,q,\mathcal{F}}(G)$ and $L^s_{p,q,\mathcal{F}}(G)$ and their quotients (because, for example, $F^s_{p,q}(G)$ is the image of the restriction operator with quotient norm). From the point of view of the results presented here, they have the same properties (including sharpness) as the spaces discussed in *d*).

2.2 Function spaces as subspaces of auxiliary spaces

In this subsection we show that the second (if not the first) major idea standing behind the introduction of the Sobolev spaces in [58], besides the generalised functions and generalised derivatives, still makes a lot of sense for various Nikol'skii-Besov and Lizorkin-Triebel spaces.

It is possible to classify all the function spaces of Nikol'skii-Besov, Lizorkin-Triebel and Sobolev types defined above into two categories: homogeneous (semi-normed) spaces and normed spaces.

Given a linear topological space W, a (quasi) Banach space Y and an injective linear operator $A:\ W \supset D(A) \to Y$, let $z_{A,Y}$ be the completion of the linear space $\{x \in W:\ Ax \in Y\}$ endowed with the (quasi) norm $\|x|z_{A,Y}\| := \|Ax\|_Y$. If $\operatorname{Ker} A \neq \{0\}$ is closed, we use $W/\operatorname{Ker} A$ instead of W. We note that, according to this definition, $z_{A,Y}$ is isometric to the closure $\overline{\operatorname{Im} A}^Y$ of the image of A in Y.

Given $\varsigma \in (0,\infty]$, (quasi) Banach spaces X and Y and a closed linear operator $A:\ X \supset D(A) \to Y$, let $Z_{A,X,Y} = Z_{A,X,Y,\varsigma}$ be the (quasi) Banach linear space $D_X(A) = D(A) \cap X$ endowed with the (quasi) norm

$$\|x|Z_{A,X,Y}\|^\varsigma := \|x\|^\varsigma_X + \|Ax\|^\varsigma_Y.$$

Note that the completeness of $Z_{A,X,Y}$ is equivalent to the closedness of A.

In what follows we shall often deduce some properties of the spaces $z_{A,Y}$ and $Z_{A,X,Y}$ from the corresponding properties of the spaces Y and the l_ς-sum $l_\varsigma(I_2,\{X,Y\})$ respectively.

The following auxiliary operators introduced in [3] are the particular choices of A corresponding to the spaces

$$B^s_{p,q,a}(G), \tilde{B}^{s,A}_{p,q,a}(G), L^s_{p,q,a}(G), \tilde{L}^{s,A}_{p,q,a}(G), b^s_{p,q,a}(G), \tilde{b}^{s,A}_{p,q,a}(G),$$

$$l^s_{p,q,a}(G), \tilde{l}^{s,A}_{p,q,a}(G), B^s_{p,q,\mathcal{F}}(G), L^s_{p,q,\mathcal{F}}(G).$$

Definition 2.12. *For $s \geq 0$, $a \in (0,\infty]^n$, $A \in \mathbb{N}_0^n$, $|A| < +\infty$, let*

$$\Upsilon_{t,z,A} = \begin{cases} t^{-s} Q_{A,1,0} \circ \sigma_t^{-1} \circ \tau_z^{-1} & \text{for} \quad Q_t(z) \subset G, \\ 0 & \text{for} \quad Q_t(z) \not\subset G, \end{cases}$$

where $Q_{A,1,0} : L_a(Q_1(0)) \to L_a(Q_1(0))/\mathcal{P}_A$ is the quotient map.

$$\Xi_{i,t,z} = t^{-s} \tau_{\gamma u}^{-1} \circ \Delta_i^{m_i}(u, G_{kt})) \circ \sigma_t^{-1} \circ \tau_z^{-1}.$$

Let us designate, by means of $\Upsilon_{\tilde{B}}$ and $\Upsilon_{\tilde{L}}$, the operators

$$\Upsilon_{\tilde{B}} : \tilde{b}^{s,A}_{p,q,a}(G) \longrightarrow L_{*q}\left(\mathbb{R}_+, L_p\left(G, L_a(Q_1(0))/\mathcal{P}_A\right)\right),$$

$$\Upsilon_{\tilde{L}} : \tilde{l}^{s,A}_{p,q,a}(G) \longrightarrow L_p\left(G, L_{*q}\left(\mathbb{R}_+, L_a(Q_1(0))/\mathcal{P}_A\right)\right),$$

where

$$\Upsilon_{\tilde{L}} = \Upsilon_{\tilde{B}} = \{\Upsilon_{t,z,A}\}_{z\in G}^{t\in\mathbb{R}_+}.$$

And by means of Ξ_B and Ξ_L, we designate the operators

$$\Xi_B : b^s_{p,q,a}(G) \longrightarrow \prod_{i=1}^n L_{*q}\left(\mathbb{R}_+, L_p\left(G, L_{a_i}([-1,1])\right)\right),$$

$$\Xi_L : l^s_{p,q,a}(G) \longrightarrow \prod_{i=1}^n L_p\left(G, L_{*q}\left(\mathbb{R}_+, L_{a_i}([-1,1])\right)\right),$$

where

$$\Xi_L = \Xi_B = \{\Xi_{i,t,z}\}_{z\in G,\ t\in\mathbb{R}_+}^{i\in I_n}.$$

In the case of $B^s_{p,q,\mathcal{F}}(G)$ and $L^s_{p,q,\mathcal{F}}(G)$, the corresponding closed operator A is defined by

$$A: \ f \mapsto \{F_k f\}_{k\in\mathbb{N}_0}.$$

2.3 Independently generated spaces

The purpose of this subsection is to introduce a wide class of auxiliary spaces containing not only almost all the auxiliary spaces related to the function spaces defined above but also the Lebesgue spaces with mixed norm and l_p-sums of various spaces that can naturally appear, for example, in initial and boundary value problems in linear and non-linear PDE.

Definition 2.13. Independently generated spaces
Let $\mathcal{S}$ be a set of ideal (quasi-Banach) spaces, such that every element $Y \in \mathcal{S}$ is either a sequence space $Y = Y(I)$ with a finite or countable I, or a space $Y = Y(\Omega)$, where (Ω, μ) is a measure space with a countably additive measure μ without atoms.

By means of the **leaf growing process (step)** *from some $Y \in \mathcal{S}$, we shall call the substitution of Y with:*
(Type A) either $Y(I, \{Y_i\}_{i\in I})$ for some $\{Y_i\}_{i\in I} \subset \mathcal{S}$ if $Y = Y(I)$, or
(Type B) $Y(\Omega, Y_0)$ for some $Y_0 \in \mathcal{S}$ if $Y = Y(\Omega)$.

Here the quasi-Banach space $Y(I, \{Y_i\}_{i\in I})$ is the linear subset of $\prod_{i\in I} Y_i$ of the elements $\{x_i\}_{i\in I}$ with the finite quasi-norm

$$\|\{x_i\}_{i\in I}|Y(I, \{Y_i\}_{i\in I})\| := \|\{\|x_i\|_{Y_i}\}_{i\in I}\|_Y .$$

Note that a type B leaf (i.e. of the form $Y = Y(\Omega)$) can grow only one leaf of its own.

We shall also refer to either $\{Y_i\}_{i\in I}$, or Y_0 as to the leaves growing from Y, which could have been a leaf itself before the tree growing process. Let us designate by means of $IG(\mathcal{S})$ the class of all spaces obtained from an element of $\mathcal{S}$ in a **finite** *number of the tree growing steps consisting of the tree growing processes for some or all of the current leaves.*

Thus, there is a one-to-one correspondence between $IG(\mathcal{S})$ and the trees of the finite depth with the vertices from $\mathcal{S}$, such that every vertex of the form $Y(I)$ has at most I branches and every vertex of the form $Y(\Omega)$ has at most one branch. The tree corresponding to a space $X \in IG(\mathcal{S})$ is designated by $T(X)$. The appearance of the space forming the root of $T(X)$ is counted as the first step of the tree-growing process. The minimal number of steps necessary to "grow" X is designated by $N_{\min}(X)$.

The set of all vertices (elements of $\mathcal{S}$) of the tree corresponding to some $X \in IG(\mathcal{S})$ will be denoted by means of $\mathcal{V}(X)$.

We shall always assume that the generating set $\mathcal{S}$ of $IG(\mathcal{S})$ is minimal in the sense that there does not exist a proper subset $\mathcal{Q} \subset \mathcal{S}$, such that $\mathcal{S} \subset IG(\mathcal{Q})$.

If the set $\mathcal{S}$ includes only the spaces described by (different) numbers of parameters from $[1, \infty]$ and $X \in IG(\mathcal{S})$, we assume that $I(X)$ is the set of all the parameters of the spaces at the vertices of the tree T corresponding to X and

$$p_{\min}(X) := \inf I(X) \text{ and } p_{\max}(X) := \sup I(X).$$

For the sake of brevity, we also assume that

$$IG := \{X \in IG(l_p, L_p, lt_{p,q}, lt^*_{p',q'}) : \ [p_{\min}(X), p_{\max}(X)] \subset (1, \infty)\},$$
$$IG_0 := \{X \in IG(l_p, L_p) : \ [p_{\min}(X), p_{\max}(X)] \subset (1, \infty)\}.$$

We say that two IG-spaces are of the same tree type if their trees are congruent and the spaces at the corresponding vertices are both either l_p-spaces, or $L_p(\Omega)$-spaces on two measure spaces (Ω, μ_0) and (Ω, μ_1) with their (non-negative) measures μ_0 and μ_1 being absolutely continuous with respect to σ-additive and σ-finite a measure μ on Ω, or $lt_{p,q}$-spaces, or $lt^*_{p,q}$-spaces.

It is convenient to think about the parameters of the spaces with the same tree type in terms of the parameter functions $p : P \to [1,\infty]$ *defined on the same parameter position set* $P = \mathcal{V}(X)$ *(here we slightly abuse the notation in the sense that the vertices that are* $lt_{r,q}$ *and* $lt^*_{r,q}$ *are multiplied to cover all their parameters, or that the value of* p *on them is multi-dimensional with the vector operations as in the beginning of this section) determined by* T *and the spaces at its vertices.*

For an IG-space X *and some parameter functions* $p : P \to [1,\infty]$ *defined on the same parameter position set* $P = \mathcal{V}(X)$ *as the function* p_X *describing* X*, we shall denote by* X_p *the IG-space with the same tree* $T(X)$ *and the parameter function* p.

Every IG space X *can be interpreted as a space of functions defined on some set* $\Omega = \Omega(X)$ *that is obtained by (repeated) combinations of the operations of taking sums and Cartesian products from a family of measurable spaces and index sets.*

We also assume that an abstract L_p *is an IG space with* $I(L_p) = \{p\}$.

Let us also **define** *the class* IG_+. *A space* X *belongs to* IG_+ *if it is either in IG, or obtained from a space* $X_- \in IG$ *by means of the leaf growing process, in the course of which some leaves (or just one leaf) of* X_- *have grown some new leaves from*

$$\{S_p,\ S_p^n,\ L_p(\mathcal{M},\tau)\}_{p\in(1,\infty)}^{n\in\mathbb{N}},$$

where $(\mathcal{M},\tau)$ *is a semifinite von Neumann algebra (defined below in subsection* 2.4*). The set of the parameters* p *of these "last noncommutative leaves" is included into* $I(X)$ *and they are part of the corresponding tree* $T(X)$. *Let also*

$$IG_{0+} = \{Y \in IG_+ :\ Y_- \in IG_0\}.$$

Remark 2.6. *a*) As shown [41] for $q > 1$, every Banach lattice X (as a set) can be endowed with the new addition $x \oplus y = \left(x^{1/p} + y^{1/p}\right)^p$ and multipliction by the scalars $\alpha \odot x = \alpha|\alpha|^{q-1}x$ for $x, y \in X$ and $\alpha \in \mathbb{R}$ to become the Banach lattice $X^{(q)}$ with the norm $\|x\|_{X^{(q)}} := \|x\|_X^{1/q}$ that is lattice-q-convex in the sense

$$\left\|(|x|^q \oplus |y|^q)^{1/q}\right\|_{X^{(q)}}^q \le \|x\|_{X^{(q)}}^q + \|y\|_{X^{(q)}}^q \text{ for } x, y \in X^{(q)}$$

(equivalent to $\||x| + |y|\|_X \le \|x\|_X + \|y\|_X$ for $x, y \in X$). Moreover, the identity mapping $\phi_q : X \longrightarrow X^{(q)}$, $x \longmapsto x$ behaves exactly as the Mazur mapping $M_q : L_1(\Omega) \longrightarrow L_q(\Omega)$, $f \longmapsto f|f|^{q-1}$ [15] (see Theorem 6.1 below).

b) Interpreting an IG-space X with the parameter function p as a Banach function space (and a Banach lattice), we observe that its q-convexification $X^{(q)}$ is isometric to the IG-space X_{qp} with the natural addition and multiplication by the scalars. Section 5 is dedicated to the detailed study of the Mazur mapping $M_q : X \longrightarrow X_{qp}$, $f \longmapsto f|f|^{q-1}$ and its generalisations

Remark 2.7. *a*) We shall deal with the set $\{l_p, L_p, lt_{p,q}, lt^*_{p',q'}\}$, where l_p, L_p, $lt_{p,q}$ and $lt^*_{p',q'}$ designate, respectively, the classes $\{l_p(I)\}_{p\in[1,\infty]}$, $\{L_p(\Omega)\}_{p\in[1,\infty]}$, where (Ω,μ) is a measure space with a countably additive and not purely atomic measure μ, $\{lt_{p,q}\}_{p\in[1,\infty]^n}^{n\in\mathbb{N},q\in[1,\infty]}$ and $\{lt^*_{p',q'}\}_{p\in[1,\infty]^n}^{n\in\mathbb{N},q\in[1,\infty]}$.

b) The subclass of l_p-spaces can formally be excluded from the definition of the class IG because l_p is isometric to $lt_{p,p} = lt^*_{p,p}$ but is left there for the sake of technical convenience. The subclass of $lt^*_{p,q}$-spaces is included to make IG closed with respect to passing to dual spaces.

c) The Lebesgue or sequence spaces with mixed norm and the l_p-sums of them are particular elements of $IG(l_p, L_p)$.

2.4 Noncommutative spaces

Definition 2.14. *Let $\mathcal{M}$ be a von Neumann algebra, and $\mathcal{M}_+$ be its positive part (cone). A* trace *on $\mathcal{M}$ is a map $\tau: \ \mathcal{M}_+ \to [0,\infty]$ satisfying*

a) $\tau(x+y) = \tau(x) + \tau(y)$ *for every $x, y \in \mathcal{M}_+$ (additivity)*;

b) $\tau(\lambda x) = \lambda\tau(x)$ *for every $\lambda \in [0,\infty)$ and $x \in \mathcal{M}_+$ (positive homogeneity)*;

c) $\tau(u^*u) = \tau(uu^*)$ *for every $u \in \mathcal{M}$.*

If a function $\phi: \ \mathcal{M}_+ \to [0,\infty]$ satisfies all the properties of the trace except for c), then it is called weight*. The trace τ (weight ϕ) is said to be:*
normal *if* $\sup_\alpha \tau(x_\alpha) = \tau(\sup_\alpha x_\alpha)$ *for every bounded increasing net $\{x_\alpha\} \subset \mathcal{M}_+$,*
semifinite *if for every non-zero $x \in \mathcal{M}_+$ there exists a non-zero $y \in \mathcal{M}_+$ satisfying $y \le x$ and $\tau(y) < \infty$,*
faithful *if $\tau(x) = 0$ implies $x = 0$, and*
finite *if $\tau(1) < \infty$ (without loss of generality one assumes $\tau(1) = 1$).*

A von Neumann algebra $\mathcal{M}$ is said to be semifinite (finite) if it admits a normal semifinite (finite) faithful (n. s.(f.) f.) trace. Let also $|x| = (x^*x)^{1/2}$ for $x \in \mathcal{M}$.

Every von Neumann algebra admits a normal semifinite faithful weight. The ways of defining a general noncommutative $L_p(\mathcal{M}) = L_p(\mathcal{M}, \phi)$ for a von Neumann algebra $\mathcal{M}$ with a normal semifinite faithful weight ϕ are described in [53]. We provide the definition of $L_p(\mathcal{M}, \phi)$ when ϕ is a normal semifinite faithful trace.

Definition 2.15. *Let $\mathcal{M}$ be a semifinite algebra, and $x \in \mathcal{M}_+$. The support* supp x *is the least projection in $\mathcal{M}$ satisfying $px = x$ (or, equivalently, $xp = x$). Assume also that $\mathcal{S}$ is the linear span of the set $\mathcal{S}_+$ of all $x \in \mathcal{M}_+$ with $\tau(\text{supp } x) < \infty$. For $p \in (0,\infty)$ and $x \in \mathcal{S}$, let the* $\min(p,1)$*-norm of x be defined by*

$$\|x\|_p = \tau\left(|x|^p\right)^{1/p}.$$

The corresponding noncommutative Lebesgue space $L_p(\mathcal{M}, \tau)$ is the closure of $\mathcal{S}$ with respect to $\|\cdot\|_p$; $L_\infty(\mathcal{M}, \tau)$ is $\mathcal{M}$ endowed with the operator norm.

The linear manifold $\mathcal{S}$ is a w^*-dense $*$-subalgebra of $\mathcal{M}$. A trace τ admits a continuous extension to $\mathcal{S}$ and, hence, to $L_1(\mathcal{M}, \tau)$.

Examples 2.1. [30] Let H_1, H_2 be Hilbert spaces and $p \in [1,\infty)$. The Schatten-von Neumann class $S_p = S_p(H_1, H_2)$ is the Banach space of all compact operators $A \in \mathcal{L}(H_1, H_2)$ with the finite norm

$$\|A\|_{S_p} = \|A|S_p(H_1, H_2)\| := \left(tr(A^*A)^{p/2}\right)^{1/p}.$$

The class $S_\infty(H_1, H_2)$ is the space of all compact operators with the norm inherited from $\mathcal{L}(H_1, H_2)$. The trace of a projector P, $\tau(P) = tr(P) = \dim(\operatorname{Im} P)$ corresponds to the counting measure and

$$S_p(H_1, H_2) = L_p\left(\mathcal{L}(H_1, H_2), tr\right) \text{ and } S_p = S_p(H_1, H_1) \text{ for } \dim(H_1) = \infty.$$

This trace is normal, semifinite and faithful. When $\dim(H_1) = \dim(H_2) = n < \infty$, the elements of the Schatten-von Neumann classes can be represented by matrixes with a dense subset of the invertible matrixes, and the trace is finite. In this case we designate $S_p^n = S_p(H_1, H_2)$.

The next theorem from [53] (see also [27]) implies the superreflexivity and, thus, Radon-Nykodim property of the noncommutative Lebesgue spaces. It also implies that the spaces from the class IG_+ are UMD spaces (see [12] for this and more properties of IG_+ spaces).

Theorem 2.1. ([27], Corollary 7.7 in [53]). *Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful weight ϕ and $p \in (0, \infty))$. Then $L_p(\mathcal{M}, \phi)$ is a UMD-space.*

2.5 Groups of spaces under consideration

For the sake of convenience and brevity we divide the spaces that we will consider most often into the following 6 numbered groups of spaces. Let J be a convex subset of $[1, \infty]$, and let G be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

$$\Gamma_1(J) = \Big\{ B^s_{p,q,a}(G), \tilde{B}^{s,A}_{p,q,a}(G), L^s_{p,q,a}(G), \tilde{L}^{s,A}_{p,q,a}(G),$$
$$b^s_{p,q,a}(G), \tilde{b}^{s,A}_{p,q,a}(G), l^s_{p,q,a}(G), \tilde{l}^{s,A}_{p,q,a}(G), B^s_{p',q',a'}(G)^*,$$
$$\tilde{B}^{s,A}_{p',q',a'}(G)^*, L^s_{p',q',a'}(G)^*, \tilde{L}^{s,A}_{p',q',a'}(G)^*,$$
$$b^s_{p',q',a'}(G)^*, \tilde{b}^{s,A}_{p',q',a'}(G)^*, l^s_{p',q',a'}(G)^*, \tilde{l}^{s,A}_{p',q',a'}(G)^* \Big\},$$

where $p, a \in J^n$, $q, \varsigma \in J$, $s \in (0, \infty)$ and $A \subset \mathbb{N}_0^n$ with $|A| < \infty$.

$$\Gamma_2(J) = \{W^s_p(G),\ W^s_{p'}(G)^* :\ p \in J^n,\ s \in \mathbb{N}_0^n\}.$$

$$\Gamma_3(J) = \{B^s_{p,q}(\mathbb{R}^n)_w,\ L^s_{p,q}(\mathbb{R}^n)_w,\ B^s_{p',q'}(\mathbb{R}^n)^*_w,\ L^s_{p',q'}(\mathbb{R}^n)^*_w :\ p \in J^n,\ q \in J,\ s \in (0, \infty)\}.$$

$$\Gamma_4(J) = \{B^s_{p,q,\mathcal{F}}(G), L^s_{p,q,\mathcal{F}}(G), B^s_{p',q',\mathcal{F}}(G)^*, L^s_{p',q',\mathcal{F}}(G)^* :\ p \in J^n,\ q, \varsigma \in J,\ s \in \mathbb{R}\}.$$

$$\Gamma_5(J) = \{L_p(\mathcal{M}, \tau),\ S_p,\ S^n_p :\ p \in J\},$$

where $(\mathcal{M}, \tau)$ is a von Neumannn algebra with a normal semifinite faithful trace τ.

For the sake of further convenience we also assume that $G \subset \mathbb{R}^n$ if it is not stated otherwise, and that

$$\Gamma_0 = IG_+ \text{ and } \Gamma_i := \Gamma_i\left((1, \infty)\right) \text{ for } i \in I_5.$$

Remark 2.8. *a*) Note that the idea of §2.2 and the closedness of the IG_+-class with respect to taking l_p-sums permit us to cover (in the sense of the applicability of the results presented) also the l_p-sums of the spaces from these groups and their weighted and variable smoothness counterparts, or the spaces of the same type with dominating derivatives, as well as the l_p-sums of their subspaces, quotients and other related spaces, that may be convenient to deal with for a particular problem. For example, it could be an l_p-sum of the initial part, boundary part and the right hand side of a partial differential equation (or ΨDE), or some divergence-free subspace, or a closure of a linear span and so on. Same weighted variants of the spaces could be of use for the application of fixed point theorems.
b) The spaces from all groups Γ_i are reflexive ([12]). In particular, the proof of the reflexivity of the function spaces under consideration from [2] demonstrates the presence of UMD property as well. This observation suggests the approach via the variants of the asymmetric representation of the uniform convexity and smoothness below.

2.6 Asymmetric uniform convexity and smoothness

To be able to obtain explicit estimates of the constants in the following sections, we need the detailed description of the asymmetric uniform convexity and smoothness in the following homogeneous form. The description of the background, the full statements including the sharpness, the proofs and various applications can be found in [9, 12]. More applications are in [5, 6, 7, 10] and below.

Definition 2.16. ([12]). *Let X be a Banach space and $2 \in [q,p] \subset [1,\infty]$.*

We say that the space X is (p,h_c)-uniformly convex if, for every $x, y \in X$ and $\mu = 1-\nu \in (0,1)$, we have

$$\mu\|x\|_X^p + \nu\|y\|_X^p \geq \|\mu x + \nu y\|_X^p + h_c(\mu)\mu\nu\|y-x\|_X^p.$$

We say that the space X is (q,h_s)-uniformly smooth if, for every $x, y \in X$ and $\mu = 1-\nu \in (0,1)$, we have

$$\mu\|x\|_X^q + \nu\|y\|_X^q \leq \|\mu x + \nu y\|_X^q + h_s(\mu)\mu\nu\|y-x\|_X^q.$$

Having in mind the non-improvable estimates

$$\max\left(\|\mu x + \nu y\|_X, \|x-y\|_X/2\right) \leq \max(\|x\|_X, \|y\|_X) \text{ and}$$

$$\mu\|x\|_X + \nu\|y\|_X \leq \|\mu x + \nu y\|_X + 2\mu\nu\|x-y\|_X,$$

valid for every Banach space X, we shall refer to them as to $(\infty,1)$-uniform convexity and $(1,1)$-uniform smoothness respectively for the sake of the convenience, explained in the proof of Theorem 3.15, a) in [12] (see also [9]) and the preservation of the duality.

To formulate the main theorems of this section determining the (p,h_c)-uniform convexity and (q,h_s)-uniform smoothness, we define two functions. For $s,t \in (1,\infty)$ and $\mu \in [0,1/2]$, let

$$\omega_c(\mu,s,t) = \begin{cases} (s-1)2^{2-t} & \text{for} \quad s \leq 2, \\ \psi_s(\mu)2^{s-t} & \text{for} \quad s \geq 2; \end{cases}$$

$$\omega_s(\mu,s,t) = \begin{cases} (\psi_s(\mu))^{\frac{t-1}{s-1}}\, 2^{\frac{s-t}{s-1}} & \text{for} \quad s \le 2, \\ (s-1)^{t-1} 2^{2-t} & \text{for} \quad s \ge 2, \end{cases}$$

where

$$\psi_s(\mu) = \frac{1+(z_s(\mu))^{s-1}}{(1+z_s(\mu))^{s-1}} \text{ and}$$

$$z_s(\mu) = \begin{cases} \text{positive root of } \nu z^{s-1} - \mu = (\nu z - \mu)^{s-1} & \text{for} \quad s \ne 2 \text{ and } \mu \ne 0, \\ \text{positive root of } (s-2)z^{s-1} + (s-1)z^{s-2} = 1 & \text{for} \quad s \ne 2 \text{ and } \mu = 0, \\ 1 & \text{for} \quad s = 2. \end{cases}$$

Theorem 2.2. ([9, 12]). *Let $1-\nu=\mu\in[0,1]$, and let X be either a subspace or a quotient of a space $Y \in IG$, or finitely represented in Y with $[p_{\min}(Y), p_{\max}(Y)] \cup \{2\} \subset [r,q] \subset (1,\infty)$. Then, for $f, g \in X$, we have*

a) $\|\mu f + \nu g\|_X^q + \mu\nu\omega_c(\min(\mu,\nu), p_{\min}(Y), q)\|f-g\|_X^q \le \mu\|f\|_X^q + \nu\|g\|_X^q$;

b) $\mu\|f\|_X^r + \nu\|g\|_X^r \le \|\mu f + \nu g\|_X^r + \mu\nu\omega_s(\min(\mu,\nu), p_{\max}(Y), r)\|f-g\|_X^r$.

Remark 2.9. *a*) The finite representability of $l_{p_{\min}(Y)}$ and/or $l_{p_{\max}(Y)}$ in $Y \in IG$ is relatively easy to check. The “worst" cases may look like $l_{p_0}(\mathbb{N}, \{l_{p_i}(I_{n_i})\}_{i\in\mathbb{N}})$ for some unbounded $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ with $\inf_{i\in\mathbb{N}_0} p_i \notin \{p_i\}_{i\in\mathbb{N}_0}$ and/or $\sup_{i\in\mathbb{N}_0} p_i \notin \{p_i\}_{i\in\mathbb{N}_0}$.
b) The sharpness of the range for r_s is important, for example, for estimating the atomic Lyapunov and Kadets constants in Corollary 5.4 and Theorem 5.17 in [12] and for the variety of results in section 6 and section 7 in [12] (see also [9]).

Before presenting the next theorem describing the (q, h_c)-uniform convexity and (r, h_s)-uniform smoothness properties of noncommutative Lebesgue spaces (ex. Schatten-von Neumann classes), we recall that, for $s, t \in [1,\infty]$, one has

$$\omega_c(1/2, s, t) = (\min(s,2)-1)\, 2^{2-t};$$

$$\omega_s(1/2, s, t) = (\max(s,2)-1)^{t-1}\, 2^{2-t}.$$

The midpoint case $\mu=\nu=1/2$ of the next theorem was considered by Dixmier [31] in 1953, Simon [56] in 1987 and Ball, Carlen and Lieb [14] in 1994. Namely (for $\mu = 1/2$), the cases of both Part *a*) with $p=q\in[2,\infty]$ and Part *b*) with $p=r\in[1,2]$ were treated in [31, 56], while the cases of both Part *a*) with $p=r\in[1,2]$ and Part *b*) with $p=q\in[2,\infty]$ were considered in [14].

Theorem 2.3. ([9, 12]). *Let $1-\nu=\mu\in[0,1]$, $r,q\in[1,\infty]$ and $p\in(1,\infty)$, and let X be either a subspace or a quotient of $Y \in \{L_p(\mathcal{M},\tau), S_p\}$, or finitely represented in Y, where $(\mathcal{M},\tau)$ is a von Neumannn algebra with a normal semifinite faithful weight τ. Assume also that $\{p,2\}\subset[r,q]$. Then, for $f,g\in X$, we have*

a) $\|\mu f + \nu g\|_X^q + \mu\nu\omega_c(1/2, p, q)\|f-g\|_X^q \le \mu\|f\|_X^q + \nu\|g\|_X^q$;

b) $\mu\|f\|_X^r + \nu\|g\|_X^r \le \|\mu f + \nu g\|_X^r + \mu\nu\omega_s(1/2, p, r)\|f-g\|_X^r$.

Theorem 2.4. ([9, 12]). *Let $1-\nu=\mu\in[0,1]$, and let X be either a subspace or a quotient of a space $Y\in IG_+$, or finitely represented in Y with $[p_{\min}(Y),p_{\max}(Y)]\cup\{2\}\subset[r,q]\subset(1,\infty)$. Then, for $f,g\in X$, we have*

a) $\|\mu f+\nu g\|_X^q+\mu\nu\omega_c(1/2,p_{\min}(Y),q)\|f-g\|_X^q\le\mu\|f\|_X^q+\nu\|g\|_X^q$;

b) $\mu\|f\|_X^r+\nu\|g\|_X^r\le\|\mu f+\nu g\|_X^r+\mu\nu\omega_s(1/2,p_{\max}(Y),r)\|f-g\|_X^r$.

Remark 2.10. To illustrate the reason for the sharpness of some constants in Theorems 2.2 − 2.7, let us notice, in particular, that the space $l_p(A)$ with $A=\prod_{i=1}^n A_i$ and $p\in(1,\infty)^n$ for $\cup_{i=1}^n A_i\subset\mathbb{N}$, we necessarily have $|A_i|>1$. Otherwise, it would be just the space $l_{\tilde p}\left(\prod_{j\in I_n}^{j\neq i}A_i\right)$ with the corresponding $\tilde p\in(1,\infty)^{n-1}$.

Theorem 2.5. ([9, 12]). *Let $G\subset\mathbb{R}^n$, $p,a\in(1,\infty)^n$, $q,\varsigma\in(1,\infty)$, $s\in(0,\infty)$ and*

$$[\min(p_{\min},q,a_{\min},2),\max(p_{\max},q,a_{\max},2)]\subset[r_s,r_c]\subset(1,\infty).$$

Assume also that

$$Y\in\Big\{B^s_{p,q,a}(G),\tilde B^{s,A}_{p,q,a}(G),L^s_{p,q,a}(G),\tilde L^{s,A}_{p,q,a}(G),$$

$$b^s_{p,q,a}(G),\tilde b^{s,A}_{p,q,a}(G),l^s_{p,q,a}(G),\tilde l^{s,A}_{p,q,a}(G),B^s_{p',q',a'}(G)^*,$$

$$\tilde B^{s,A}_{p',q',a'}(G)^*,L^s_{p',q',a'}(G)^*,\tilde L^{s,A}_{p',q',a'}(G)^*,b^s_{p',q',a'}(G)^*,\tilde b^{s,A}_{p',q',a'}(G)^*,l^s_{p',q',a'}(G)^*,\tilde l^{s,A}_{p',q',a'}(G)^*\Big\},$$

and X is either a subspace, or a quotient, or almost isometrically finitely represented in Y. Then

a) the space X is (r_c,h_c)-uniformly convex and (r_s,h_s)-uniformly smooth with $h_c(\mu)=\omega_c(\mu,\min(p_{\min},q,a_{\min}),r_c)$ and $h_s(\mu)=\omega_s(\mu,\max(p_{\max},q,a_{\max}),r_s)$ for some $\mu\in[0,1]$.

b) If a is admissible for Y that is (β_c,h_c)-uniformly convex with $h_c(\mu)>0$ for some $\mu\in(0,1)$ and (β_s,h_s)-uniformly smooth with $h_s(\mu)$ for some $\mu\in(0,1)$, then

$$[\min(p_{\min},q,2),\max(p_{\max},q,2)]\subset[\beta_s,\beta_c].$$

Theorem 2.6. ([9, 12]). *Let $Y\in\{W^s_p(G),\ W^s_{p'}(G)^*\}$ for $G\subset\mathbb{R}^n$, $p\in(1,\infty)^n$, $\varsigma\in(1,\infty)$, $s\in\mathbb{N}_0^n$ and*

$$[\min(p_{\min},2),\max(p_{\max},2)]\subset[r_s,r_c]\subset(1,\infty).$$

Assume also that X is either a subspace, or a quotient, or almost isometrically finitely represented in Y. Then
a) the space X is (r_c,h_c)-uniformly convex and (r_s,h_s)-uniformly smooth with $h_c(\mu)=\omega_c(\mu,p_{\min},r_c)$ and $h_s(\mu)=\omega_s(\mu,p_{\max},r_s)$ for $\mu\in[0,1]$.
b) If Y is (β_c,h_c)-uniformly convex with $h_c(\mu)>0$ for some $\mu\in(0,1)$ and (β_s,h_s)-uniformly smooth with $h_s(\mu)<\infty$ for some $\mu\in(0,1)$, then

$$[\min(p_{\min},2),\max(p_{\max},2)]\subset[\beta_s,\beta_c].$$

Theorem 2.7. ([9, 12]). *Let* $Y \in \{B^s_{p,q}(\mathbb{R}^n)_w,\ L^s_{p,q}(\mathbb{R}^n)_w,\ B^s_{p',q'}(\mathbb{R}^n)^*_w,\ L^s_{p',q'}(\mathbb{R}^n)^*_w\}$ *for* $G \subset \mathbb{R}^n$, $p \in (1,\infty)^n$, $q, \varsigma \in (1,\infty)$, $s \in (0,\infty)$ *and*

$$[\min(p_{\min}, q, 2), \max(p_{\max}, q, 2)] \subset [r_s, r_c] \subset (1,\infty).$$

Assume also that X *is either a subspace, or a quotient, or almost isometrically finitely represented in* Y. *Then*
a) the space X *is* (r_c, h_c)*-uniformly convex and* (r_s, h_s)*-uniformly smooth with* $h_c(\mu) = \omega_c(\mu, \min(p_{\min}, q), r_c)$ *and* $h_s(\mu) = \omega_s(\mu, \max(p_{\max}, q), r_s)$ *for* $\mu \in [0,1]$.
b) If Y *is* (β_c, h_c)*-uniformly convex with* $h_c(\mu) > 0$ *for some* $\mu \in (0,1)$ *and* (β_s, h_s)*-uniformly smooth with* $h_s(\mu) < \infty$ *for some* $\mu \in (0,1)$, *then*

$$[\min(p_{\min}, q, 2), \max(p_{\max}, q, 2)] \subset [\beta_s, \beta_c].$$

Theorem 2.8. ([9, 12]). *Let* $Y \in \{B^s_{p,q,\mathcal{F}}(G), L^s_{p,q,\mathcal{F}}(G), B^s_{p',q',\mathcal{F}}(G)^*, L^s_{p',q',\mathcal{F}}(G)^*\}$ *for* $G \subset \mathbb{R}^n$, $p \in (1,\infty)^n$, $q, \varsigma \in (1,\infty)$, $s \in \mathbb{R}$. *Assume also that*

$$[\min(p_{\min}, q, 2), \max(p_{\max}, q, 2)] \subset [r_s, r_c] \subset (1,\infty),$$

and X *is either a subspace, or a quotient, or almost isometrically finitely represented in* Y. *Then the space* X *is* (r_c, h_c)*-uniformly convex and* (r_s, h_s)*-uniformly smooth with* $h_c(\mu) = \omega_c(\mu, \min(p_{\min}, q), r_c)$ *and* $h_s(\mu) = \omega_s(\mu, \max(p_{\max}, q), r_s)$ *for* $\mu \in [0,1]$.

Remark 2.11. *a*) For some combinations of the parameters (and parameter functions), sharper estimates for the midpoint ($\mu = 1/2$) uniform convexity and smoothness constants $\omega_s(1/2,\cdot,\cdot)$ and $\omega_c(1/2,\cdot,\cdot)$ (and, thus, also the Rademacher type and cotype constants for all the spaces under consideration) for all the groups of spaces under consideration are established in subsection 4.6 of [12] (see also [9]).
b) Let us note that $\omega_s(\mu, s, t)$ is non-decreasing in μ, and $\omega_c(\mu, s, t)$ is non-increasing in μ thanks to Theorem 4.13 from [12] (see also [9]).

3 Hölder-Lipschitz mappings: basic mappings and properties. Part I

In this section we introduce some notation and auxiliary mappings, discuss the basic properties of Hölder-Lipschitz mappings and investigate, in an explicit quantitative manner, our such new tools as the homogeneous Hölder-smooth right inverses of closed (linear) surjections and the Hölder-smooth version of the Kalton-Pełczyńki decomposition approached by means of solving the three-space problem for homogeneous Hölder homeomorphisms of Banach spaces.

3.1 Hölder-Lipschitz mappings and related parameters

Definition 3.1. *For a metric space* X, $x, y \in X$, $B \subset X$ *and a bounded* $A \subset X$, *let* $d_X(x,y)$ *be the distance between* x *and* y *and* $d_X(y, B) = \inf_{z \in B} d_X(y,z)$. *Assume also* *that the* Chebyshev radius of A relative to B *and the* asymmetric error of A relative to B *are, correspondingly,*

$$r(A,B) = r_X(A,B) = \inf_{x\in B}\sup_{y\in A} d_X(x,y) \text{ and } æ(A,B) = \sup_{y\in A}\inf_{x\in B} d_X(x,y).$$

For the sake of brevity, let also

$$r_X(A,x) = r_X(A,\{x\}) \text{ and } r_X(A) = r_X(A,X).$$

The diameter of the set A is

$$d(A) = \sup_{x,y\in A} d_X(x,y) = \sup_{x\in A} r_X(A,x).$$

Note that $r_X(\{x\},B) = d_X(x,B)$.
The next definition provides an important example of a metric space.

Definition 3.2. *Let X be a metric space and $B \subset X$. The metric space $H(B)$ is the set of all closed bounded subsets of B endowed with the Hausdorff metric*

$$d_H(F,G) = \max\left(æ(F,G), æ(G,F)\right) = \max\left(\sup_{x\in F} d_X(x,G), \sup_{y\in G} d_X(y,F)\right)$$

for $F\cup G \subset B$.
The (closed) ε-neighborhood F_ε of a subset $F\subset X$ in X is $\{x\in X:\ d_X(x,F)\le\varepsilon\}$.

Note that, if A and B are subsets of a normed space X and $r>0$, then

$$d_X(A_r) = d_X(A) + 2r \text{ and } r_X(A_r,B) = r_X(A,B) + r.$$

Definition 3.3. *Assume that X and Y are metric spaces and $\alpha\in(0,1]$.*
For $f:\ X\to Y$, its (first order) modulus of continuity on a subset $A\subset X$ is defined by

$$\omega(t,f,A) = \sup\left\{d_Y\left(f(x),f(y)\right):\ x,y\in A,\ d_X(x,y)<t\right\} \text{ for } t>0;$$
$$\omega(t,f) = \omega(t,f,X).$$

The mapping f is uniformly continuous on A if

$$\omega(t_0,f,A)<\infty \text{ for some } t_0>0 \text{ and } \lim_{t\to 0}\omega(t,f,A)=0.$$

By means of $H^\alpha(X,Y)$, we designate the family of all mappings $f:\ X\to Y$ satisfying:

$$\|f|H^\alpha(X,Y)\| := \sup\left\{d_Y\left(f(x),f(y)\right)/d_X(x,y)^\alpha : x,y\in X,\ x\ne y\right\}$$

$$= \sup_{t>0}\frac{\omega(t,f,X)}{t^\alpha} < \infty.$$

Note that $H^\alpha(X,Y)$ is a seminormed space if Y is a (complete) linear metric space, and that $f\in H^\alpha(X,Y)$ is a Hölder (Lipschitz for $\alpha=1$) mapping.

Remark 3.1. If X is a convex subset of a normed space endowed with the inherited metric, Y is a metric space, and $f:\ X \to Y$ with a finite $\omega(t_0, f, X)$ for some $t_0 > 0$, then f is *Lipschitz for large distances*: for every $d > 0$,

$$\omega(t, f) \le 2\omega(d, f)t/d \text{ for } t \ge d.$$

Therefore, we will primarily be interested in the dependence on $d \in (0, \infty)$ of the constant c_α and the smoothness parameter α in

$$\omega(t, f) \le c_\alpha t^\alpha \text{ for } t < d.$$

Corollary 3.1. *a) Let X, Y, Z be metric spaces and $\phi \in H^\alpha(X, Y)$, $\psi \in H^\beta(Y, Z)$. Then one has*

$$\|\psi \circ \phi | H^{\alpha\beta}(X, Z)\| \le \|\phi | H^\alpha(X, Y)\|^\beta \|\psi | H^\beta(Y, Z)\|.$$

b) If X is a bounded metric space with the diameter $d = d_X(X)$ and $\emptyset \ne [\beta, \alpha] \subset (0, 1]$, then the norm of the embedding $H^\alpha(X, Y) \hookrightarrow H^\beta(X, Y)$ is equal to $d^{\alpha-\beta}$.
c) If X and Y are normed spaces, a bounded $F \cup G \subset X$, and $A \in \mathcal{L}(X, Y)$, then

$$d_{H(Y)}(A(F), A(G)) \le \|A|\mathcal{L}(X, Y)\| d_{H(X)}(F, G) \text{ and } r(A(F), A(G)) \le \|A|\mathcal{L}(X, Y)\| r(F, G).$$

Retractions, metric projections and homogeneous inverses of linear operators between Banach spaces are important examples of Hölder mappings dealt with in Sections 3.2 and 9.

Definition 3.4. *Let X and Y be (quasi) Banach spaces and $\alpha, \beta \in (0, 1]$. We say that the unit spheres S_X and S_Y are (α, β)-Hölder homeomorphic, or that X and Y are homogeneously (α, β)-Hölder homeomorphic, and write $X \overset{(\alpha,\beta)}{\longleftrightarrow} Y$ if there exists a homeomorphism $\phi : S_X \leftrightarrow S_Y$ satisfying*

$$\phi \in H^\alpha(S_X, Y) \text{ and } \phi^{-1} \in H^\beta(S_Y, X).$$

The (positive) homogeneous extensions $\|x\|_X \phi(x/\|x\|_X)$ and $\|y\|_Y \phi^{-1}(y/\|y\|_Y)$ will be designated by ϕ and ϕ^{-1} as well. Let also

$$\alpha(X, Y) = \sup \left\{ \alpha \in (0, 1] : X \overset{(\alpha,\beta)}{\longleftrightarrow} Y \text{ for some } \beta \in (0, 1] \right\}.$$

For example, the properties of the classical Mazur mapping [46, 69] mean that $L_p \overset{\min(\frac{p}{q},1),\min(\frac{q}{p},1)}{\longleftrightarrow} L_q$.

As in the case of the uniform homeomorphisms of unit spheres (see subsection 9.1 in [15]), it is easier to check an equivalent condition.

Lemma 3.1. *Let X and Y be Banach spaces and $\alpha, \beta \in (0, 1]$. Then $X \overset{(\alpha,\beta)}{\longleftrightarrow} Y$ if, and only if, there exists a (positive) homogeneous bijective and surjective mapping $\psi : X \leftrightarrow Y$ satisfying $\psi \in H^\alpha(B_X, Y)$ and $\psi^{-1} \in H^\beta(B_Y, X)$.*

The proof of Lemma 3.1. The necessity of the condition is clear since (the homogeneous extension of) ϕ from Definition 3.4 can be taken as such ψ. The same argument that leads to the boundedness of a continuous linear operator provides the boundedness of ψ and ψ^{-1} on the spheres (i.e. $\|\psi(x)\|_X/\|x\|_X \in [c, C] \subset (0, \infty)$ for $x \in X \setminus \{0\}$). Thus, Corollary 3.1, a) and the estimate $\|\pi|H^1(Z\setminus\sigma B_Z, Z)\| \le 2/\sigma$ for $\pi : z \to z/\|z\|_Z$ and a normed Z imply the sufficiency. □

Remark 3.2. Corollary 3.1, a) provides the transitivity: if $X \overset{(\alpha_0,\beta_0)}{\longleftrightarrow} Y$ and $Y \overset{(\alpha_1,\beta_1)}{\longleftrightarrow} Z$, then $X \overset{(\alpha_0\alpha_1,\beta_0\beta_1)}{\longleftrightarrow} Z$.

3.2 Homogeneous right inverses: regularity and sharpness

In different branches of mathematics, one needs to find a solution of an equation of the form $Ax = y$ with a closed operator A from a quasi-Banach space X onto a quasi-Banach space Y, while the solution is not unique. Moreover, the solution x is better depend continuously on y and possess the norm that either minimal, or comparable to it. For example, A could consist of both the linear partial (pseudo) differential operator and the initial and/or boundary value trace operators, and every solution operator is a right inverse to A. The next theorem from [5, 12] shows that a linear or Liptschitz right inverse to A does not always exist.

Theorem 3.1. ([12, 5]). *Let X be a Banach space. Then the following properties are equivalent.*

a) The space X is isomorphic to a Hilbert space.

b) For every bounded linear operator A from X onto a Banach space Y, there exists its right inverse $B \in \mathcal{L}(Y, X)$.

c) For every closed linear operator A from $D(A) \subset X$ onto a Banach space Y, there exists its right inverse $B \in \mathcal{L}(Y, X)$.

d) The space X is reflexive, and for every bounded linear operator A from X onto a Banach space Y, there exists its right inverse B, that is Lipschitz on Y and $B0 = 0$.

e) The space X is reflexive, and for every closed linear operator A from $D(A) \subset X$ onto a Banach space Y, there exists its right inverse B, that is Lipschitz on Y and $B0 = 0$.

The last theorem relies on the deep result due to Lindenstrauss and Tzafriri [42] that the only Banach spaces possessing only complemented subspaces are those that are isomorphic to the Hilbert spaces (see also Theorem 7.1 below). It was shown by Skaletskiy [57], that, if X has a uniform normal structure, then there exists a bounded homogeneous right inverse B that is uniformly continuous on every bounded subset of Y (see also [54, 15]). Moreover, Tsar'kov [66] had proven that, whenever the kernel of a surjective $A \in \mathcal{L}(X, Y)$ is reflexive, the existence of the linear (bounded) right inverse to A is equivalent to the existence of a Lipschitz right inverse defined on a neighborhood of the origin of Y. He had also provided a counterexample showing that a surjective $A \in \mathcal{L}(X, Y)$ does not even have to possess a uniformly continuous right inverse without additional restrictions.

This subsection is devoted to the existence of the right homogeneous (but non-additive) inverses for the closed linear surjective operators from a (p, h_c)-uniformly convex (and, possibly, (q, h_s)-uniformly smooth) Banach space X onto a Banach space Y that are also Hölder-regular mappings on bounded subsets of Y.

It appears that the ordinary continuity can be characterised in therm of Vlasov's condition. Let us recall that a Banach space X is said to satisfy *Vlasov's condition* [68] (see also Theorem 9.2) if every subsequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ with $\|x_n\|_X = 1$ for every n, satisfying the condition $\lim_{k\to\infty} f(x_k) = 1$ for some $f \in X^*$ with $\|f\|_{X^*} = 1$, is convergent in X.

Theorem 3.2. ([12]). *Let X and Y be (quasi) Banach spaces, and let X be isomorphic to a Banach space Z satisfying Vlasov's condition and $d_{BM}(X, Z) \le d$. Then, for every (linear) closed surjective operator A from $D(A) \subset X$ onto Y, their exists a homogeneous right-inverse operator $B:\ Y \to X$ satisfying*

$$A \circ B = I,\ \ B\lambda x = \lambda Bx \ \text{and} \ \sup_{y\in B_Y} \|By\|_X \le d\|\tilde{A}^{-1}|\mathcal{L}(Y, \tilde{X})\|.$$

The following results are extracts from the corresponding results in [12] (their counterparts in [5] are less precise in the general setting but still lead to the same numerical estimates for the spaces under consideration). According to Theorem 6.16 from [12], the Hölder-Lipschitz regularity exponent of the homogeneous right inverse given in the next theorem and corollary are sharp for $X \in IG_+$ under the restriction that, if $p_{\min}(X) < 2$, X (Y) contains isometric 1-complemented copies of $\{l_{p_k}\}_{k\in\mathbb{N}}$ with $p_k \in I(X)$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} p_k = p_{\min}(X)$, and, if $p_{\max}(X) > 2$, X contains isometric 1-complemented copies of $\{l_{q_k}\}_{k\in\mathbb{N}}$ with $q_k \in I(X)$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} q_k = p_{\max}(X)$.

Theorem 3.3. ([5, 12]). *For $2 \in [q, p] \subset (1, \infty)$, let X and Y be quasi-Banach spaces, and let X be isomorphic to a (p, h_c)-uniformly convex and (q, h_s)-uniformly smooth Banach space Z with $d_{BM}(X, Z) < d$. Assume that A is a closed linear surjective operator from $D(A) \subset X$ onto Y, and that a bounded $F \subset Y$ and*

$$c_c = \sup_{\mu\in(0,1/2]} (1-\mu)h_c(\mu) \ \text{and} \ c_s = \inf_{\mu\in(0,1/2]} (1-\mu)^{1-q} h_c(\mu).$$

Then there exists a homogeneous right-inverse operator $B:\ Y \to X$ satisfying $A \circ B = I$,

$$B\lambda x = \lambda Bx,\ \sup_{y\in B_Y} \|By\|_X \le d\|\tilde{A}^{-1}|\mathcal{L}(Y, \tilde{X})\|,$$

$$\begin{aligned}&\|By - Bx\|_X \\ &\le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\|\left(\|y-x\|_Y + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p}\left(\|x\|_Y^q + c_s c_c^{-q/p}\|y-x\|_Y^q\right)^{1/q-1/p}\|y-x\|_Y^{q/p}\right)\end{aligned}$$

for every $x, y \in Y$ *and*

$$\|B|H^{q/p}(F,X)\| \\ \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \left(d(F)^{1-q/p} + \left(\frac{pc_s}{qc_c^{1+q/p}} \right)^{1/p} \left(r(F,\{0\})^q + c_s c_c^{-q/p} d(F)^q \right)^{1/q-1/p} \right),$$

where
$\tilde{X} = X/\operatorname{Ker} A$ *and* $\tilde{A}: \ \tilde{X} \to Y$ *is defined by the canonical factorisation* $A = \tilde{A} \circ Q_{\operatorname{Ker} A}$.

If, in addition, $p = q = 2$*, then we also have*

$$\|B|H^{q/p}(Y,X)\| \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \left(1 + \frac{c_s^{1/2}}{c_c} \right).$$

Moreover, if X *is a* (p, h_c)*-uniformly convex and a* (q, h_s)*-uniformly smooth Banach space itself, one takes* $d = 1$ *in these estimates and* $\|By\|_X = \min\{\|x\|_X : Ax = y\}$ *for* $y \in Y$*.*

Corollary 3.2. *Under the conditions of Theorem* 3.3*, one has*

$$\|B|H^{q/p}(F,X)\| \\ \le dr(F,\{0\})^{1-q/p}\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \left(2^{1-q/p} + \left(\frac{pc_s}{qc_c^{1+q/p}} \right)^{1/p} \left(1 + c_s c_c^{-q/p} 2^q \right)^{1/q-1/p} \right).$$

Using also $\|I|\mathcal{L}(l_1, l_q)\| = 1$ for $q \ge 1$ and $d(F) \le 2r(F, 0)$, we obtain the important estimates in Section 4.

3.3 Hölder-smooth Kalton-Pełczyńki decomposition and three-space problem

The three-space problem for a property A of a Banach space X is the relation between X possessing A and both its subspace Z and its quotient X/Z possessing A. In our setting, A means the existence of a homogeneous Hölder homeomorphism (or a Hölder homeomorphism extended by homogeneity) with a given Banach space Y (especially a Hilbert $Y = H$). A natural technical question is the validity of the principle of "two policemen" for A: knowing that Y is a subspace or quotient of a Banach space X possessing A, and Z possessing A is a subspace or a quotient of Y, can one conclude that Y is possessing A? To answer this question in the setting of the uniformly continuous homeomorphisms of spheres, Nigel J. Kalton [15] found an abstract approach in the style of the Pełczyńki decomposition method for complemented subspaces in the linear setting, leading to the uniform classification of spheres of wide classes of spaces (see [15]). In this subsection, we develop its counterpart in the Hölder-continuous setting.

The next two lemmas are our sharpened versions of Lemmas 9.10 and 9.9 from [15]. The first lemma solves the three-space problem for homogeneous Hölder homeomorphisms of uniformly convex and smooth Banach spaces in an explicit quantitative manner.

Lemma 3.2. *For $2 \in [q,p] \subset (1,\infty)$, let X be a quasi-Banach space isomorphic to a Banach space Y that is (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth. Assume also that $Z \subset X$ is a subspace. Then one has*

$$X \overset{(q/p,q/p)}{\longleftrightarrow} Z \oplus X/Z.$$

The proof of Lemma 3.2. Let $Q_Z : X \to X/Z$ be the quotient map. Theorem 3.3 and Corollary 3.2 provide us with its right homogeneous inverse operator $B_Z \in H^{q/p}(B_{X/Z}, X)$ and

$$\|B_Z | H^{q/p}(B_{X/Z}, X)\| \leq C = 2^{1-q/p} + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(1 + c_s c_c^{-q/p} 2^q\right)^{1/q-1/p}, \qquad (1)$$

$$c_c = \sup_{\mu \in (0,1/2]} (1-\mu) h_c(\mu) \text{ and } c_s = \inf_{\mu \in (0,1/2]} (1-\mu)^{1-q} h_s(\mu).$$

Then the mapping $\phi_Z : X \longrightarrow Z \oplus X/Z$ defined by $\phi_Z : x \longmapsto (x - B_Z(Q_Z x), Q_Z x)$ is (positive) homogeneous and

$$\phi_Z \in H^{q/p}(B_X, Z \oplus X/Z). \qquad (2)$$

Moreover, it has the inverse

$$\phi_Z^{-1} : Z \oplus X/Z \longrightarrow X : (y,z) \longmapsto y + B_Z(z) \text{ with } \phi_Z^{-1} \in H^{q/p}\left(B_{Z \oplus X/Z}, X\right). \qquad (3)$$

The combination of (2) and (3) is what is claimed in the lemma. □

The proof of Part *b*) of the next lemma is a typical application of Corollary 5.3 and Lemma 5.4 in combination with Lemma 5.2 (Hölder inequality). Its variations with different combinations of the homogenization and smoothness can be found in Lemma 5.5. The main conclusion is that the β-homogenization defines the summability and can worsen the smoothness.

Lemma 3.3. *Let X, Y, X_1, Y_1 be Banach spaces, $\alpha, \beta, \alpha_1, \beta_1 \in (0,1]$ and $p \in [1,\infty]$. Then*

a) $X \overset{(\alpha,\beta)}{\longleftrightarrow} Y$ *and* $X_1 \overset{(\alpha_1,\beta_1)}{\longleftrightarrow} Y_1$ *imply* $X \oplus X_1 \overset{(\min(\alpha,\alpha_1),\min(\beta,\beta_1))}{\longleftrightarrow} Y \oplus Y_1$;

b) $X \overset{(\alpha,\beta)}{\longleftrightarrow} Y$ *implies* $l_p(\mathbb{N}, X) \overset{(\alpha,\beta)}{\longleftrightarrow} l_p(\mathbb{N}, Y)$.

The proof of Lemma 3.3. Part *a*) is an immediate consequence of the definition. To establish Part *b*), we assume that $\phi : X \to Y$ is a homogeneous homeomorphism realizing $X \overset{(\alpha,\beta)}{\longleftrightarrow} Y$ and construct its point-wise extension $\psi = \phi \otimes I_{l_p} : \{x_i\} \longmapsto \{\phi(x_i)\}$. For $x, y \in B_{l_p(\mathbb{N},X)}$ and $i \in \mathbb{N}$, we have

$$\|\phi(y_i) - \phi(x_i)\|_Y \leq \|\phi | H^\alpha(B_X, Y)\| \left(\|x_i\|_X^{1-\alpha} + \|y_i\|_X^{1-\alpha}\right) \|y_i - x_i\|_X^\alpha. \qquad (1)$$

Hence, Lemma 5.2 and the triangle inequality for $l_{p/(1-\alpha)}(\mathbb{N}, X)$ imply

$$\|\psi(y) - \psi(x) | l_p(\mathbb{N}, Y)\| \leq 2^{1-\alpha} \|\phi | H^\alpha(B_X, Y)\| \|y_i - x_i\|_X^\alpha. \qquad (2)$$

We finish the proof by changing the roles of X and Y (and ϕ and ϕ^{-1}) and applying Lemma 3.1. □

Theorem 3.4. *For $2 \in [q,p] \subset (1,\infty)$ and a (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth Banach space X, let Y be either a subspace or a quotient of X, and let Z be either a subspace or a quotient of Y. For $\alpha,\beta,\alpha_1,\beta_1 \in (0,1]$ and a Hilbert space H, assume that*

$$X \overset{(\alpha,\beta)}{\longleftrightarrow} H \text{ and } Z \overset{(\alpha_1,\beta_1)}{\longleftrightarrow} H.$$

Then we have

$$Y \overset{(\delta,\delta)}{\longleftrightarrow} H \text{ for } \delta = \alpha\beta\alpha_1\beta_1(q/p)^4.$$

Moreover, $\delta = \alpha\beta\alpha_1\beta_1(q/p)^2$ if either Z is a complemented subspace, or a quotient with respect to a complemented subspace of Y, or Y is a complemented subspace, or a quotient with respect to a complemented subspace of X. One also has $\delta = \alpha\beta\alpha_1\beta_1$ if they both have these properties.

The proof of Theorem 3.4. Considering finite sums as l_2-sums, we have the isometries $H = H \oplus H = l_2(\mathbb{N}, H)$ and $l_2(\mathbb{N}, Y) = l_2(\mathbb{N}, Y) \oplus Y$. Thanks to Lemmas 3.2 and 3.3, they imply the following chain of Hölder homeomorphisms

$$Y \overset{(q/p,q/p)}{\longleftrightarrow} Z \oplus Z_1 \overset{(\alpha_1,\beta_1)}{\longleftrightarrow} H \oplus H \oplus Z_1 \overset{(\beta_1,\alpha_1)}{\longleftrightarrow} H \oplus Z \oplus Z_1 \overset{(q/p,q/p)}{\longleftrightarrow} H \oplus Y = l_2(\mathbb{N}, H) \oplus Y$$
$$\overset{(\beta,\alpha)}{\longleftrightarrow} l_2(\mathbb{N}, X) \oplus Y \overset{(q/p,q/p)}{\longleftrightarrow} l_2(\mathbb{N}, Y_1) \oplus l_2(\mathbb{N}, Y) \oplus Y \overset{(q/p,q/p)}{\longleftrightarrow} l_2(\mathbb{N}, X) \overset{(\alpha,\beta)}{\longleftrightarrow} l_2(\mathbb{N}, H) = H,$$

where $Z_1 = Y/Z$ if Z is a subspace of Y, or $Z = Y/Z_1$, and also $Y_1 = X/Y$ if Y is a subspace of X, or $Y = X/Y_1$. We finish the proof in the general case by employing the transitivity property in Remark 3.2. The complementability means the (linear) Lipschitz version of Lemma 3.2. $\square$

4 Homogeneous Hölder homeomorphisms: abstract approaches

4.1 Duality mapping: quantitative monotonicity and Hölder regularity

In this section, we provide an explicit quantitative description of the Hölder regularity and monotonicity of the duality mapping that is the simplest homogeneous Hölder homeomorphism available even in the setting of classes of spaces without the local unconditional structure (such as S_p and many noncommutative L_p-spaces) or nice complex interpolation properties.

Let us recall that the duality mapping $J_X : S_X \to S_{X^*}$ is correctly defined by $\langle J_X x, x\rangle = 1$ in the case of a smooth X thanks to the Hahn-Banach theorem. It has its natural inverse $J_X^{-1} = J_{X^*}$ if X is also strictly convex and reflexive.

The following lemma, describing the monotonicity of the duality mapping in a quantitative manner, is a particular case of Part a) of Lemma 4.1 in [12] (see also [9]) combined with the quantitative duality of our notions of smoothness and convexity established in Theorem 4.5 in [12] (with two different proofs).

Lemma 4.1. ([9, 12]). *For $2 \in [q,p] \subset (1,\infty)$, let X be a (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth Banach space and $x,y \in S_X$ with $f_x, f_y \in S_{X^*}$ satisfying $\langle f_x, x\rangle = \langle f_y, y\rangle = 1$. Then we have*

$$\sup_{\mu\in(0,1)} h_c(\mu)\|x-y\|^p/p \leq \langle f_x - f_y, x-y\rangle \text{ and}$$

$$\left(\inf_{\mu\in(0,1)} h_s(\mu)\right)^{1-q'} \|f_x - f_y\|^{q'}/q' \leq \langle f_x - f_y, x-y\rangle.$$

$$\|f_x - f_y\|_{X^*} \leq pp'^{p/p'} \inf_{\mu\in(0,1)} h_s(\mu)\|x-y\|_X^{p-1} \leq pe^{p/e} \inf_{\mu\in(0,1)} h_s(\mu)\|x-y\|_X^{p-1}.$$

Moreover, the corresponding relation holds if X is either (p,h_c)-uniformly convex, or (q,h_s)-uniformly smooth.

We reformulate an immediate corollary to this lemma in terms of the following explicit estimates for the Hölder noms of the duality mapping. Theorems in subsection 2.6 provide explicit expressions for the quantities in brackets in terms of the parameters of the spaces from the groups Γ_i.

Theorem 4.1. *For $2 \in [q,p] \subset (1,\infty)$, let X be a (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth Banach space and $J_X: S_X \to S_{X^*}$ the duality mapping. Then we have $X \overset{(q-1,p'-1)}{\longleftrightarrow} X^*$ with $J_X^{-1} = J_{X^*}$,*

$$\left\|J_X|H^{q-1}(S_X, S_{X^*})\right\| \leq q'^{q/q'} \inf_{\mu\in(0,1)} h_s(\mu) \text{ and}$$

$$\left\|J_X^{-1}|H^{(p-1)^{-1}}(S_{X^*}, S_X)\right\| \leq p^{p'/p}\left(\sup_{\mu\in(0,1)} h_c(\mu)\right)^{(1-p)^{-1}}.$$

One has a related uniform bound $\max\left(q'^{q/q'}, p^{p'/p}\right) < e^{2/e} < e$.

Remark 4.1. Let us note that the exponents of the Hölder regularity can be the same as with the constructive approach. For example, the duality map for the pair of Lebesgue spaces $(L_p, L_{p'})$ is just the Mazur map m_{p-1}, and either m_{p-1} or its inverse is Lipschitz.

4.2 Lozanovskii factorisation: quantitative Hölder-regular version

In this subsection, we rely on Theorem 4.1 and Lemma 4.1 from the preceding subsection to establish, in an explicit quantitative manner, the Hölder regularity of the Lozanovskii factorisation mapping introduced and studied in [43, 44] (see [15, 26, 49, 33] for more references and related applications), that had become the major tool of study in the uniform setting thanks to E. Odell and Th. Schlumprecht [49].

The following theorem contains strengthened versions of Lemma 9.5 and Corollary 9.6 from [15] where the same mapping was shown to be a uniform homeomorphism.

Theorem 4.2. *For $2 \in [q,p] \subset (1,\infty)$, let X be a (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth Banach space that is also a lattice. Then there exist Lozanovskii mappings $L_X: X \to L_1(E,\mu)$ and $L_{X^*}: X^* \to L_1(E,\mu)$ for some measure space (E,μ) that are correctly defined and realise the equivalences*

$$X \overset{(q-1,1/p)}{\longleftrightarrow} L_1(E,\mu) \overset{(1/q',p'-1)}{\longleftrightarrow} X^*.$$

Moreover, we also have the estimates

$$\left\|L_X|H^{q-1}(S_X, S_{L_1(E,\mu)})\right\| \le 2^{2-q} + q'^{q/q'} \inf_{\mu\in(0,1)} h_s(\mu),$$

$$\left\|L_X^{-1}|H^{1/p}(S_{L_1(E,\mu)}, S_X)\right\| \le (2p)^{1/p}\left(\sup_{\mu\in(0,1)} h_c(\mu)\right)^{-1/p}, \text{ and}$$

$$\left\|L_{X^*}^{-1}|H^{1/q'}(S_{L_1(E,\mu)}, S_{X^*})\right\| \le (2q')^{1/q'}\left(\inf_{\mu\in(0,1)} h_s(\mu)\right)^{1/q}.$$

One has a related uniform bound $\max\left(q'^{q/q'}, (2q')^{1/q'}, (2p)^{1/p}\right) < e^{2/e} < e$.

Remark 4.2. Let us note that the exponents of the Hölder regularity provided for L_X^{-1} are sharp for $X = L_p$ for $p \ge 2$.

The proof of Theorem 4.2. According to the representation theorem for the lattices (see [41, 15]), X and X^* are (linearly) isometric to some Banach function or sequence spaces on a measure space (Ω,μ) with $\langle h, x\rangle = \int_\Omega h(t)x(t)d\mu(t)$ for $h \in X^*$ and $x \in X$. It is known (for example, see the proof of Theorem 9.7 and subsection 9.6 in [15]) that the mapping

$$L_X: \ X \longrightarrow L_1(\Omega,\mu): \ x \longmapsto |x(t)|J_X x(t) \tag{L}$$

is the uniform homeomorphism of the unit spheres but we only need to know that it is a homeomorphism of dense subsets (such as the union of the finite-dimensional subspaces spanned by simple functions). The "onto" property for the restrictions onto the finite-dimensional subspaces follows from the continuity (particularly, the Hölder estimates) and the Brouwer theorem (the argument from [23, 15]). Thus, we only establish the estimates for the Hölder norms assuming that $\|\cdot\|_1$ is the $L_1(E,\mu)$-norm.

Let $x, y \in S_X$, $f = L_X x = |x|J_X x \in S_{L_1(E,\mu)}$, $g = L_X y = |y|J_X y \in S_{L_1(E,\mu)}$ and $\|f-g\|_1 = \varepsilon$. Assume that $G = \{t \in E: \ x(t)y(t) > 0\}$ and

$$h = \chi_G \frac{x}{|x|} \min(|f|,|g|). \tag{1}$$

Comparing $|f|+|g|$ with $|f-g|$ on G and $E \setminus G$, we see that $\|h\|_1 = 1 - \varepsilon/2$. For $\lambda > 2$, let

$$B_\lambda = \left\{t \in G: \ \frac{x(t)}{y(t)} + \frac{y(t)}{x(t)} \ge \lambda\right\} \text{ and } D_\lambda = G \setminus B_\lambda. \tag{2}$$

The lattice properties also suggest that $|x|, |y| \in S_X$ and $|J_X x|, |J_X y| \in S_{X^*}$ meaning $\langle |J_X x|, |y|\rangle \le 1$, $\langle |J_X x|, |y|\rangle \le 1$, and

$$2 \ge \langle |J_X x|, |y|\rangle + \langle |J_X y|, |x|\rangle \ge \left\|h\left(\frac{x}{y} + \frac{y}{x}\right)\right\|_1 \ge 2\|h\|_1 + (\lambda-2)\left\|\chi_{B_\lambda} h\right\|_1. \tag{3}$$

Therefore, comparing with the norm of h, we see that

$$\|\chi_{B_\lambda} h\|_1 \le \varepsilon/(\lambda-2) \text{ and } \|\chi_{D_\lambda} h\|_1 \ge 1-\varepsilon/2-\varepsilon/(\lambda-2). \tag{4}$$

Now remembering the definition of D_λ, we deduce from (4) that

$$\begin{aligned}\langle |J_X x|, |y|\rangle + \langle |J_X y|, |x|\rangle \ge \langle \chi_{D_\lambda} J_X x, y\rangle + \langle \chi_{D_\lambda} J_X y, x\rangle \ge \\ \ge \left\| \chi_{D_\lambda} h \left(\frac{x}{y}+\frac{y}{x}\right)\right\|_1 \ge 2\,\|\chi_{D_\lambda} h\|_1 \ge 2-\varepsilon\frac{\lambda}{\lambda-2}.\end{aligned} \tag{5}$$

Combined with the first inequality in (3), this gives us

$$\left|\langle \chi_{\bar D_\lambda} J_X x, y\rangle\right| + \left|\langle \chi_{\bar D_\lambda} J_X y, x\rangle\right| \le \varepsilon\frac{\lambda}{\lambda-2}, \tag{6}$$

where $\bar D_\lambda = E\setminus D_\lambda$. Now (5) and (6) naturally imply the key estimate

$$\langle J_X x, y\rangle + \langle J_X y, x\rangle \ge 2-2\varepsilon\frac{\lambda}{\lambda-2}, \tag{7}$$

where we take the limit $\lambda\to\infty$ and obtain

$$\langle J_X x - J_X y, x-y\rangle \le 2\varepsilon. \tag{8}$$

Combining (8) with Lemma 4.1, we arrive at the desirable estimates for the Hölder norms of L_X^{-1} and $L_{X^*}^{-1}$. To finish the proof, we deduce the first estimate of the theorem from Theorem 4.1, the triangle inequality and the representation

$$\begin{aligned}&L_X x - L_X y = (|x|-|y|)\,J_X x + |y|\,(J_X x - J_X y):\\ &\|L_X x - L_X y\|_1 \le \|x-y\|_X + \|J_X x - J_X y\|_{X^*} \le \left(2^{2-q} + q'^{q/q'} \inf_{\mu\in(0,1)} h_s(\mu)\right)\|x-y\|_X^{q-1}.\end{aligned}$$

□

4.3 Homogeneous Hölder homeomorphisms via complex interpolation method

Relying on the results on the Hölder continuity of homogeneous inverses from Section 3.2, we employ the complex interpolation method to construct the Hölder-smooth homeomorphisms between the spheres of couples of Banach spaces from the classes closed with respect to the complex interpolation and establish their Hölder continuity with explicit estimates and occasionally sharp exponents. As an example, we apply our results to the scale of the noncommutative $L_p(\mathcal{M},\tau)$, where $\mathcal{M}$ is asemifinite von Neumann algebra. Further applications are in Sections 5.2 and 6.2.

Let the boundary ∂S of the strip $S=\{z\in\mathbb{C} : \mathrm{Re}z\in[0,1]\}$ consist of $\partial S_j = \{z\in\mathbb{C} :\ \mathrm{Re}z = j\}$ for $j=0,1$.

Definition 4.1. ([24]). *Let $\bar{A} = (A_0, A_1)$ be a compatible pair of Banach spaces, $p \in [1, \infty)$ and $\mathrm{Re}z \in (0,1)$. The symbol $\mathcal{F}_p(z) = \mathcal{F}_p(z, \bar{A})$ denotes the completion of $\mathcal{F} = \mathcal{F}(\bar{A})$ in the l_p-sum $l_p\left(\{0,1\}, \{L_p(\partial S_j, \omega_{z,j}, A_j)\}_{j\in\{0,1\}}\right)$ of the weighted A_j-valued Lebesgue-Bochner spaces $L_p(\partial S_j, \omega_{z,j}, A_j)$ (with respect to the Lebesgue measure on the lines $\{\partial S_j\}$) for $j = 0, 1$, where the weights $\{\omega_{z,j}\}_{j=0}^1$ correspond to the (probability) harmonic measure on ∂S (see [16]):*

$$\omega_{z,j}(\tau) = \frac{e^{\pi(\mathrm{Im}z-\tau)}\sin(\pi\mathrm{Re}z)}{\sin^2(\pi\mathrm{Re}z) + \left(\cos(\pi\mathrm{Re}z) - e^{\pi(ij+\mathrm{Im}z-\tau)}\right)^2} \text{ for j=0,1.}$$

The next lemma is Proposition I.2 in [15].

Lemma 4.2. ([24, 15]). *Let (A_0, A_1) be a compatible pair, $\theta \in (0,1)$ and $p \in [1, \infty)$. Then one has*

$$\|x\|_{\bar{A}_{[\theta]}} = \inf\left\{\|f\|_{\mathcal{F}_p(\theta)} :\ f(\theta) = x\right\}.$$

The next lemma is a slight variation of Proposition I.3 in [15]. There is only one difference in the proof: one uses the strict convexity of A_j and A_j^* (thanks to V.L. Shmul'yan's duality) instead of the uniform convexity assumed in [15].

Lemma 4.3. *Let (A_0, A_1) be a compatible pair, $\theta \in (0,1)$ and $p \in [1, \infty)$. Assume also that both A_0 and A_1 are reflexive. Then, for every $x \in \bar{A}_{[\theta]}$, there exists $g_x \in \mathcal{F}_p(\theta)$ satisfying $\|x\|_{\bar{A}_{[\theta]}} = \|g_x\|_{\mathcal{F}_p(\theta)}$. Moreover, if both A_0 and A_1 are also strictly convex, the bijection $J_{p,\theta} :\ x \mapsto g_x$ is correctly defined and $\|g_x\|_{A_j} = \|x\|_{\bar{A}_{[\theta]}}$ a.e on ∂S_j for $j = 0, 1$. If, in addition, both A_0 and A_1 are smooth, then $\|g_x(z)\|_{\bar{A}_{[\mathrm{Re}z]}} = \|x\|_{\bar{A}_{[\theta]}}$ for every z with $\mathrm{Re}z \in (0,1)$.*

The following theorem is the Hölder counterpart of Theorem 9.12 in [15] due to M. Daher [28] and N.J. Kalton [15].

Theorem 4.3. *Let (A_0, A_1) be a compatible pair, $\theta, \eta \in (0,1)$, $2, r \in [q,p] \subset (1, \infty)$ and a bounded $A \subset \bar{A}_{[\theta]}$. Assume also that the l_r-sum $Y = l_r\left(\{0,1\}, \{L_r([0,1], A_j)\}_{j\in\{0,1\}}\right)$ is (p, h_c)-uniformly convex, and the mapping $m_{\theta,\eta} :\ \bar{A}_{[\theta]} \to \bar{A}_{[\eta]}$ is defined by $m_{\theta,\eta}x = g_x(\eta)$, where $g_x = J_{r,\theta}x$ with $J_{r,\theta}$ from Lemma 4.3. Then $m_{\theta,\eta}$ is a homogeneous Hölder homeomorphism of $\bar{A}_{[\theta]}$ and its unit sphere onto $\bar{A}_{[\eta]}$, its unit sphere respectively, satisfying $m_{\theta,\eta}^{-1} = m_{\eta,\theta}$,*

$$\|m_{\theta,\eta}y - m_{\theta,\eta}x\|_{\bar{A}_{[\eta]}} \le \|y - x\|_{\bar{A}_{[\theta]}}(1 + (p/c_c)^{1/p}) + (p/c_c)^{1/p}\|y - x\|_{\bar{A}_{[\theta]}}^{1/p}\|x\|_{\bar{A}_{[\theta]}}^{1/p'}$$

for every $x, y \in \bar{A}_{[\theta]}$ and

$$\|m_{\theta,\eta}|H^{1/p}(A, \bar{A}_{[\eta]})\| \le r(A,0)^{1/p'}\left(2^{1/p'} + p^{1/p}c_c^{-1/p}3^{1/p'}\right).$$

Moreover, if Y is also (q, h_s)-uniformly smooth, then we also have

$$\|m_{\theta,\eta}y - m_{\theta,\eta}x\|_{\bar{A}_{[\eta]}} \le \left(\left(\frac{pc_s^{1+p/q}}{qc_c^{2+q/p}}\right)^{1/p} + 1\right)\|y - x\|_{\bar{A}_{[\theta]}} + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p}\|y - x\|_{\bar{A}_{[\theta]}}^{q/p}\|x\|_{\bar{A}_{[\theta]}}^{1-q/p}$$

for every $x, y \in \bar{A}_{[\theta]}$, *and*

$$\|m_{\theta,\eta}|H^{q/p}(A, \bar{A}_{[\eta]})\| \le r(A,0)^{1-q/p}\left(2^{1-q/p} + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p}\left(1 + c_s c_c^{-q/p}2^q\right)^{1/q-1/p}\right).$$

The constants c_c *and* c_s *are defined by*

$$c_c = \sup_{\mu\in(0,1/2]}(1-\mu)h_c(\mu) \text{ and } c_s = \inf_{\mu\in(0,1/2]}(1-\mu)^{1-q}h_s(\mu).$$

The proof of Theorem 4.3. Note that $Z_{r,\theta} = l_r\left(\{0,1\}, \{L_r(\partial S_j, \omega_{\theta,j}, A_j)\}_{j\in\{0,1\}}\right)$ is isometric to $l_r\left(\{0,1\}, \{L_r([0,1], A_j)\}_{j\in\{0,1\}}\right)$. We provide the proof in the case of $Z_{r,\theta}$ that is both (p, h_c)-uniformly convex and (q, h_s)-uniformly smooth. When $Z_{p,\theta}$ is only (p, h_c)-uniformly convex, one uses the counterparts of Theorem 3.3 and Corollary 3.2 from [12] (Theorems 6.9 and 6.10 there) with slightly better constant c_c than in [5]. The mapping $m_{\theta,\eta}$ is the composition $R_{r,\eta}\circ I_{\theta,\eta}\circ J_{r,\theta}$, where $R_{r,\eta}: \mathcal{F}_r(\eta) \to \bar{A}_{[\eta]}, f \mapsto f(\eta)$ and $I_{\theta,\eta}: \mathcal{F}_r(\theta) \to \mathcal{F}_r(\eta), f \mapsto f$ are the restriction and identity mappings respectively. The former is linear with the norm 1 thanks to Lemma 4.2. While the latter is linear with the norm dominated by $\max_{j=0,1}\|\omega_{\eta,j}/\omega_{\theta,j}|L_\infty(\partial S_j)\|^{1/r}$, we need its restriction to the image $J_{r,\eta}(\bar{A}_{[\theta]})$ of $J_{r,\eta}$ only, where it is isometric. In turn, $J_{r,\eta}$ itself is exactly the homogeneous inverse dealt with in Theorem 3.3 and Corollary 3.2 because it provides the unique minimal norm pre-image according to Lemmas 4.3 and 4.2. The application of Corollary 3.1, a) provides the desirable estimate of the Hölder seminorm. Since $Z_{r,\theta}$ and $Z_{r,\eta}$ are isometric (just different weights), $Z_{r,\eta}$ is both (p, h_c)-uniformly convex and (q, h_s)-uniformly smooth, and the mappings $J_{r,\eta}$ and $m_{\eta,\theta}$ are well-defined and possesses the same properties as $J_{r,\theta}$ and $m_{\theta,\eta}$. To see that $m_{\eta,\theta} = m_{\theta,\eta}^{-1}$, we note that, thanks to Lemma 4.3, g_x also satisfies

$$\|g_x(m_{\theta,\eta}x)\|_{\bar{A}_{[\eta]}} = \|m_{\theta,\eta}x\|_{\bar{A}_{[\eta]}} = \|g_x\|_{\mathcal{F}_r(\eta)}$$

and minimises the $Z_{r,\eta}$-norm. The uniqueness means that $g_x = J_{r,\theta}x = J_{r,\eta}m_{\theta,\eta}x$ finishing the proof. □

As seen from the results in subsection 2.6, one or both of the parameters p and q in Theorem 4.3, can be strictly worse than the convexity and smoothness parameters of both $\bar{A}_{[\theta]}$ and $\bar{A}_{[\eta]}$ leading, for instance, to worse Hölder-Lipschitz regularity than the one provided by the Mazur mappings (see subsection 5.1) even in the setting of Lebesgue spaces. The following corollary addresses this problem.

Corollary 4.1. *Under the conditions of Theorem* 4.3, *assume that* $\{(\theta_k, \eta_k)\}_{k\in\mathbb{N}}$ *is a decreasing system of open subintervals of* $[0,1]$ *with*

$$\cap_{k\in\mathbb{N}}(\theta_k, \eta_k) = [\min(\theta,\eta), \max(\theta,\eta)].$$

Let also, for every $k \in \mathbb{N}$, *the* l_r*-sum* $Y_k = l_r\left(\{0,1\}, \{L_r([0,1], \bar{A}_{[\theta_k]}), L_r([0,1], \bar{A}_{[\eta_k]})\right)$ *be* (p_k, h_{ck})*-uniformly convex for a non-increasing* $\{p_k\}_{k\in\mathbb{N}} \subset [2,\infty)$ *with* $p_0 = \lim_{k\to\infty} p_k$

and $\lim_{k\to\infty} c_{ck} = c_{c0}$. *Then* $m_{\theta,\eta}$ *is a homogeneous Hölder homeomorphism of* $\bar{A}_{[\theta]}$ *and its unit sphere onto* $\bar{A}_{[\eta]}$, *its unit sphere respectively, satisfying* $m^{-1}_{\theta,\eta} = m_{\eta,\theta}$,

$$\|m_{\theta,\eta}y - m_{\theta,\eta}x\|_{\bar{A}_{[\eta]}} \le \left(1 + (p_0/c_{c0})^{1/p_0}\right) \|y - x\|_{\bar{A}_{[\theta]}} + (p_0/c_{c0})^{1/p_0} \|y - x\|^{1/p_0}_{\bar{A}_{[\theta]}} \|x\|^{1/p_0'}_{\bar{A}_{[\theta]}}$$

for every $x, y \in \bar{A}_{[\theta]}$, *and*

$$\|m_{\theta,\eta}|H^{1/p_0}(A, \bar{A}_{[\eta]})\| \le r(A,0)^{1/p_0'} \left(2^{1/p_0'} + p_0{}^{1/p_0} c_{c0}^{-1/p_0} 3^{1/p_0'}\right).$$

Moreover, if Y_k *is also* (q_k, h_{sk})*-uniformly smooth for a non-decreasing* $\{q_k\}_{k\in\mathbb{N}} \subset (1,2]$ *with* $q_0 = \lim_{k\to\infty} q_k$ *and* $\lim_{k\to\infty} c_{sk} = c_{s0}$, *then we have*

$$\|m_{\theta,\eta}y - m_{\theta,\eta}x\|_{\bar{A}_{[\eta]}} \le \left(\left(\frac{p_0 c_{s0}^{1+p_0/q_0}}{q_0 c_{c0}^{2+q_0/p_0}}\right)^{1/p_0} + 1\right) \|y - x\|_{\bar{A}_{[\theta]}}$$

$$+ \left(\frac{p_0 c_{s0}}{q_0 c_{c0}^{1+q_0/p_0}}\right)^{1/p_0} \|y - x\|^{q_0/p_0}_{\bar{A}_{[\theta]}} \|x\|^{1-q_0/p_0}_{\bar{A}_{[\theta]}}$$

for every $x, y \in \bar{A}_{[\theta]}$ *and*

$$\|m_{\theta,\eta}|H^{q_0/p_0}(A, \bar{A}_{[\eta]})\|$$

$$\le r(A,0)^{1-q_0/p_0} \left(2^{1-q_0/p_0} + \left(\frac{p_0 c_{s0}}{q_0 c_{c0}^{1+q_0/p_0}}\right)^{1/p_0} \left(1 + c_{s0} c_{c0}^{-q_0/p_0} 2^{q_0}\right)^{1/q_0 - 1/p_0}\right).$$

For $k \in \mathbb{N}$, *the constants* c_{ck} *and* c_{sk} *are defined by*

$$c_{ck} = \sup_{\mu\in(0,1/2]} (1-\mu) h_{ck}(\mu) \text{ and } c_{sk} = \inf_{\mu\in(0,1/2]} (1-\mu)^{1-q_k} h_{sk}(\mu).$$

The proof of Corollary 4.1. According to the reiteration theorem for complex method (see [16]), $\bar{A}_{[\theta]}$ and $\bar{A}_{[\eta]}$ are interpolation spaces for the pairs $(\bar{A}_{[\theta_k]}, \bar{A}_{[\eta_k]})$ for $k \in \mathbb{N}$. By re-scaling we see that switching from the pair (A_0, A_1) to $(\bar{A}_{[\theta_k]}, \bar{A}_{[\eta_k]})$ in the definition of $\bar{A}_{[\theta]}$ and $\bar{A}_{[\eta]}$ corresponds to switching from the construction on the strip $S = \{z \in \mathbb{C} : \ \mathrm{Re} z \in [0,1]\}$ to the same construction on the strip $S_k = \{z \in \mathbb{C} : \ \mathrm{Re} z \in [\theta_k, \eta_k]\}$. Relying on Lemma 4.3, we note that the restriction of $g_x = J_{r,\theta}x$ defined on S onto S_k satisfies the same minimisation properties as a member of $\mathcal{F}_r\left(\lambda_k, (\bar{A}_{[\theta_k]}, \bar{A}_{[\eta_k]})\right)$ (for λ_k defined by the re-scaling $\lambda_k = (\theta - \theta_k)/(\eta_k - \theta_k)$ transforming S_k onto S) and, thus, coincides with $J_{r,\lambda_k}x$ (re-scaled to S_k). This observation shows that the homeomorphism $m_{\theta,\eta} : \ \bar{A}_{[\theta]} \to \bar{A}_{[\eta]}$ does not depend on the interpolation pair that was used to construct it, or, that is equivalent, on the strip S_k used to construct the mapping. We finish the proof by taking the limit $k \to \infty$ in the estimates given in Theorem 4.3. □

The interpolation properties of the noncommutative L_p-spaces were established by V. I. Ovchinnikov [50, 51].

Theorem 4.4. ([50, 51]). *Let $\mathcal{M}$ be a semifinite von Neumann algebra with an n. s. f. trace τ, $p_0, p_1 \in [1,\infty]$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then*

$$a)\ \left(L_{p_0}(\mathcal{M},\tau), L_{p_1}(\mathcal{M},\tau)\right)_{[\theta]} = L_p(\mathcal{M},\tau)\ (isometry);$$

$$b)\ \left(L_{p_0}(\mathcal{M},\tau), L_{p_1}(\mathcal{M},\tau)\right)_{\theta,p} = L_p(\mathcal{M},\tau)\ (isomorphism).$$

Combining Corollary 4.1 with Theorems 2.4 and 4.4, we obtain the following remarkable homeomorphism.

Theorem 4.5. *Let $\mathcal{M}$ be a semifinite von Neumann algebra with an n. s. f. trace τ, a bounded $A \subset L_p(\mathcal{M},\tau)$, $p,q \in (1,\infty)$, $r_c = \max(p,q,2)$, $r_s = \min(p,q,2)$ and $\alpha = r_s/r_c$. Then there exists a homogeneous Hölder homeomorphism*

$$m_{p,q}:\ L_p(\mathcal{M},\tau) \longleftrightarrow L_q(\mathcal{M},\tau),$$

that is also a homeomorphism of the unit spheres of these spaces, satisfying $m_{p,q}^{-1} = m_{q,p}$,

$$\|m_{p,q}|H^\alpha(A, L_q(\mathcal{M},\tau))\| \le r(A,0)^{1-\alpha}\left(2^{1-\alpha} + \left(\frac{r_c c_s}{r_s c_c^{1+\alpha}}\right)^{1/r_c}\left(1 + c_s c_c^{-\alpha} 2^{r_s}\right)^{1/r_s - 1/r_c}\right),$$

$$\|m_{p,q}y - m_{p,q}x\|_q \le \left(\left(\frac{r_c c_s^{1+1/\alpha}}{r_s c_c^{2+\alpha}}\right)^{1/r_c} + 1\right)\|y-x\|_p + \left(\frac{r_c c_s}{r_s c_c^{1+\alpha}}\right)^{1/r_c}\|y-x\|_p^\alpha \|x\|_p^{1-\alpha}$$

for every $x, y \in L_p(\mathcal{M},\tau)$, where $c_c = \omega_c(1/2, r_s, r_c)$ and $c_s = \omega(1/2, r_c, r_s)$ (see subsection 2.6) and $\|\cdot\|_r = \|\cdot|L_r(\mathcal{M},\tau)\|$ for $r \in \{p,q\}$.

Remark 4.3. Note that, in the case $2 \in [p,q]$, the Hölder regularity of $m_{p,q}$ between the noncommutative L_p-spaces is the same as the regularity of the Mazur mapping between the corresponding Lebesgue spaces.

The proof of Theorem 4.5. Let us choose $A_0 = L_{p_0}(\mathcal{M},\tau)$, $A_1 = L_{p_1}(\mathcal{M},\tau)$ and $r = 2$ in Theorem 4.3 and Corollary 4.1 with $p,q \in (p_0,p_1)$. Since the class IG_+ contains the l_2-sum $Z_{p_0,p_1} = l_2\left(\{0,1\}, L_2([0,1],A_0), L_2([0,1],A_1)\right)$, we apply Theorem 2.4 to establish the $(r_c, \omega_c(1/2, r_s, r_c))$-uniform convexity and the $(r_s, \omega(1/2, r_c, r_s))$-uniform smoothness of Z_{p_0,p_1}. According to Theorem 4.4, $L_p(\mathcal{M},\tau) = \bar{A}_{[\theta]}$ and $L_q(\mathcal{M},\tau) = \bar{A}_{[\eta]}$ for some $\theta, \eta \in (0,1)$. We choose the mapping $m_{\theta,\eta}$ as our homeomorphism. We take the limits $p_0 \to \min(p,q,2)$ and $p_1 \to \max(p,q,2)$ in Corollary 4.1 finishing the proof. □

5 Homogeneous Hölder homeomorphisms: constructive approach

In this section, we develop a general homogenisation technique to construct our counterparts of the classical Mazur mapping between the compatible pairs (i.e. with the common tree) of $IG_0([1,\infty))$, IG_{0+} and IG_+ spaces. They appear to be homogeneous Hölder-Lipschitz homeomorphisms with occasionally sharp exponents of the Hölder continuity.

5.1 Simple Mazur ascent and complex Mazur descent: IG_0 setting

Let us define the simplest generalizations of the classical Mazur map [46] that we study and employ to construct more involved variants.

Definition 5.1. *For $\beta \geq 1$, a measure space Ω, an index set I, Banach spaces X and $\{X_i\}_{i\in I}$ and appropriate ideal spaces $Y_j(\Omega)$ and $Y_j(I)$ for $j \in \{0,1\}$, we define the X-valued Mazur descents $M_{\beta,X} : Y_0(\Omega, X) \to Y_1(\Omega, X)$ and $M_{\beta,\{X_i\}} : Y_0(I, \{X_i\}) \to Y_1(I, \{X_i\})$ and the homogeneous X-valued Mazur descents $m_{\beta,X} : Y_0(\Omega, X) \to Y_1(\Omega, X)$ and $m_{\beta,\{X_i\}} : Y_0(I, \{X_i\}) \to Y_1(I, \{X_i\})$ as:*

$$M_{\beta,X} : f(\tau) \longmapsto \|f(\tau)\|_X^{\beta-1} f(\tau),\ m_{\beta,X} : f(\tau) \longmapsto \left(\frac{\|f(\tau)\|_X}{\|f|Y_0(\Omega,X)\|}\right)^{\beta-1} f(\tau),$$

$$M_{\beta,\{X_i\}} : f(i) \longmapsto \|f(i)\|_{X_i}^{\beta-1} f(i) \text{ and } m_{\beta,\{X_i\}} : f(i) \longmapsto \left(\frac{\|f(i)\|_{X_i}}{\|f|Y_0(I,\{X_i\}_{i\in I})\|}\right)^{\beta-1} f(i).$$

For $X = \mathbb{R}$ and $\beta \in (0,1)$, we also define the (simple) Mazur ascent $M_{\beta,a}$ and the homogeneous (simple) Mazur ascent $m_{\beta,a}$ by means of

$$M_{\beta,a} :\ f(\tau) \longmapsto |f(\tau)|^{\beta-1} f(\tau) \text{ and } m_{\beta,a} :\ f(\tau) \longmapsto \left(\frac{|f(\tau)|}{\|f|Y_0(\Omega)\|}\right)^{\beta-1} f(\tau).$$

We also use the same notation when Ω is discrete. One also assumes that all these mappings send origin to origin.

The next lemma reveals the basic Hölder-Lipschitz regularity properties of the vector-valued Mazur descents and simple Mazur ascents.

Lemma 5.1. *For $p, q \in [1,\infty)$, $\beta = p/q > 0$, a measure space Ω, an index set I, smooth Banach spaces X and $\{X_i\}_{i\in I}$, let $M_{\beta,X}$, $m_{\beta,X}$, $M_{\beta,\{X_i\}}$ and $m_{\beta,\{X_i\}}$ for $\beta \geq 1$ and $M_{\beta,a}$ and $m_{\beta,a}$ for $\beta < 1$ be the Mazur mappings from Definition* 5.1. *Then one has*

- *a)* $\left\| M_{\beta,X} \right| H^1(A, L_q(\Omega, X)) \big\| \leq \beta r(A,0)^{\beta-1}$ *for* $A \subset L_p(\Omega, X)$;
- *b)* $\left\| m_{\beta,X} \right| H^1(L_p(\Omega, X), L_q(\Omega, X)) \big\| \leq 2\beta - 1$;
- *c)* $\left\| M_{\beta,\{X_i\}} \right| H^1(A, l_q(I, \{X_i\}_{i\in I})) \big\| \leq \beta r(A,0)^{\beta-1}$ *for* $A \subset l_p(I, \{X_i\}_{i\in I})$;
- *d)* $\left\| m_{\beta,\{X_i\}} \right| H^1(l_p(I, \{X_i\}_{i\in I}), l_q(I, \{X_i\}_{i\in I})) \big\| \leq 2\beta - 1$;
- *e)* $\left\| M_{\beta,a} \right| H^\beta(L_p(\Omega), L_q(\Omega)) \big\| \leq 2^{1-\beta}$;
- *f)* $\left\| m_{\beta,a} \right| H^\beta(A, L_q(\Omega)) \big\| \leq \left(1 + 2^{1-\beta}\right) r(A,0)^{1-\beta}$ *for* $A \subset L_p(\Omega)$.

The operators in $a) - f)$ remain bounded if X and $\{X_i\}_{i\in I}$ are not smooth.

The proof of Lemma 5.1. Let D be the Gâteaux derivative. Since the smoothness of X is equivalent to the Gâteaux differentiability of $\|\cdot\|_X$, we use the Lagrange and Lebesgue theorems to compute the Gâteaux derivatives of $M_{\beta,X}$ and $m_{\beta,X}$ at $f \in L_p(\Omega, X)$ with an increment $h \in L_p(\Omega, X)$:

$$DM_{\beta,X}(f)h(\tau) = (\beta-1)\|f(\tau)\|_X^{\beta-2}\langle g_f(\tau), h(\tau)\rangle f(\tau)+\|f(\tau)\|_X^{\beta-1}h(\tau) \text{ for a. e. } \tau \in \Omega \text{ and}$$

$$Dm_{\beta,X}(f)h(\tau) = (\beta-1)\frac{\|f(\tau)\|_X^{\beta-2}\langle g_f(\tau), h(\tau)\rangle f(\tau)}{\|f|L_p(\omega, X)\|^{\beta-1}} + \frac{\|f(\tau)\|_X^{\beta-1}h(\tau)}{\|f|L_p(\omega, X)\|^{\beta-1}}+ \\ + (1-\beta)\frac{\|f(\tau)\|_X^{\beta-1}\langle \bar{g}_f, h\rangle f(\tau)}{\|f|L_p(\omega, X)\|^{\beta}} \text{ for a. e. } \tau \in \Omega, \quad (1)$$

where $g_f(\tau) \in X^*$ is defined by $\langle g_f(\tau), f(\tau)\rangle = \|f(\tau)\|_X$ and $\bar{g}_f(\tau) = \|f(\tau)\|_X^{p-1}\|f|L_p(\omega, X)\|^{-p}g_f(\tau)$. Applying the triangle and Hölder inequalities ($1/q = \beta/p$ and $1/q - 1/p = (\beta-1)/p$) to (1), we obtain

$$\| DM_{\beta,X}(f)h|\, L_q\,(\Omega, X)\| \le \beta\, \|\, f|\, L_p\,(\Omega, X)\|^{\beta-1}\, \|\, h|\, L_p\,(\Omega, X)\| \text{ and}$$

$$\| Dm_{\beta,X}(f)h|\, L_q\,(\Omega, X)\| \le (2\beta-1)\, \|\, h|\, L_p\,(\Omega, X)\|\,, \quad (2)$$

implying Parts *a*) and *b*. In the same manner one checks the validity of *c*) and *d*).

To establish *e*), let us note that the triangle and Hölder inequalities provide the estimates

$$|M_{\beta,a}f(\tau) - M_{\beta,a}g(\tau)| \le |M_{\beta,a}(f-g)(\tau)| \text{ for } f(\tau)f(\tau) \ge 0 \text{ and} \\ |M_{\beta,a}f(\tau) - M_{\beta,a}g(\tau)| \le 2^{1-\beta}\,|M_{\beta,a}(f-g)(\tau)| \text{ for } f(\tau)g(\tau) < 0. \quad (3)$$

because of the invariance

$$\|M_{\beta,a}f|L_q(\Omega)\| = \|f|L_p(\Omega)\|^{\beta}. \quad (4)$$

The homogeneity $\|\phi(\lambda\cdot)|H^\alpha\| = \lambda^\alpha\|\phi|H^\alpha\|$, $\phi:\ Z_0 \subset A \to Z_1$ of the Hölder seminorm implies

$$\|\phi|H^\alpha(A, Z_1)\| \le \|\phi|H^\alpha(B_{Z_0}, Z_1)\|r(A, 0)^{1-\alpha}. \quad (5)$$

Assume that $\|y\|_X \le \|x\|_X = 1$ and a homogeneous $\phi_1(x) = \|x\|_X^{1-\alpha}\phi(x)$, where $\|\phi|H^\alpha(B_X, Y)\| = C$ with $\phi(B_X) \subset B_Y$. Then the triangle inequality implies, for $x, y \in B_X$,

$$\|\phi_1(x) - \phi_1(y)\|_Y \le \|\phi(x) - \phi(y)\|_Y + (1 - \|y\|_X^{1-\alpha})\|\phi(y)\|_Y \le C\|x-y\|_X^\alpha + \\ + (\|x\|_X - \|y\|_X) \le C\|x-y\|_X^\alpha + (\|x\|_X - \|y\|_X)^{1-\alpha}\,\|x-y\|_X^\alpha \le (C+1)\|x-y\|_X^\alpha.$$

The last estimates show that *e*) and *f*) hold too. To finish the proof we note that Corollary 5.3 and Lemma 5.4 below imply similar estimates with worse bounds in the case of non-smooth spaces (see the proof of Theorem 5.2). □

The preceding lemma stands behind our following definition of the complex Mazur descent and simple Mazur ascent between IG_0-spaces.

Definition 5.2. *Let $X \in IG_0 = IG \cap IG(\{L_r, l_r :\ r \in (1,\infty)\})$ with the parameter function $p_X : P = \mathcal{V}(X) \to [1,\infty)$ defined on the vertices of the corresponding tree $T(X)$. Thus, a "leaf" $Z_j \in T(X)$, grown at the kth step of the "tree-growing" process and corresponding to $j \in P$, is either $L_{p_X(j)}(\Omega)$ for a measure space Ω, or $l_{p_X(j)}(I)$ for an index set I. Together with all the "leaves" originating from Z_j during the later steps, it is $\tilde{Z}_j$ of the form of either*

$$L_{p_X(j)}(\Omega, W), \ \text{or}\ l_{p_X(j)}\left(I, \{W_i\}_{i\in I}\right),$$

where W and $\{W_i\}_{i\in I}$ are Banach spaces from IG_0 themselves.

For $\alpha \geq 1$, by means of $m_{j,\alpha} : X \longrightarrow X_{\bar{p}_j}$, where $p_X(l) = \bar{p}_j(l)$ for $j \neq l \in P$ and $\bar{p}_j(j) = p(j)/\alpha$, we designate the mapping induced by $m_{\alpha,W}$ if $Z_j = L_{p(j)}(\Omega)$, or $m_{\alpha,\{W_j\}}$ if $Z_j = l_{p(j)}(I)$, changing only the $\tilde{Z}_j$-component of every $f \in X$. (Recall that $X_{\bar{p}_j}$ is an IG_0-space with the same tree as X but different parameter position function $\bar{p}_j$.)

Assume that $\beta :\ P \to [1,\infty)$ and $Y \in IG_0$ with $T(Y) = T(X)$ and $p_Y = p_X/\beta$. Let the complex Mazur descent *$m_\beta :\ X \to Y$, be the composition*

$$m_\beta = \prod_{j\in P} m_{j,\beta(j)}.$$

Assume that $\gamma \in (0,1)$ and $Z \in IG_0$ with $p_Z = p_X/\gamma$. Let the simple Mazur ascent *$m_\beta :\ X \to Z$, be the mapping*

$$m_{\gamma,a} :\ f \longmapsto \left(\frac{|f|}{\|f\|_X}\right)^{\gamma-1} f.$$

The correctness of this definition is discussed in the next remark and theorem.

Remark 5.1. For the sake of the future usage and the matter of correctness, let us note the following algebraic properties of the Mazur mappings defined above.

a) $m_{j,\alpha} m_{l,\beta} = m_{l,\beta} m_{j,\alpha}$ and $m_{j,\alpha} m_{j,\beta} = m_{j,\alpha\beta}$ for $i, j \in P, \alpha, \beta \geq 1$;
b) $m_\beta m_\gamma = m_\gamma m_\beta = m_{\beta\gamma}$ for $\beta, \gamma :\ P \to [1,\infty)$;
c) $m_\beta m_{\alpha,a} = m_{\alpha,a} m_\beta = I$ for $\alpha \in (0,1]$ and constant $\beta :\ P \to \{1/\alpha\}$;
d) both m_β and $m_{\gamma,a}$ preserve the $IG_0 -$ norms.

Theorem 5.1. *Let $X, Y \in IG_0$ with the same parameter position set P, $A \subset X$ and $\beta = p_X/p_Y$. Then we have*

a) $\|m_\beta | H^1(X,Y)\| \leq \|2\beta - 1 | L_\infty(P)\|^{N_{\min}(X)}$ *if* $\beta :\ P \to [1,\infty)$;
b) $\|m_{\alpha,a} | H^\alpha(A,Y)\| \leq \left(1 + 2^{1-\alpha}\right) r(A,0)^{1-\alpha}$ *if* $\beta :\ P \to \{\alpha\}$ *and* $\alpha \in (0,1]$.

The proof of Theorem 5.1. Part *b*) follows immediately from Part *f*) of Lemma 5.1 applied to the last "leaves" of X and followed by the multiple usage of the identity (4) from the proof of Lemma 5.1.

To establish $a)$, we represent m_β as a finite product of the products of the groups of $m_{j,\beta(j)}$ dealing with the "leaves grown" during one and the same step of the "tree-growing" process creating X. Indeed, assume that X is created in $N_{\min}(X)$ steps, and the "leave" P_k appeared during the kth step of the process. Then Parts $b)$ and $d)$ of Lemma 5.1 provide us with the estimates

$$\left\|\prod_{j\in P_k} m_{j,\beta j}\right\|_{H^1} \le \sup_{j\in P_k}(2\beta(j)-1) \le \|2\beta-1|L_\infty(P)\| \text{ for } k\in I_{N_{\min}(X)}$$

leading to $a)$ with the aid of Corollary 3.1, $a)$. □

Corollary 5.1. *Let $X, Y \in IG_0([1,\infty))$ have the same tree $T(X)=T(Y)$ (and, thus, common P). Then*

$$X \overset{(\alpha,\beta)}{\longleftrightarrow} Y \text{ with } \alpha = \inf_{i\in P}\left\{\frac{p_X(i)}{p_Y(i)},1\right\},\ \beta = \inf_{i\in P}\left\{\frac{p_Y(i)}{p_X(i)},1\right\}.$$

The exponent α is sharp in the following cases:
a) $p_X \ge p_Y$;
b) $p_{\min}(Y)\in[p_{\min}(X),2]$ and there exists $\{i_k\}\in P$ satisfying $\lim_{k\to\infty} p_X(i_k)=p_{\min}(X)$ and $\lim_{k\to\infty} p_Y(i_k)=p_{\min}(Y)$;
c) $p_{\max}(X)\in[2,p_{\max}(Y)]$ and there exists $\{i_k\}\in P$ satisfying $\lim_{k\to\infty} p_X(i_k)=p_{\max}(X)$ and $\lim_{k\to\infty} p_Y(i_k)=p_{\max}(Y)$.

The proof of Corollary 5.1. If $\alpha=1$, we take the homeomorphism $\phi=m_u$ with $u=p_X/p_Y$ and use the representation $\phi^{-1}=m_{1/u\beta}m_{\beta,a}$ (see Lemma 5.1, Theorem 5.1 and Corollary 3.1, $a)$) to establish $\phi\in H^1(X,Y)$ and $\phi^{-1}\in H^\beta(B_Y,X)$. Switching X and Y covers the case $\beta=1$. If $\alpha,\beta<1$, we take

$$\phi = m_{u/\alpha}m_{\alpha,a} \text{ and } \phi^{-1}=m_{1/u\beta}m_{\beta,a} \tag{1}$$

Theorem 5.1 and Corollary 3.1, $a)$ provide the smoothness exponents. Eventually we deduce the sharpness in $a)$, $b)$ and $c)$ from Theorem 11.1. □

Corollary 5.2. *Let $X\in IG_0([1,\infty))$. Then (for the constant function: $\bar 2:\ P\to\{2\}$)*

$$X \overset{(\alpha,\beta)}{\longleftrightarrow} H = X_{\bar 2} \text{ with sharp } \alpha=\min(p_{\min}(X),2)/2,\ \beta = 2/\max(p_{\max}(X),2).$$

5.2 Abstract Mazur ascent and complex Mazur descent: IG_{0+} setting

The next lemma is the Hölder inequality for IG-spaces. Since IG-spaces are lattices of functions, the operation of the pointwise product is well-defined for the functions from the IG-spaces with the same tree. It can be interpreted as the Hölder inequality for convexifications of a lattice (see Remark 2.6, $a)$). Recall that, for a parameter function $q:\ P\to(1,\infty)$ defined on the same same parameter position set P as the parameter function p_X of $X\in IG_+$, the space X_q is the space with the same tree as X and the parameter function q.

Lemma 5.2. *Let $X \in IG$ with the parameter function $p = p_X$ and $\alpha \in (0,1)$. Then we have*

$$\|fg|X\| \le \|f|X_{\frac{p}{1-\alpha}}\|^{1-\alpha}\|g|X_{\frac{p}{\alpha}}\|^{\alpha}.$$

Let us define the *abstract Mazur ascent* mapping between IG_{0+} spaces.

Definition 5.3. *For $X \in IG_{0+}$ with the parameter position set P, let P_{ll} be the set of the last leaves of $T(X)$, and let $P = P_- \cup P_{nc}$ be the decomposition into the union of the parameter position set P_- of X_- (from which X was created by "growing" the noncommutative leaves) and its part P_{nc} corresponding to the noncommutative spaces that are some of the last leaves from $\{Z_i\}_{i\in P_{ll}}$, where every Z_i has the parameter $p(i)$. Thus, every $f \in X$ is defined on some set $\Omega = \Omega(X)$ and takes values in $\cup_{i\in P_{ll}} Z_i$.*

Assume that $q: P \to (1,\infty)$, $\beta \in (0,\infty)$, $1 < \inf\{p(i), q(i): i \in P_{nc}, p(i) \ne q(i)\}$ and

$$\alpha = \alpha(X,q) = \inf\left\{1, \frac{\min(p(i),q(i),2)}{\max(p(i),q(i),2)}: i \in P_{nc}, p(i) \ne q(i)\right\}.$$

For every $i \in P_{nc}$ and $m_{p(i),q(i)}$ provided by Theorem 4.5, let $M_{p(i),q(i),\beta,a}$ be the β-homogenisation of $m_{p(i),q(i)}$ defined by

$$M_{p(i),q(i),\beta,a}: x \longmapsto \|x\|_{Z_i}^{\beta} m_{p(i),q(i)}\left(\frac{x}{\|x\|_{Z_i}}\right). \qquad (Aa_1)$$

Now let the abstract Mazur ascent *mapping $M_{p,q,\beta,Aa}$ be defined, for every $\omega \in \Omega$ and $i = i(\omega)$, by the relation*

$$\left(M_{p,q,\beta,Aa}f\right)(\omega) = M_{p(i),q(i),\beta,a}\left(f(\omega)\right)$$

if $p(i) \ne q(i)$ and Z_i is noncommutative, defined by the relation

$$\left(M_{p,q,\beta,Aa}f\right)(\omega) = M_{p(i),q(i),\beta,a}\left(f(\omega)\right) = \|f(\omega)\|_{Z_i}^{\beta-1} f(\omega) \qquad (Aa_2)$$

on the corresponding noncommutative leaf Z_i if $p(i) = q(i)$, and defined as the classical Mazur ascent or simple descent (see Definition 5.1)

$$\left(M_{p,q,\beta,Aa}f\right)(\omega) = M_{\beta,a}\left(f(\omega)\right) \qquad (Aa_3)$$

on the corresponding last leaf Z_i if it is "commutative" (not noncommutative).

We shall also define the homogeneous abstract Mazur ascent *mapping*

$$m_{p,q,\beta,Aa}: f \longmapsto \|f\|_X^{1-\beta} M_{p,q,\beta,Aa} f \text{ for } X \in X. \qquad (aa)$$

We also continue to use the notation m_u for the complex Mazur descent acting on the "commutative" vertices of X (i.e. on the vertices of X_-).

Remark 5.2. The abstract ingredient of the proof of Lemma 5.1, f) shows that, for $\beta \in (0,1]$, one has

$$\left\| m_{p,q,\beta,Aa} \right| H^{\beta}(B_X,Y) \| \le 1 + \left\| M_{p,q,\beta,Aa} \right| H^{\beta}(B_X,Y) \| .$$

The corresponding re-homogenisation (or β-homogenisation) is the subject of Corollary 5.3 and Lemma 5.4. Together with Theorem 4.5, they also imply

$$M^{-1}_{p,q,\beta,Aa} = M_{q,p,1/\beta,Aa} \text{ and } m^{-1}_{p,q,\beta,Aa} = m_{q,p,1/\beta,Aa}.$$

Lemma 5.3. *For Banach spaces X and Y with the unit spheres S_X and S_Y and $\beta \in (0, \alpha] \subset (0, 1]$, let $\phi \in H^\alpha(S_X, Y)$. Assume also that $\psi \in H^\beta((0, \infty), \mathbb{R})$ satisfying $0 \le \psi(t) \le C_\psi t^\beta$ for $t \in (0, \infty)$ and*

$$\phi_\psi : X \longrightarrow Y,\ x \longmapsto \psi(\|x\|_X)\,\phi\left(\frac{x}{\|x\|_X}\right), \phi_\psi(0) = 0.$$

Then we also have

$$\|\phi_\psi | H^\beta(X, Y)\| \le \left\|\psi | H^\beta((0, \infty), \mathbb{R})\right\| \|\phi | C(S_X, Y)\| + C_\psi 2^\alpha \|\phi | H^\alpha(S_X, Y)\|.$$

Moreover, $\phi_\psi(X) = Y$ if $\phi(S_X) = S_Y$ and $\psi((0, \infty)) = (0, \infty)$, and also

$$\phi_\psi^{-1}(y) = \psi^{-1}(\|y\|_Y)\,\phi^{-1}\left(\frac{y}{\|y\|_Y}\right) \text{ if } \phi^{-1} \text{ and } \psi^{-1} \text{ exist.}$$

The proof of Lemma 5.3. Let us start by noting that

$$\|\phi | H^\beta(S_X, Y)\| \le 2^{\alpha-\beta} \|\phi | H^\alpha(S_X, Y)\|. \tag{1}$$

For $x, y \in X$ with $\|x\|_X \ge \|y\|_X$, we use the representation

$$\phi_\psi(x) - \phi_\psi(y) = \phi\left(\frac{y}{\|y\|_X}\right)(\psi(\|x\|_X) - \psi(\|y\|_X)) + \psi(\|x\|_X)\left(\phi\left(\frac{x}{\|x\|_X}\right) - \phi\left(\frac{y}{\|y\|_X}\right)\right) \tag{2}$$

and the observation (used, particularly, in [5])

$$\left\|\frac{x}{\|x\|_X} - \frac{y}{\|y\|_X}\right\|_X \le \frac{\|x - y\|_X}{\|x\|_X} + \frac{\|x\|_X - \|y\|_X}{\|x\|_X} \le \frac{2}{\|x\|_X}\|x - y\|_X, \tag{3}$$

helping us with estimating the Y-norm of the second summand in (2) in the view of (1), to conclude with

$$\begin{aligned} &\|\phi_\psi(x) - \phi_\psi(y)\|_Y \\ &\quad\le \left(\left\|\psi | H^\beta((0, \infty))\right\| \|\phi | C(S_X, Y)\| + C_\psi 2^\beta \|\phi | H^\beta(S_X, Y)\|\right) \|x - y\|_X^\beta \\ &\quad\le \left(\left\|\psi | H^\beta((0, \infty))\right\| \|\phi | C(S_X, Y)\| + C_\psi 2^\alpha \|\phi | H^\alpha(S_X, Y)\|\right) \|x - y\|_X^\beta, \end{aligned}$$

where we have also used the triangle inequality in X. Using the polar decomposition $Y = S_Y \times (0, \infty))$ in X and Y, one finishes the proof. □

Corollary 5.3. *For Banach spaces X and Y with the unit spheres S_X and S_Y and $\beta \in (0, \alpha] \subset (0, 1]$, let $\phi \in H^\alpha(S_X, Y)$ and*

$$\phi_\beta : X \longrightarrow Y,\ x \longmapsto \|x\|_X^\beta \phi\left(\frac{x}{\|x\|_X}\right), \quad \phi_\beta(0) = 0.$$

Then we also have

$$\|\phi_\beta | H^\beta(X, Y)\| \le \|\phi | C(S_X, Y)\| + 2^\alpha \|\phi | H^\alpha(S_X, Y)\|.$$

Moreover, $\phi_\beta(X) = Y$ if $\phi(S_X) = S_Y$, and

$$\phi_\beta^{-1}(y) = \|y\|_Y^{1/\beta}\phi^{-1}\left(\frac{y}{\|y\|_Y}\right) \text{ if } \phi^{-1} \text{ exists.}$$

Lemma 5.4. *For Banach spaces X and Y with the unit spheres S_X and S_Y, $\alpha \in (0,1]$ and $\beta \in (0,\infty)$, let $\phi \in H^\alpha(S_X,Y)$ and*

$$\phi_\beta : X \longrightarrow Y,\, x \longmapsto \|x\|_X^\beta \phi\left(\frac{x}{\|x\|_X}\right), \quad \phi_\beta(0) = 0.$$

Then we also have

$$\|\phi_\beta | H^{\min(\alpha,\beta)}(B_X,Y)\| \le \max(\beta,1) 2^{(\min(\beta,1)-\alpha)_+} \|\phi|C(S_X,Y)\| + 2^\alpha \|\phi|H^\alpha(S_X,Y)\|.$$

Moreover, $\phi_\beta(X) = Y$ if $\phi(S_X) = S_Y$, and

$$\phi_\beta^{-1}(y) = \|y\|_Y^{1/\beta} \phi^{-1}\left(\frac{y}{\|y\|_Y}\right) \text{ if } \phi^{-1} \text{ exists.}$$

The proof of Lemma 5.4. The case $\beta \in (0,\alpha]$ is Corollary 5.3. For $x, y \in X$ with $1 \ge \|x\|_X \ge \|y\|_X$, we use either the fact that X is a linear metric space with the metric $\|x-y\|^\beta$ for $\beta \in (\alpha,1]$, or the Lagrange theorem for $\beta > 1$ to obtain

$$\|x\|_X^\beta - \|y\|_X^\beta \le \max(\beta,1)\|x\|_X^{(\beta-1)_+}\|x-y\|_X^{\min(\beta,1)} \le 2^{\min(\beta,1)-\alpha}\max(\beta,1)\|x-y\|_X^\alpha. \quad (1)$$

This estimate, along with the representation (2) and relation (3) from the proof of Lemma 5.3 with $\psi(t) = t^\beta$, implies the estimate for the H^α-norm of ϕ_β sought for. The proof is finished by using the polar representation in X and Y. □

The next lemma demonstrates the roles playing by the homogenisation parameter β and the smoothness α for the boundedness of basic Mazur mappings on the model case of the Bochner-Lebesgue spaces: the lower homogenisation provides global regularity at the expense of, possibly, worse smoothness, while upper homogenisation preserves the smoothness.

Lemma 5.5. *For $p, q \in [1,\infty]$, let (E,μ) be a measure space with countably additive μ, and X a Banach space. Under the conditions of Lemma* 5.4*, let $\beta = p/q$ and*

$$\psi_\beta : L_p(E,X) \longrightarrow L_q(E,Y),\, f(\tau) \longmapsto \phi_\beta(f(\tau)) \text{ for a.e. } \tau \in E.$$

Then we have

a) $\left\|\psi_\beta | H^\beta(L_p(E,X), L_q(E,Y))\right\| \le \|\phi|C(S_X,Y)\| + 2^\alpha\|\phi|H^\alpha(S_X,Y)\|$ *if* $\beta \in (0,\alpha]$;

b) $\left\|\psi_\beta | H^\alpha\left(B_{L_p(E,X)}, L_q(E,Y)\right)\right\|$
$\le \max(\beta,1)2^{\beta+\min(\beta,1)-2\alpha}\|\phi|C(S_X,Y)\| + 2^\beta\|\phi|H^\alpha(S_X,Y)\|$
if $\beta \in (\alpha,\infty)$.

The proof of Lemma 5.5. Let $f, g \in L_p(E,X)$. In the case of Part *a*), we just take L_q-norm of the both sides of the following estimate provided by Corollary 5.3:

$$\|\psi_\beta f(\tau) - \psi_\beta g(\tau)\|_Y \le (\|\phi|C(S_X,Y)\| + 2^\alpha\|\phi|H^\alpha(S_X,Y)\|)\,\|f(\tau)-g(\tau)\|_X^\beta. \quad (1)$$

In the case of Part *b*), we have from Lemma 5.4

$$\|\psi_\beta f(\tau)-\psi_\beta g(\tau)\|_Y \le \big(\max(\beta,1)2^{\min(\beta,1)-\alpha}\|\phi|C(S_X,Y)\| + 2^\alpha\|\phi|H^\alpha(S_X,Y)\|\big) \\ \times\left(\|f(\tau)\|_X+\|g(\tau)\|_X\right)^{\beta-\alpha}\|f(\tau)-g(\tau)\|_X^\alpha, \quad (2)$$

and assume also that f,g are in the unit ball of $L_p(E,X)$. Applying Lemma 5.2 with $X=L_q(E)$ and the parameter α/β (Hölder inequality) and the triangle inequality for $L_p(E)$ to (2), we obtain

$$\|\psi_\beta f-\psi_\beta g|L_q(E,Y)\| \le \\ \le 2^{\beta-\alpha}\big(\max(\beta,1)2^{\min(\beta,1)-\alpha}\|\phi|C(S_X,Y)\|+2^\alpha\|\phi|H^\alpha(S_X,Y)\|\big)\,\|f-g|L_p(E,X)\|^\alpha, \quad (3)$$

finishing the proof. □

The following theorem describes the Hölder smoothness of the abstract Mazur ascent mapping that forms the complete system of the Hölder homeomorphisms of the spheres of IG_{0+} spaces in composition with the complex Mazur descent (see Remark 5.3, *c*)).

Theorem 5.2. *Let $X,Y\in IG_{0+}([1,\infty))$ be spaces with the same tree $T(X)=T(Y)$, $\beta>0$ and $p_X(i)=\beta p_Y(i)$ for $i\in P\setminus P_I$ and $P_I=\{i\in P_{nc}:\ p_X(i)=p_Y(i)\}$. Assume also that*

$$1<\inf_{i\in P_{nc}\setminus P_I}\{p_X(i),p_Y(i)\}\ \text{and}\ 1/\alpha=\begin{cases}\left\|\frac{\max(p_X,p_Y,2)}{\min(p_X,p_Y,2)}\Big|l_\infty(P_{nc}\setminus P_I)\right\| & \text{if } P_I\ne P_{nc},\\ 1 \text{ if } P_I=P_{nc}\end{cases}.$$

Then $m^{-1}_{p_X,p_Y,\beta,Aa}=m_{p_Y,p_X,1/\beta,Aa}$ and there exists a constant C depending on $\{p_X(i),p_Y(i)\}_{i\in P_{nc}\setminus P_I}$ and β, such that

$$\|m_{p_X,p_Y,\beta,Aa}|H^{\min(\alpha,\beta)}(B_X,Y)\|\le C.$$

Remark 5.3. *a*) Theorem 5.2 is naturally extended (with the same bounds) to the setting of $IG(S)_+$ classes, where S is the class of Banach function and sequence lattices, while the noncommutative leaves are still noncommutative L_p-spaces with the set of exponents strictly separated from 1 and ∞.

b) The results involving the complex Mazur descent (applied to X_- of $X\in IG_{0+}$) are transferred to the setting of IG_{0+} spaces without any changes.

c) Note that in the degenerated case, when the parameters of all noncommutative leaves of X and Y coincide ($P_I=P_{nc}$), the construction in the proof of Theorem 5.2 gives the inverse of the complex Mazur descent $m_{1/\beta}$ with the constant descent parameter $1/\beta$ (naturally acting on the vertices of X_-) if $\beta\in(0,1)$ and the descent itself (that is simple this time) if $\beta>1$.

d) The major advantage of Theorem 5.2 is that α depends only on the parameters of the noncommutative spaces that we want to change. Further reduction to β permits to change also the extreme "commutative" parameters.

The proof of Theorem 5.2. The identity follows from Corollary 5.3 and Theorem 4.5. For $f, g \in X$ and $\omega \in \Omega = \Omega(X) = \Omega(Y)$, let $f(\omega), g(\omega) \in Z_i$ for some $i \in P_{nc}$, where $Z_i = L_{p_X(i)}(\mathcal{M}_i, \tau_i)$. Then the corresponding mapping $m_{p_X(i),p_Y(i)}$ from Theorem 4.5 maps Z_i onto the corresponding leaf $L_{p_Y(i)}(\mathcal{M}_i, \tau_i)$ of Y and satisfies, for $\alpha_i = \min(p(i), q(i), 2)/\max(p(i), q(i), 2) \geq \alpha$ and $C_i > 0$ (provided in Theorem 4.5)

$$\left\| m_{p(i),q(i)} \right| H^{\alpha_i}\big(B_{Z_i}, L_{p_Y(i)}(\mathcal{M}_i, \tau_i)\big)\big\| \leq C_i. \tag{1}$$

Let us start with the case $\beta \in (0, \alpha]$. Applying Corollary 5.3 to (1) if $p_X(i) \neq p_Y(i)$, or to the identity mapping (with $C_i = 1$) if $p_X(i) \neq p_Y(i)$, we see that

$$\|M_{p(i),q(i),\beta,a} f(\omega) - M_{p(i),q(i),\beta,a} g(\omega)\|_{q(i)} \leq (1 + 2^{\alpha_i} C_i) \|f(\omega) - g(\omega)\|^{\beta}_{p(i)} \tag{2}$$

for all noncommutative leaves $i \in P_{nc}$. If, in turn, Z_i $(i \in P_{ll})$ is a leaf that is not noncommutative, then $M_{p(i),q(i),\beta,a}$ is just the classical Mazur map $M_{\alpha,a}$ (see Lemma 5.1, e)), and one has

$$\|M_{p(i),q(i),\beta,a} f(\omega) - M_{p(i),q(i),\beta,a} g(\omega)\|_{q(i)} \leq 2^{1-\beta} \|f(\omega) - g(\omega)\|^{\beta}_{p(i)}. \tag{3}$$

The explicit expressions for C_i given in Theorem 4.5 and the definition of IG_{0+} class imply the uniform boundedness

$$\sup_{i \in P_{ll}} \{2^{1-\beta}, 1 + 2^{\alpha_i} C_i, 3\} = C < \infty.$$

Now we combine the relations (1), (2) and (3) (with the common C) and finish the proof for $\beta \in (0, \alpha]$ by means of the multiple usage of the identity

$$\||h|^{\alpha}\|_{p/\alpha} = \|h\|^{\alpha}_p \text{ for } p \in (0, \infty]. \tag{4}$$

Assume now that $\beta > \alpha$ and $f, g \in B_X$. Keeping in mind Theorem 4.5, we apply Lemma 5.4 to (1) if $p_X(i) \neq p_Y(i)$, or to the identity mapping (with $C_i = 1$ and $\alpha_i = 1$) if $p_X(i) \neq p_Y(i)$, we see that

$$\begin{aligned} &\|M_{p(i),q(i),\beta,a} f(\omega) - M_{p(i),q(i),\beta,a} g(\omega)\|_{q(i)} \leq \\ &\leq \left(\max(\beta,1) 2^{\min(\beta,1)-\alpha_i} + 2^{\alpha_i} C_i\right) \left(\|f(\omega)\|_{p(i)} + \|g(\omega)\|_{p(i)}\right)^{\beta-\alpha_i} \|f(\omega) - g(\omega)\|^{\alpha_i}_{p(i)} \leq \\ &\leq \left(\max(\beta,1) 2^{\min(\beta,1)-\alpha_i} + 2^{\alpha_i} C_i\right) \left(\|f(\omega)\|_{p(i)} + \|g(\omega)\|_{p(i)}\right)^{\beta-\alpha} \|f(\omega) - g(\omega)\|^{\alpha}_{p(i)} \end{aligned} \tag{5}$$

for all noncommutative leaves $i \in P_{nc}$. If, in turn, Z_i $(i \in P_{ll})$ is a leaf that is not noncommutative, then $M_{p(i),q(i),\beta,a}$ is just the classical Mazur map (see the scalar-valued case of Lemma 5.1, e)), and one has

$$\begin{aligned} &\|M_{p(i),q(i),\beta,a} f(\omega) - M_{p(i),q(i),\beta,a} g(\omega)\|_{q(i)} \leq \\ &\quad \leq \max(\beta,1) 2^{(1-\beta)_+} \left(\|f(\omega)\|_{p(i)} + \|g(\omega)\|_{p(i)}\right)^{(\beta-1)_+} \|f(\omega) - g(\omega)\|^{\min(\beta,1)}_{p(i)} \leq \\ &\qquad \leq \max(\beta,1) 2^{(1-\beta)_+} \left(\|f(\omega)\|_{p(i)} + \|g(\omega)\|_{p(i)}\right)^{\beta-\alpha} \|f(\omega) - g(\omega)\|^{\alpha}_{p(i)} \end{aligned} \tag{6}$$

Theorems 4.5 and 2.4 (2.3) demonstrate the uniform boundedness of the constants in (5) and (6). Hence we apply the Hölder inequality

$$\left\|u^{\beta-\alpha}v^{\alpha}|Z\right\| \leq \left\|u^{\beta}|Z_{\frac{p_Z\beta}{\beta-\alpha}}\right\|^{1-\frac{\alpha}{\beta}} \left\|v^{\beta}|Z_{\frac{p_Z\beta}{\alpha}}\right\|^{\frac{\alpha}{\beta}}$$

from Lemma 5.2 for the space $Z \in IG_0$, whose tree $T(Z)$ is the subtree of Y obtained from $T(Y)$ by eliminating (not "growing") all the last leaves with p_Z being the restriction of p_Y, the identity (4) and the triangle inequality for X to establish the key estimate

$$\|M_{p_X,p_Y,\beta,Aa}f - M_{p_X,p_Y,\beta,Aa}g\|_Y \leq C\left(\|f\|_X + \|g\|_X\right)^{\beta-\alpha}\|f-g\|_X^{\alpha}. \tag{7}$$

To finish the proof with the aid of (7), we apply Lemma 5.4 to the restriction of $M_{p_X,p_Y,\beta,Aa}$ onto the unit sphere S_X to obtain

$$\|m_{p_X,p_Y,\beta,Aa}|H^{\alpha}(B_X,Y)\| \leq \max(\beta,1)2^{(\min(\beta,1)-\alpha)_+} + 2^{\beta}C.$$

□

Corollary 5.4. *Let $X, Y \in IG_{0+}([1,\infty))$ have the same tree $T(X) = T(Y)$ (and, thus, common P) and $P_I = \{i \in P_{nc} : \ p_X(i) = p_Y(i)\}$. Assume also that*

$$1 < \inf_{i\in P_{nc}\setminus P_I}\{p_X(i), p_Y(i)\} \ \textit{and}\ \alpha_{nc} = \inf\left\{1, \frac{\min\left(p_X(i), p_Y(i), 2\right)}{\max\left(p_X(i), p_Y(i), 2\right)} : \ i \in P_{nc}\setminus P_I\right\}.$$

Then we have

$$X \overset{(\alpha,\beta)}{\longleftrightarrow} Y \ \textit{with}\ \alpha = \inf_{i\in P}\left\{\frac{p_X(i)}{p_Y(i)}, 1, \alpha_{nc}\right\}.$$

The exponent α is sharp in the following cases:
a) $p_X \geq p_Y$ and $P_{nc} = P_I$;
b) $2 \in \{p_X(i), p_Y(i)\}$ and $p_X(i) \leq p_Y(i)$ for every $i \in P_{nc}$ and either
(i) $p_{\min}(Y) \in [p_{\min}(X), 2]$ and there exists $\{i_k\} \in P$ satisfying $\lim_{k\to\infty} p_X(i_k) = p_{\min}(X)$ and $\lim_{k\to\infty} p_Y(i_k) = p_{\min}(Y)$, or
(ii) $p_{\max}(X) \in [2, p_{\max}(Y)]$ and there exists $\{i_k\} \in P$ satisfying $\lim_{k\to\infty} p_X(i_k) = p_{\max}(X)$ and $\lim_{k\to\infty} p_Y(i_k) = p_{\max}(Y)$.

The proof of Corollary 5.4. With respect to X_- and Y_-, it is conducted exactly as the proof of Corollary 5.1: the abstract Mazur ascent parameter is chosen to, at least, elevate all the (commutative) $p_X(i)$ above $p_Y(i)$ and, then, adjust with the aid of the appropriate complex (but Lipschitz) Mazur descent. The only difference is that changing the parameters $p_X(i)$ of those noncommutative leaves that have to be changed (i.e. $i \in P_{nc} \setminus P_I$) may require the further decrease in the abstract Mazur ascent parameter. Moreover, we need to start with the abstract Mazur ascent if any of the noncommutative parameters $p_X(i)$ need to be changed (even to be decreased). The sharpness follows from Theorem 11.1. □

Theorem 6.7 extends the following immediate corollary.

Corollary 5.5. *Let* $X \in IG_{0+}([1,\infty))$ *with* $1 < \inf\{p_X(i) :\ i \in P_{nc} \setminus P_I\}$ *if* $P_{nc} \neq P_I$ *and*

$$\alpha_{nc} = \inf\left\{1, \frac{\min(p_X(i),2)}{\max(p_X(i),2)} :\ i \in P_{nc} \setminus P_I\right\}.$$

Then we have

$$X \xleftrightarrow{(\alpha,\beta)} H = X_{\bar{2}} \text{ with } \alpha = \min(p_{\min}(X)/2, 1, \alpha_{nc}) \text{ and } \beta = \min(2/p_{\max}(X), 1, \alpha_{nc}).$$

The parameters α *and/or* β *are sharp if, respectively,* $\alpha_{nc} \geq p_{\min}(X)/2$ *and/or* $\alpha_{nc} \geq 2/p_{\max}(X)$.

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