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HAUSDORFF OPERATORS ON HARDY SPACES

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Abstract. Hausdorff operators (Hausdorff summability methods) appeared long ago aiming to solve certain classical problems in analysis. Modern theory of Hausdorff operators started with the work of Siskakis in complex analysis setting and with the work of Liflyand-Móricz in the Fourier transform setting. In this paper a survey of the main results on Hausdorff operators in various settings is given. Many open problems in the subject are formulated.

1 Introduction

The main goal of this survey paper is to give a picture of the status of a modern topic which links old and classical notions of Hausdorff summability with modern theories of Hardy spaces as well as with other related spaces. To make the picture comprehensive, we are going to briefly overview all the main constituents of this topic. These and the relations between them look as follows.

Hausdorff summability of number series $\Longleftrightarrow$ *(little) moment problem*
$\Downarrow$
Hausdorff summability of Fourier series
$\Downarrow$

Hausdorff summability of power series $\Longleftrightarrow$ *Hardy spaces, BMO, ...*
$\Downarrow$
Hausdorff summability of Fourier transforms $\Longleftrightarrow$ *Hardy spaces, BMO, ...*
$\Downarrow$ $\Downarrow$
One-dimensional case *Multidimensional case*

The history of what is called Hausdorff summability methods goes back to 1917 when Hurwitz and Silverman in [47] studied a family of methods (see Section 2 below) within the classical framework of summability of number series where regularity (consistency) and comparison of various methods are the main goal. However, a "genuine" history had started with the paper [42] in 1921, in which Hausdorff not only rediscovered the same summability methods but enriched their study by associating them with the famous and important moment problem for a finite interval. Under the popularity of summability theory in those times and later period, Hausdorff methods took their place and continued to be of interest. Let us mention some sources where enough attention is paid to them, like the monographs [41], [83], [67], and [9] - unfortunately never translated, or a survey paper [25]. We will give a brief overview of this topic in Section 2 with the emphasis on the connection to the moment problem. Certain details and proofs are omitted there but the picture as a whole seems to be clear enough, both on its own and as a starting point for consequent study.

We would like to mention that many applications of Hausdorff summability were made to Fourier series in one and several dimensions (just a couple of random examples: [45, 31, 63, 80, 36]); we will not touch this topic here, mostly because it has nothing in common with Hardy spaces so far.

What did affect the situation, rather than the Hausdorff summability of Fourier series, was the Hausdorff summability of power series of analytic functions, which started with the work of Siskakis [73] on composition operators and the Cesàro means in H^p spaces and his nice short proof for H^1 in [74]. General Hausdorff summability was not considered in [23] and [24] for analytic functions in Hardy spaces (as well as in some other spaces) until after [60] had appeared. We fill that giving a clear picture of this subject is a must, to what Section 3 is devoted. The last and pretty big amount of this section is devoted to the multidimensional case, more precisely to recent results in [4], where the estimates for Hausdorff operators are obtained as an application of a new approach to the theory of Hardy spaces for several complex variables.

The next natural step was an extension of the results from [73, 74] to the Fourier transform setting on the real line. It was done in [28] and in a slightly different manner in [35]. Moreover, it was [28] that inspired Móricz and the author to try a more general averaging than the Cesàro (Hardy) one. The paper [60] opened, in a sense, a new stage in the study of Hausdorff summability. Besides the mentioned progress it inspired in the analytic functions setting, it also led to a number of new open problems, first of all in several dimensions. This growing interest in these problems was mainly connected not with the type of summability itself - it had already been actually known (see, e.g., [32] or [27]), but with involving various more sophisticated spaces than L^p in consideration, first of all Hardy spaces, and, correspondingly, different techniques. In Section 4 we discuss the initial proof in [60] and certain related problems.

A natural passage from dimension one to several dimensions was made almost in parallel. The paper [61] was written immediately after [60]. Then other papers followed, [56] is pretty recent. In Section 5, a key point is Theorem 5.2 from [58]. In that section we discuss various definitions of Hardy spaces and, correspondingly, various existing and possible proofs of the boundedness of Hausdorff operators on Hardy spaces. In the end of the section we give some results for BMO.

In Section 6 we present a collection of open problems. The number of open problems corresponding to the multidimensional case is larger than that for other sections and subsections, and similarly their discussion is more detailed.

Thus, there are two main “personages" in these notes: Hausdorff summability and Hardy space, both in various settings and versions. Their interplay is the main feature of this work. Of course, there are “personages" of less importance, at least in our scenario; for instance, [55], [48] or [69] are random examples of such type, though quite representative. On the other hand, the topic actively runs its natural course. To wit, let us mention [17], where the use of atomic approach, like in [58], to the boundedness of the Hausdorff operator in the local Hardy space h^1 is undertaken. This type of spaces is rather important (see [33] or [19]), and the results of that paper are apparently the first attempt to develop a theory of Hausdorff operators in h^1. The authors also deal with the Herz type spaces (see also [15]). Also, estimates for commutators of multidimensional Hausdorff operators begin to attract attention, see, e.g., [13].

Correspondingly, it was not our aim to give a complete list of references, we give only those immediately involved in the study. Many others can be found in the papers referred to.

In what follows $a \ll b$ means $a \le Cb$, with C being an absolute constant in this and any other occurrence. In our study we are not interested in explicit indication of these constants. If a constant depends on certain parameters, they will be indicated as subscripts, like C_p. Correspondingly, $a \asymp b$ means $a \le C_1 b$ and $b \le C_2 a$ with different constants C_1 and C_2.

2 Hausdorff summability of number series

The Hausdorff means, the Cesàro means among them, are known long ago in connection with summability of number series. Let us briefly describe this subject. We follow the nice way it is presented in [83, Ch. III] (see also [25]).

Let the sequence $s_0, s_1, s_2, \ldots$ be represented by the infinite matrix S in which it is the first column, while the rest of the entries are zeros. Similarly, the sequence $t_0, t_1, t_2, \ldots$ is represented by the matrix T. Explicitly,

$$S = \begin{pmatrix} s_1 & 0 & 0 & \ldots \\ s_2 & 0 & 0 & \ldots \\ s_3 & 0 & 0 & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} t_1 & 0 & 0 & \ldots \\ t_2 & 0 & 0 & \ldots \\ t_3 & 0 & 0 & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{pmatrix}.$$

Let M be the infinite diagonal matrix with the sequence $\mu_0, \mu_1, \mu_2, \ldots$ as its diagonal entries:

$$M = \begin{pmatrix} \mu_1 & 0 & 0 & \ldots \\ 0 & \mu_2 & 0 & \ldots \\ 0 & 0 & \mu_3 & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{pmatrix}.$$

Let, finally,

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & ... \\ 1 & -1 & 0 & 0 & 0 & ... \\ 1 & -2 & 1 & 0 & 0 & ... \\ 1 & -3 & 3 & -1 & 0 & ... \\ . & . & . & . & . & ... \\ . & . & . & . & . & ... \end{pmatrix}$$

be the difference matrix. The latter, with the entries $r_{mn} = (-1)^n \binom{m}{n}$ for $n = 0, 1, 2, ..., m$, and zeros otherwise, is self-reciprocal, that is, $R = R^{-1}$. The reason for the term *difference matrix* becomes clear if one observes that

$$\sum_{n=0}^{\infty} r_{mn} s_n = \sum_{n=0}^{m} (-1)^n \binom{m}{n} s_n = \Delta^m s_0,$$

with $\Delta s_k = s_k - s_{k+1}$ and $\Delta^m = \Delta(\Delta^{m-1})$.

Now, given a matrix $A = (a_{mn})$, $m, n = 0, 1, 2, ...$, the operation

$$T = AS \tag{2.1}$$

transforms the matrix S into T or, if we consider only the first columns of S and T, the sequence $\{s_n\}$ into the sequence $\{t_n\}$. Then the sequence $\{s_n\}$ is summable by the matrix A to the sum s if the sequence $\{t_n\}$ is defined by (2.1) and if

$$\lim_{m\to\infty} t_m = s. \tag{2.2}$$

More explicitly, this means that all series

$$t_m = \sum_{n=0}^{\infty} a_{mn} s_n, \tag{2.3}$$

where $m = 0, 1, 2, ...$, and the numbers a_{mn} are the entries of the matrix A, converge and (2.2) holds.

One of the basic examples is that when

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & ... \\ 1/2 & 1/2 & 0 & 0 & ... \\ 1/3 & 1/3 & 1/3 & 0 & ... \\ . & . & . & . & ... \\ . & . & . & . & ... \end{pmatrix},$$

then

$$t_m = \frac{s_0 + s_1 + \cdots + s_m}{m+1},$$

which leads to the Cesàro summability.

A method of summability is called *regular* (the term *consistent* is used sometimes) if every convergent sequence is summable by it to the actual limit of the sequence.

We are now in a position to define Hausdorff summability.

Definition 2.1. The matrix A is a *Hausdorff matrix* corresponding to the sequence $\{\mu_n\}$, $n = 0, 1, 2, ...$, if $A = RMR^{-1}$.

It is easily seen that multiplication of Hausdorff matrices is commutative.

What is important is to determine which sequences $\{\mu_n\}$ lead to *regular* Hausdorff matrices. The celebrated Toeplitz theorem is a natural tools to test this.

Theorem 2.1. *Summability by the matrix A is regular if and only if a constant K exists such that*

$$\sum_{n=0}^{\infty} |a_{mn}| < K, \qquad m = 0, 1, 2, ...; \tag{2.4}$$

$$\lim_{m\to\infty} a_{mn} = 0, \qquad n = 0, 1, 2, ...; \tag{2.5}$$

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1. \tag{2.6}$$

Before applying this theorem to the Hausdorff summability method we first find the Hausdorff matrix in terms of the given sequence $\{\mu_n\}$. By definition,

$$T = RMR^{-1}S. \tag{2.7}$$

We wish to determine the elements a_{mn} so that (2.7) will be of the form (2.3). Employing the identity

$$\binom{m}{j}\binom{j}{n} = \binom{m}{n}\binom{m-n}{j-n}$$

for $n \le j \le m$, we get

$$a_{mn} = \binom{m}{n}\Delta^{m-n}\mu_n$$

for $n = 0, 1, ..., m$, and zero otherwise. This is the *necessary and sufficient* condition for the matrix A to correspond to the sequence $\{\mu_n\}$.

With this and Theorem 2.1 in hand, we are in a position to obtain a criterion for regularity of the Hausdorff method.

Theorem 2.2. *The Hausdorff summability method corresponding to the sequence* $\{\mu_n\}$ *is regular if and only if*

$$\mu_n = \int_0^1 t^n d\mu(t), \qquad n = 0, 1, ..., \tag{2.8}$$

where the function μ *is of bounded variation in* $(0,1)$, $\mu(0) = \mu(0+) = 0$, *and* $\mu(1) = 1$.

To prove this criterion, we have only to apply Theorem 2.1 to the matrix A. To show that the main condition (2.4) reduces to the boundedness of variation needs certain efforts, which lead to the fact that conditions (2.5) and (2.6) of Theorem 2.1 become, respectively,

$$\lim_{m\to\infty} \binom{m}{n} \int_0^1 t^n (1-t)^{m-n} d\mu(t) = 0, \quad n = 0, 1, 2, ...,$$

and

$$\lim_{m\to\infty} \int_0^1 d\mu(t) = 1,$$

from which the rest of the conditions is easily derived.

It was, in fact, the study of the summability of divergent series which led Hausdorff to investigation of the (little) moment problem: given a sequence $\{\mu_n\}$, under what conditions it is possible to determine a function $\mu(t)$ of bounded variation in $(0,1)$ such that (2.8) holds. Such $\{\mu_n\}$ is called a *moment sequence*. The solution is as follows: the *necessary and sufficient* condition for $\{\mu_n\}$ to be a moment sequence is that a constant K exists such that for $a_{mn} = \binom{m}{n} \Delta^{m-n} \mu_n$

$$\sum_{n=0}^{m} |a_{mn}| < K, \qquad m = 0, 1, 2, ...$$

It might be of interest to consider various multidimensional generalizations of this subject, with respect to various types of ordering the elements of the series, or in other words with respect to various types of summation. Certain partial results do exist (see, e.g., [1]), but to the best of our knowledge the topic is not generalized in full.

3 Hausdorff summability of power series

It was the analytic functions setting where connections of Hardy spaces as well as certain related ones to Hausdorff summability historically came into play. We follow the way the subject and relevant results are given in [73, 74, 23, 24]. We first give all needed prerequisites.

3.1 Hardy spaces

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. For $1 \le p < \infty$ the Hardy space H^p is the space of all analytic functions $f : D \to \mathbb{C}$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

With this norm H^p is a Banach space (and Hilbert for $p = 2$). If $1 \le p \le q < \infty$ then $H^1 \supset H^p \supset H^q$. Functions $f \in H^p$ possess boundary values (non-tangential limits) $f(e^{i\theta})$ which are p-integrable on ∂D. Identifying f with its boundary function provides an isometric embedding of H^p into $L^p(\partial D)$, the norm in the latter will be denoted by $\|\cdot\|_p$. If $f \in H^p$ then for $z \in D$

$$|f(z)| \le 2^{1/p} \frac{\|f\|_p}{(1 - |z|)^{1/p}},$$

see [20, p. 36].

For each function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$, Hardy's inequality (see, e.g., [20, p. 48]) holds true

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \le \pi ||f||_{H^1}. \tag{3.1}$$

Every analytic function $a(z) : D \to D$, that is, mapping the unit disk into itself, induces a bounded composition operator

$$W_a f(z) = f(a(z))$$

on the Hardy space H^p; see [20, p. 29]. In addition, if $b(z)$ is a bounded analytic function on D then the weighted composition operator

$$W_{a,b} f(z) = b(z) f(a(z))$$

is bounded on H^p as well.

3.2 The Cesàro means for power series

The Cesàro means for power series from the Hardy space H^1 in the unit disk were considered by Siskakis. Such Cesàro means are constructed by replacing the coefficients a_k in the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

with $f \in H^1$, by their Hardy transform

$$\frac{1}{k+1} \sum_{p=0}^{k} a_p,$$

which results in

$$Cf(z) = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{p=0}^{k} a_p \right) z^k. \tag{3.2}$$

An elegant proof of the boundedness of the corresponding operator on H^1 is given in [74]. It relies on the following Hardy-Littlewood result (see [39]). For $0 < r < 1$ and $f \in H^1$, we denote by

$$M_q(f; r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(rte^{i\theta})|^q \, d\theta \right)^{1/q}$$

the integral means on $|z| = r$ of an analytic f.

Lemma 3.1. *If $f \in H^1$ and $q > 1$, then*

$$\int_0^1 M_q(f; s)(1 - s)^{-1/q} ds \le C_q \|f\|_{H^1},$$

where the constant C_q depends only on q.

It is worth mentioning that the proof from [74] is applicable to H^p for no p except $p = 1$. More general approach is used in [73] but we stop the discussion here and proceed to the general Hausdorff means of which the Cesàro means is a (simple) partial case.

3.3 The Hausdorff means for power series

Later, already after appearance of the paper [60], general Hausdorff matrices were considered in [23] (for a continuation, see [24]) as follows.

Let, as in the previous section, Δ be the forward difference operator defined on scalar sequences $\mu = (\mu_n)_{n=0}^{\infty}$ by $\Delta \mu_n = \mu_n - \mu_{n+1}$ and $\Delta^k \mu_n = \Delta(\Delta^{k-1} \mu_n)$ for $k = 1, 2, ...$ with $\Delta^0 \mu_n = \mu_n$.

Setting

$$c_{n,k} = \binom{n}{k}\Delta^{n-k}\mu_k, \qquad k \le n,$$

we define the *Hausdorff matrix* $H = H_\mu$ with generating sequence μ to be the lower triangular matrix with the entries

$$H_\mu(i,j) = \begin{cases} 0, & i < j \\ c_{i,j}, & i \ge j \end{cases}.$$

It induces two operators on spaces of power series; they are formally given by

$$\mathcal{H}_\mu f(z) = \mathcal{H}_\mu\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_{n,k}a_k\right) z^n,$$

which is obtained by letting the matrix H_μ to act on the Taylor coefficients of f, and

$$\mathcal{A}_\mu f(z) = \mathcal{A}_\mu\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} c_{n,k}a_n\right) z^k,$$

which is obtained by letting the transposed matrix $A_\mu = H_\mu^*$ to act on the Taylor coefficients of f. Such a matrix A_μ is called a *quasi-Hausdorff matrix.* The convergence of the power series $\mathcal{A}_\mu f$ is more delicate than that of $\mathcal{H}_\mu f$. However, it is clear that if f is a polynomial then $\mathcal{A}_\mu f$ is also a polynomial. If the space under investigation contains polynomials, we may ask whether $\mathcal{A}_\mu$ extends to a bounded operator on the whole space.

An important special case of such matrices occurs when μ_n is the moment sequence of a finite (positive) Borel measure μ on $(0,1]$:

$$\mu_n = \int_0^1 t^n d\mu(t), \qquad n = 0, 1, ...$$

In this case for $k \le n$

$$\begin{aligned} c_{n,k} &= \binom{n}{k}\Delta^{n-k}\int_0^1 t^k\, d\mu(t) \\ &= \binom{n}{k}\int_0^1 [t^k - \binom{n-k}{1}t^{k+1} + ... + t^n]\, d\mu(t) \\ &= \binom{n}{k}\int_0^1 t^k(1-t)^{n-k} d\mu(t). \end{aligned}$$

All this becomes more transparent if one recalls the structure and properties of the Hausdorff means from the previous section.

It follows from the work of Hardy [40] that if the measure μ satisfies

$$\int_0^1 t^{-1/p} d\mu(t) < \infty,$$

then $\mathcal{H}_\mu$ determines a bounded linear operator

$$\mathcal{H}_\mu : \{a_n\} \to \{A_n\}, \qquad A_n = \sum_{k=0}^{n} c_{n,k} a_k, \quad n = 0, 1, ...,$$

on the sequence space l^p, $1 < p < \infty$, whose norm is exactly the last integral.

Various choices of the measure μ give rise to well known classical matrices. For example, when μ is the Lebesgue measure one has the Cesàro matrix. Indeed, since

$$\sum_{k=0}^{n} \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} dt = \int_0^1 dt = 1,$$

it suffices to prove that all $c_{n,k}$ are equal to each other in this case. Integrating by parts, we obtain

$$\binom{n}{k+1} \int_0^1 t^{k+1} (1-t)^{n-k-1} dt = \binom{n}{k+1} \frac{k+1}{n-k} \int_0^1 t^k (1-t)^{n-k} dt.$$

Since

$$\binom{n}{k+1} \frac{k+1}{n-k} = \binom{n}{k},$$

we have $c_{n,k+1} = c_{n,k}$, which completes the proof.

3.4 Hausdorff matrices and composition operators

The study of the Hausdorff means for analytic functions is based on relating them with certain families of composition operators. The latter under certain conditions, as we have seen above, bring us into the Hardy space.

For $t \in (0, 1]$ and $z \in D$ the two families of mappings of the disk into itself

$$\phi_t(z) = \frac{tz}{(t-1)z + 1}$$

and

$$\psi_t(z) = tz + 1 - t,$$

and the family of bounded functions on D

$$w_t(z) = \frac{1}{(t-1)z+1}$$

are used to construct composition operators associated with Hausdorff matrices.

We define then

$$S_\mu f(z) = \int_0^1 w_t(z) f(\phi_t(z))\, d\mu(t).$$

The integral is finite. There is also a need in the integral

$$T_\mu f(z) = \int_0^1 f(\psi_t(z))\, d\mu$$

for those analytic functions f and points z for which it is defined.

Lemma 3.2. *Let μ be a finite positive Borel measure on $(0,1]$ and let f be analytic in D. Then the power series $\mathcal{H}_\mu f(z)$ absolutely converges in D and $\mathcal{H}_\mu f(z) = S_\mu f(z)$ for every $z \in D$.*

The following trivial lemma is a counterpart of Lemma 3.2 for $\mathcal{A}_\mu f$. The reason the two lemmas are different both in formulation and assertion is that $\sum\limits_{n=k}^{\infty} c_{n,k} a_n$ may diverge.

Lemma 3.3. *Let μ be a finite positive Borel measure on $(0,1]$. Then for each polynomial f the function $\mathcal{A}_\mu f(z)$ is also a polynomial and $\mathcal{A}_\mu f(z) = T_\mu f(z)$ for every $z \in D$.*

These give one a possibility to derive a criterion for the boundedness of the Hausdorff means on H^1.

Theorem 3.1. *Let μ be a finite positive Borel measure on $(0,1]$. Then $\mathcal{H}_\mu : H^1 \to H^1$ is a bounded operator if and only if*

$$L_1 = \int_0^1 \Big(1 + \ln\frac{1}{t}\Big)\, d\mu(t) < +\infty. \tag{3.3}$$

In this case $||\mathcal{H}_\mu||_{H^1} \asymp L_1$.

The idea of the proof is as follows. Since $\mathcal{H}_\mu = S_\mu$, we have

$$\|S_\mu\|_{H^1} \le \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} |w_t(e^{i\theta})\, f(\phi_t(e^{i\theta}))|\, d\theta\, d\mu(t).$$

Fixing $t \in (0,1]$, we estimate the inner integral

$$A(t) = \int_{-\pi}^{\pi} \frac{1}{|1-(1-t)e^{i\theta}|}\left| f\left(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right)\right| \frac{d\theta}{2\pi}$$

and obtain the desired bound.

As for the necessity, let $\mathcal{H}_\mu$ be bounded on H^1. Observing that

$$\mathcal{H}_\mu(1)(z) = \sum_{n=0}^{\infty}\int_0^1 (1-t)^n\, d\mu(t)\, z^n$$

and using Hardy's inequality (3.1), we get

$$\int_0^1 \Big(1+\ln\frac{1}{t}\Big)\, d\mu(t) \quad \ll \quad \int_0^1 \frac{1}{1-t}\ln\frac{1}{t}\, d\mu(t).$$

From this

$$\begin{aligned}\int_0^1 \Big(1+\ln\frac{1}{t}\Big)\, d\mu(t) &\ll \int_0^1 \frac{1}{1-t}\ln\frac{1}{t}\, d\mu(t)\\ &\ll \sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 (1-t)^n\, d\mu(t) \ll \|\mathcal{H}_\mu(1)\|_{H^1} \ll \|\mathcal{H}_\mu\|_{H^1},\end{aligned}$$

as required.

In a similar manner one can prove (see [23] and [24])

Theorem 3.2. *Let μ be a finite positive Borel measure on $(0,1]$. Then $\mathcal{A}_\mu : H^1 \to H^1$ defines a bounded operator if and only if*

$$||\mathcal{A}_\mu||_{H^1} = \int_0^1 t^{-1}\, d\mu(t) < +\infty.$$

In this case $\mathcal{A}_\mu f = T_\mu f$ for every $f \in H^1$.

We mention that in the cited papers conditions were given and proved not only for H^1 but for all H^p, $1 \le p < \infty$, with corresponding dependence on p (in the case of H^∞ conditions are also found and look different from those for $p < \infty$).

Theorem 3.3. *Let μ be a finite positive Borel measure on $(0,1]$. If $1 < p \leq \infty$, then $\mathcal{H}_\mu : H^p \to H^p$ is a bounded operator if and only if*

$$||\mathcal{H}_\mu||_{H^p \to H^p} = \int_0^1 t^{1/p-1} d\mu(t) < \infty. \tag{3.4}$$

If $1/p + 1/p' = 1$, then, under the above conditions, $\mathcal{H}_\mu : H^p \to H^p$ and $\mathcal{A}_\mu : H^{p'} \to H^{p'}$ are adjoint.

Theorem 3.4. *Let μ be a finite positive Borel measure on $(0,1]$ and $1 \leq p < \infty$. Then $\mathcal{A}_\mu : H^p \to H^p$ defines a bounded operator if and only if*

$$||\mathcal{A}_\mu||_{H^p \to H^p} = \int_0^1 t^{-1/p}\, d\mu(t) < +\infty. \tag{3.5}$$

Furthermore, $\mathcal{A}_\mu$ is bounded on H^∞ if and only if

$$\lim_{n\to\infty} \ln n \int_0^1 (1-t)^n d\mu(t) = 0. \tag{3.6}$$

In this case

$$||\mathcal{A}_\mu||_{H^\infty \to H^\infty} = \mu(0,1].$$

3.5 Multidimensional Hardy spaces

The above one-dimensional results are generalized to several dimensions in [14], the first multidimensional generalization, where the sole case of the polydisk is considered, while the spaces are H^p, $1 \leq p < \infty$.

In [4], these results are extended to Hardy spaces on a rather wide class of domains – the Reinhardt domains. Sufficient conditions for the boundedness of Hausdorff type operators turn out to be necessary for a smaller subclass of domains, still quite wide. Of course, the polydisk is among them.

The approach is different from that in dimension one in [23, 24] and in several dimensions in [14], where estimates of special composition operators ensured the desired results. The point is that estimating (and sometimes even finding) such composition operators might be an extremely difficult task in the multivariate setting. The approach in [4] is based on an inductive argument (see Main Lemma below) where one-dimensional results can be directly applied.

Hardy classes of holomorphic functions of several complex variables are usually defined on bounded domains $D \subset \mathbb{C}^n$ as follows. If the boundary ∂D is smooth, then the class $H^p(D)$ consists of the functions f holomorphic in D such that

$$\overline{\lim_{\varepsilon\to 0}} \int_{\partial D} |f(z-\varepsilon\nu_z)|^p d\sigma(z) < \infty, \tag{3.7}$$

where ν_z is the external unit normal vector to ∂D at the point z, and $d\sigma(z)$ is an element of the $(2n-1)$-dimensional surface ∂D (see, e.g., [76, 51]). However, the definition for the polydisk $U^n = \{z : |z_j| < 1, j = 1, ..., n\}$ usually differs from the general definition and is defined by the following condition instead of (3.7)

$$\overline{\lim_{r\to 1}} \int_{\mathbb{T}^n} |f(rz)|^p \left|\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\right| < \infty, \tag{3.8}$$

where $\mathbb{T}^n = \{z : |z_j| = 1, j = 1, ..., n\}$ and $0 < r < 1$ (see, e.g., [70, 14]).

Let us consider bounded complete Reinhardt domains $D \subset \mathbb{C}^n$. They appear naturally as domains of convergence of multidimensional power series

$$\sum_{|\alpha|\ge 0} c_\alpha z^\alpha, \tag{3.9}$$

where $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, ..., \alpha_n)$, with all α_j nonnegative integers. Here $z^\alpha = z_1^{\alpha_1}...z_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + ... + \alpha_n$. We now define the Hardy class $H^p(D)$ to be that of all functions $f(z)$ holomorphic in D and satisfying

$$\lim_{r\to 1-} \int_{\partial D_r} |f(z)|^p d\sigma(z) < \infty, \tag{3.10}$$

where $0 < r < 1$, $D_r = rD$ is the r-th homothety of D, and $d\sigma(z)$ is an element of the $(2n-1)$-dimensional surface ∂D_r. Since the integral (3.10) can be representable by integrating first over the circles $\ell \cap \partial D_r$, where each ℓ is a complex line passing through the origin, and then by integrating over the set of such lines with respect to the corresponding positive measure, it is also a non-decreasing function of r. This explains why the usual limit is used in (3.10) instead of the upper limit; by the way, the usual limit can analogously be written in (3.8) in place of the upper limit.

Let us consider the family of parallel complex lines

$$m_k = \{z = (z_1, ..., z_{k-1}, t, z_{k+1}, ..., z_n), t \in \mathbb{C}\} \tag{3.11}$$

crossing the domain D. The intersection of each of these lines with D is a disk. For $0 < r < 1$, let us consider the set

$$\bigcup_{\{m_k\}} (m_k \cap \partial D_r). \tag{3.12}$$

We will say that the domain D is *k-tame* if the limit as $r \to 1-$ of the set (3.12) is exactly the whole ∂D. For example, the ball $\{z : |z| < 1\}$, where $|z| = (|z_1|^2 + ... +$

$|z_n|^2)^{1/2}$, is a k-tame domain for each k, $1 \le k \le n$, while the polydisk U^n is not k-tame for any k. However, it is true for U^n that

$$\overline{\lim_{r\to 1-} \bigcup_{1\le k\le n} \bigcup_{\{m_k\}} (m_k \cap \partial D_r)} = \partial D. \tag{3.13}$$

There is a need to define additional types of Reinhardt domains. First, we will call the domain D *quasi-tame* if (3.13) holds true.

Lemma 3.4. *Every bounded complete Reinhardt domain is quasi-tame.*

Further, a complete bounded Reinhardt domain D is called *k-cylindric*, $1 \le k \le n$, if $D \subset \{z : |z_k| < \rho\}$ for some $\rho > 0$, and ∂D contains a piece of the hyper-surface $\Gamma_k(\rho) = \{z : |z_k| = \rho\}$, that is, $\Gamma_k(\rho) \cap \{z : (z_1, ..., z_{k-1}, z_{k+1}, ..., z_n) \in R\} \subset \partial D$, where R is a domain in $\mathbb{C}^{n-1}$.

If $f \in H^p(D)$, then in almost every section $D \cap \alpha$ this function will have angular boundary values on the boundary of this section, that is, on the circle. Therefore, boundary values of f almost everywhere on ∂D can be understood in the sense of $(2n-1)$-dimensional measure. We will denote these boundary values by f as well.

Theorem 3.5. *If* $0 < p < \infty$, *then*

$$||f||_p^p = \int_{\partial D} |f(\zeta)|^p \, d\sigma(z)$$

and

$$\lim_{r\to 1-} \int_{\partial D} |f(r\zeta) - f(\zeta)|^p \, d\sigma(z) = 0.$$

This theorem immediately yields

Corollary 3.1. *If* $f \in H^p(D)$ *and* $0 < p < \infty$, *then for* $\zeta \in \partial D$

$$\lim_{r\to 1-} ||f(r\zeta) - f(\zeta)||_p = 0,$$

and polynomials are dense in $H^p(D)$.

We first consider the case when $0 < p < \infty$.

Main Lemma for $0 < p < \infty$. *Let D be a bounded complete Reinhardt domain, k-tame with k being a fixed integer, $1 \le k \le n$. For a function f holomorphic in D to belong to the class $H^p(D)$, it is necessary and sufficient that*

1) *for almost all complex lines m_k the restriction of the function f to the disk $m_k \cap D = Q_k$ belongs to the Hardy class $H^p(Q_k)$*
and

2) *the function f is L^p summable on ∂D, that is,*

$$\int_{\partial D} |f(z)|^p \, d\sigma(z) < \infty. \tag{3.14}$$

Here we understand the values $f(z)$, $z \in \partial D$, as angular boundary values on the circles $m_k \cap \partial D$ which by **1)** exist almost everywhere for almost all m_k, that is, almost everywhere on ∂D.

Lemma 3.5. *Let D be a bounded complete Reinhardt domain. For a function f holomorphic in D to belong to the class $H^p(D)$, it is necessary and sufficient that condition* **2)** *holds true and condition* **1)** *in Main Lemma holds true for all k, $1 \le k \le n$.*

Contrary to many other situations, here the case $p = \infty$ is easier and less restrictive.

Main Lemma for $p = \infty$. *Let D be a bounded complete Reinhardt domain and k be a fixed integer, $1 \le k \le n$. For a function f holomorphic in D to belong to the class $H^\infty(D)$, it is sufficient that*

1) *for almost all complex lines m_k, the restriction of the function $f(z)$ to the disk $m_k \cap D = Q_k$ belongs to the Hardy class $H^\infty(Q_k)$*

and

2) *the function f is $L^\infty(\partial D)$, that is,*

$$\operatorname{ess\,sup}_{\partial D} |f(z)| = B < \infty; \tag{3.15}$$

and it is necessary that

1′) *for all complex lines m_k the restriction of the function f to Q_k belongs to the Hardy class $H^\infty(Q_k)$*

and

2′) *there holds*

$$\sup_{\partial D} |f(z)| = B < \infty. \tag{3.16}$$

For a function f holomorphic in D to belong to the class $H^\infty(D)$, it is necessary and sufficient that **2)** *holds true and* **1′)** *holds true for all m_k.*

3.6 Hausdorff operators for multidimensional power series

Let us consider a natural multidimensional analogue of the Hausdorff type operators by defining it on power series (3.9), representing functions holomorphic in D, as

$$(\mathcal{H}_\mu f)(z) = \sum_{|\alpha| \ge 0} \left(\sum_{\beta \le \alpha} \prod_{j=1}^{n} h_{\alpha_j,\beta_j}(\mu_j) c_\beta \right) z^\alpha \tag{3.17}$$

for the Hausdorff operator, while for the quasi-Hausdorff operator

$$(\mathcal{A}_\mu f)(z) = \sum_{|\alpha|\ge 0} \left(\sum_{\beta\ge\alpha} \prod_{j=1}^{n} h_{\alpha_j,\beta_j}(\mu_j) c_\beta \right) z^\alpha, \tag{3.18}$$

where $\beta \le \alpha$ and $\beta \ge \alpha$ means that $\beta_j \le \alpha_j$ and $\beta_j \ge \alpha_j$, respectively, for all $j = 1, ..., n$. Here, as above in dimension one, $h_{\alpha_j,k_j}(\mu_j) = \binom{\alpha_j}{k_j} \Delta^{\alpha_j - k_j} \mu_j(k_j)$, $k_j \le \alpha_j$, with $\mu_j(k_j)$ being the moment sequence of a finite (positive) Borel measure μ_j on $(0, 1]$:

$$\mu_j(k_j) = \int_0^1 t^{k_j} d\mu_j(t), \qquad k_j = 0, 1, ...$$

In this case for $k_j \le \alpha_j$

$$h_{\alpha_j,k_j}(\mu_j) = \binom{\alpha_j}{k_j} \int_0^1 t^{k_j} (1-t)^{\alpha_j - k_j} d\mu_j(t).$$

The extension of the one-dimensional Hausdorff operators to several dimensions in this way has first been suggested in [14].

While (3.17) is well defined for $p \le \infty$, for (3.18) the definition is correct when $p < \infty$.

Various choices of the measures μ_j give rise to the well-known classical matrices. For example, when all μ_j are the Lebesgue measures one has the multidimensional Cesàro matrix, of the classical form in the case when D is the polydisk.

We also mention that (3.17) can be considered as a repeated one-dimensional Hausdorff operator in each of the n variables. This feature is pivotal in the proofs of the following results.

We will consider sufficient and necessary conditions for the boundedness of Hausdorff type operators on Hardy spaces separately, since they coincide only on a special subclass of Reinhardt domains.

As often happens, sufficient conditions hold true for a wider class of objects. We will start with results of maximal generality. By $\mathcal{H}_{\mu_k} := (\mathcal{H}_{\mu_k})(z_k)$ we will denote the operator $\mathcal{H}_{\mu_k}$ with respect to the k-th variable z_k with all other variables fixed; the same for $\mathcal{A}_{\mu_k}$.

Theorem 3.6. *Let a complete bounded Reinhardt domain D be k-tame. The Hausdorff operator $\mathcal{H}_{\mu_k}$ is bounded on $H^p(D)$ for $1 < p < \infty$ provided that*

$$\int_0^1 s^{1/p-1} d\mu_k(s) < \infty \tag{3.19}$$

and for $p = 1$ provided that

$$\int_0^1 (1 + \ln(1/s))\, d\mu_k(s) < \infty. \tag{3.20}$$

Theorem 3.7. *Let a complete bounded Reinhardt domain D be k-tame. The quasi-Hausdorff operator $\mathcal{A}_{\mu_k}$ is bounded on $H^p(D)$ for $1 \le p < \infty$ provided that*

$$\int_0^1 s^{-1/p} d\mu_k(s) < \infty. \tag{3.21}$$

Corollary 3.2. *Let D be a bounded complete Reinhardt domain D, k-tame for all k, $1 \le k \le n$. Then the Hausdorff operator $\mathcal{H}_\mu f$ is bounded in $H^p(D)$ for $1 < p < \infty$ provided that*

$$\prod_{k=1}^n \int_0^1 s^{1/p-1} d\mu_k(s) < \infty, \tag{3.22}$$

and for $p = 1$ provided that

$$\prod_{k=1}^n \int_0^1 (1 + \ln(1/s))\, d\mu_k(s) < \infty. \tag{3.23}$$

The Hausdorff operator $\mathcal{A}_\mu f$ is bounded in $H^p(D)$ for $1 \le p < \infty$ provided that

$$\prod_{k=1}^n \int_0^1 s^{-1/p} d\mu_k(s) < \infty. \tag{3.24}$$

Corollary 3.3. *Let D be a bounded complete Reinhardt domain. Then the Hausdorff operator $\mathcal{H}_\mu f$ is bounded in $H^p(D)$ for $1 < p < \infty$ provided* (3.22) *holds and for $p = 1$ provided* (3.23) *holds, while the Hausdorff operator $\mathcal{A}_\mu f$ is bounded in $H^p(D)$ for $1 \le p < \infty$ provided* (3.24) *holds.*

Let us consider an interesting particular case of the polydisk U^n. Corollary 3.3 concerns the Hardy class $H_1^p(U^n)$, when the integral over the whole boundary ∂D is involved. However, in the study of boundary properties of holomorphic functions in the polydisk, the Hardy class $H_2^p(U^n)$ used is defined by means of the integral (3.8). It can be shown that $H_1^p(U^n)$ is wider than $H_2^p(U^n)$ when $p < \infty$. However, for $p = \infty$ these classes coincide, since maximum of the absolute value of a function holomorphic in the polydisk is attained on its skeleton $\mathbb{T}^n$.

It turns out that sufficient conditions for the boundedness of Hausdorff type operators are also necessary for a smaller class of Reinhardt domains, the above defined k-cylindric domains.

Theorem 3.8. *Let a complete bounded Reinhardt domain D be k-cylindric. If an operator $\mathcal{H}_{\mu_k}$ is bounded in $H^p(D)$, $1 < p < \infty$ ($p = 1$ respectively), then* (3.19) *holds true (*(3.20) *for $p = 1$, respectively).*

Theorem 3.9. *Let a complete bounded Reinhardt domain D be k-cylindric. If an operator $\mathcal{A}_{\mu_k}$ is bounded in $H^p(D)$, $1 < p < \infty$, then* (3.21) *holds true.*

As above, extending the assumed restriction to all k yields a general result, this time necessary.

Corollary 3.4. *If a complete bounded Reinhardt domain D is k-cylindric for all k, $1 \le k \le n$. Then the condition* (3.22) *(or, relatively,* (3.23)*) is necessary for the boundedness of $\mathcal{H}_\mu$ in $H^p(D)$ when $1 < p < \infty$ (or, correspondingly, when $p = 1$).*

The necessary condition for the boundedness of $\mathcal{A}_\mu$ when $1 \le p < \infty$ is (3.24).

And, finally, let us give, as a corollary, a very special partial result earlier obtained in [14] for a smaller Hardy class.

Corollary 3.5. *If a domain D is the polydisk, then the condition* (3.22) *(or, relatively,* (3.23)*) is necessary and sufficient for the boundedness of $\mathcal{H}_\mu$ in $H^p(D)$ when $1 < p < \infty$ (or, correspondingly, when $p = 1$).*

In conclusion, the necessary and sufficient condition for the boundedness of $\mathcal{A}_\mu$ when $1 \le p < \infty$ is (3.24).

Since the version of the Main Lemma for $p = \infty$ is less restrictive, so is its application to Hausdorff operators as well.

Theorem 3.10. *Let D be a complete bounded Reinhardt domain. The Hausdorff operator $\mathcal{H}_{\mu_k}$ is bounded on $H^\infty(D)$ if and only if*

$$\int_0^1 s^{-1} d\mu_k(s) < \infty. \tag{3.25}$$

We are now in a position to present the multidimensional result in full generality.

Corollary 3.6. *Let D be a bounded complete Reinhardt domain. Then the Hausdorff operator $\mathcal{H}_\mu f$ is bounded in $H^\infty(D)$ if and only if*

$$\prod_{k=1}^{n} \int_0^1 s^{-1} d\mu_k(s) < \infty. \tag{3.26}$$

To summarize the obtained results, sufficient results for the boundedness of Hausdorff operators on Hardy spaces turn out to be necessary for a smaller class of Reinhardt domains. For the polydisk necessary and sufficient conditions exist, which were obtained earlier for a smaller Hardy class. On the other hand, for the unit ball in $\mathbb{C}^n$ only sufficient conditions are known.

4 Hausdorff operators on the real line

We have now arrived at consideration of Hausdorff operators in the Fourier transform setting. We first study a bundle of problems in dimension one, on the real axis. We will further go on to those in several dimensions.

4.1 Preliminaries

We recall that the Fourier transform $\widehat{f}$ of a (complex-valued) function f in $L^1(\mathbb{R})$ is defined by

$$\widehat{f}(t) := \int_{\mathbb{R}} f(x)e^{-itx}dx, \quad t \in \mathbb{R},$$

while its Hilbert transform $\widetilde{f}$ is defined by

$$\begin{aligned}\widetilde{f}(x) &:= \frac{1}{\pi}(\text{P.V.})\int_{\mathbb{R}} f(x-u)\frac{du}{u} \\ &= \frac{1}{\pi}\lim_{\delta\downarrow 0}\int_{\delta}^{\infty}\{f(x-u)-f(x+u)\}\frac{du}{u}, \quad x \in \mathbb{R}.\end{aligned}$$

As is well known, this limit exists for almost all x in $\mathbb{R}$, and the real Hardy space $H^1(\mathbb{R})$ is defined to be

$$H^1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \widetilde{f} \in L^1(\mathbb{R})\},$$

endowed with the norm

$$\|f\|_{H^1} := \|f\|_{L^1} + \|\widetilde{f}\|_{L^1}, \quad \text{where} \quad \|f\|_{L^1} := \int_{\mathbb{R}} |f(x)|\,dx.$$

This space is a Banach algebra under point-wise addition, scalar multiplication, and convolution.

The identity

$$(\widetilde{f})^{\wedge}(t) = (-i\,\text{sign}t)\widehat{f}(t), \quad t \in \mathbb{R}, \tag{4.1}$$

is valid for all f in $H^1(\mathbb{R})$ and plays an important role in the sequel.

For example, it implies immediately that if $f \in H^1(\mathbb{R})$, then $\widehat{f}(0) = 0$ (this mean zero property was first pointed out in [52]) and, by the uniqueness of Fourier transform, almost everywhere

$$(\widetilde{f})^{\sim}(t) = -f(t). \tag{4.2}$$

In particular, if $f \in H^1(\mathbb{R})$, then $\widetilde{f} \in H^1(\mathbb{R})$ and

$$\|\widetilde{f}\|_{H^1} = \|f\|_{H^1}.$$

A question here is how $(\widetilde{f})^\wedge$ is defined. As often happens, the distributional approach is the most general and natural. If we introduce an appropriate principal value distribution, then the Fourier transform $(\widetilde{f})^\wedge$ can be defined as a tempered distribution in such a way that (4.1) holds true.

In the previous section, we have considered Hardy spaces of analytic functions in the unit disk. Let us briefly discuss how these two cases are related to each other. First, let us consider Hardy spaces in the upper half-plane instead of those in the unit disk. Both settings are related via a conformal mapping. The Hardy space $H^1(\mathbb{C}_+)$ of analytic functions $F(z)$ in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re} z > 0\}$ is defined by the condition

$$\|F\|_{H^1} = \sup_{y>0} \int_{\mathbb{R}} |F(x+iy|\, dx < \infty.$$

This space is complete with respect to the indicated norm. It is known that each such F has a finite limit

$$\lim_{y\to 0+} F(x+iy) = f(x) + ig(x)$$

almost everywhere on the real axis; in addition, the real-valued functions f and g belong to the space $L^1(\mathbb{R})$. Moreover, $g(x) = \widetilde{f}(x)$. On the other side, it is known that if f is a real-valued function in $L^1(\mathbb{R})$ such that its Hilbert transform $\widetilde{f}$ also belongs to $L^1(\mathbb{R})$, then $f(x) + i\widetilde{f}(x)$ coincides with the limit values as $y \to 0+$ of some function $F(z) = F(x+iy) \in H^1(\mathbb{C}_+)$ almost everywhere in $\mathbb{R}$.

4.2 Definition and basic properties

The Hausdorff operator $\mathcal{H}$ generated by a function φ in $L^1(\mathbb{R})$ as introduced in [60], can be defined both directly and via the Fourier transform. The latter reads as follows:

$$(\mathcal{H}f)^\wedge(t) = (\mathcal{H}_\varphi f)^\wedge(t) := \int_{\mathbb{R}} \widehat{f}(tx)\varphi(x)\, dx, \quad t \in \mathbb{R}, \tag{4.3}$$

where f is also in $L^1(\mathbb{R})$. The existence of such a function $\mathcal{H}f$ in $L^1(\mathbb{R})$ is established in the proof of Theorem 4.1. In fact, one can find a close definition already in [41, Ch.XI, 11.18], along with its summability properties. Later on the Hausdorff mean (of a Fourier-Stieltjes transform) was studied on $L^1(\mathbb{R})$ in [27] (see also [32]).

We note that if $\varphi(x) := \chi_{(0,1)}(x)$, the indicator function of the unit interval $(0,1)$, then (4.3) is of the following form:

$$(\mathcal{H}f)^\wedge(t) := \int_0^1 \widehat{f}(tx)dx = \frac{1}{t}\int_0^t \widehat{f}(u)du, \quad t \neq 0.$$

In this case, $\mathcal{H}$ is the well-known Cesàro operator; its properties were studied in [28].

The objective is to determine the Fourier analytic properties of $\mathcal{H}$ on the Hardy space. One of the points is as follows. Since, generally speaking, the inverse formula

$$f(x) = (2\pi)^{-1} \int_{\mathbb{R}} \widehat{f}(t) e^{ixt} dt$$

does not take place for $f \in L^1(\mathbb{R})$ as well as for $f \in H^1(\mathbb{R})$, it is expected that

$$\int_{\mathbb{R}} (\mathcal{H}f)\widehat{\ }(y) e^{ixy} dy \tag{4.4}$$

behaves better and characterizes f properly, in a sense.

But first we establish two properties, so essential that without them further study is meaningless.

Theorem 4.1. *If $\varphi \in L^1(\mathbb{R})$, then the Hausdorff operator $\mathcal{H} = \mathcal{H}_\varphi : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is bounded and*

$$\|\mathcal{H}_\varphi\| = \sup_{\|f\|_{L^1(\mathbb{R})} \le 1} \|\mathcal{H}_\varphi\|_{L^1(\mathbb{R})} \le \|\varphi\|_{L^1(\mathbb{R})}. \tag{4.5}$$

The proof of this result comes as a by-product of the following theorem, in which additional facts are contained. But before this we need an auxiliary result in which equivalent representations for $(\mathcal{H}f)^\wedge$ are given.

Lemma 4.1. *If f and φ both belong to $L^1(\mathbb{R})$, and $\mathcal{H}f$ is defined in* (4.3)*, then*

$$(\mathcal{H}f)^\wedge(t) = \frac{1}{|t|} \int_{\mathbb{R}} \widehat{f}(u) \varphi(u/t)\, du, \quad t \ne 0, \tag{4.6}$$

and

$$(\mathcal{H}f)^\wedge(t) = \int_{\mathbb{R}} f(u) \widehat{\varphi}(tu)\, du, \quad t \in \mathbb{R}. \tag{4.7}$$

The proof is routine: integrate by substitution and make use of Fubini's theorem.

Theorem 4.2. *The function $\mathcal{H}f$ defined, for $x \in \mathbb{R}$, by*

$$\mathcal{H}f(x) = \int_{\mathbb{R}} \frac{f(t)}{|t|} \varphi\left(\frac{x}{t}\right) dt \tag{4.8}$$

is in $L^1(\mathbb{R})$ and satisfies (4.3).

In fact, (4.8) can be considered as a direct definition of the Hausdorff operator, the argument around (4.4) is one of the basic reasons to define it via the Fourier transform; of course, it might also be convenient technically.

4.3 Boundedness of the Hausdorff operator on the Hardy space

Let us now proceed to the boundedness of the Hausdorff operator on the Hardy space.

Theorem 4.3. *If $\varphi \in L^1(\mathbb{R})$, then the Hausdorff operator $\mathcal{H} = \mathcal{H}_\varphi : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is bounded.*

The first proof in [60] is not the only one and is probably not the best possible. But it leads to another interesting problem and is of interest by itself. We need two auxiliary well known results (cf. [44, §3.6]).

Lemma 4.2. *If $f \in L^1(\mathbb{R})$ is such that*

$$\widehat{f}(t) = 0 \quad \text{for} \quad t < 0, \tag{4.9}$$

then $f \in H^1(\mathbb{R})$.

The symmetric counterpart of this lemma says that if $f \in L^1(\mathbb{R})$ is such that $\widehat{f}(t) = 0$ for $t > 0$, then $f \in H^1(\mathbb{R})$.

Lemma 4.3. *If $f \in H^1(\mathbb{R})$, then there exist two functions f_1 and f_2, both in $H^1(\mathbb{R})$ such that $f = f_1 + f_2$, and $\widehat{f_1}(t) = 0$ for $t < 0$, while $\widehat{f_2}(t) = 0$ for $t > 0$.*

Now, the proof of Theorem 4.3 is based on these auxiliary results and the closed graph theorem.

Once more, this proof does not seem to be the strongest one. For example, it provides no bound for the norm of the operator, or more precisely, does not state a strong type boundedness inequality. Any multidimensional proof given below provides that in dimension one as well. However, as is mentioned, the above proof leads to an interesting problem we will study in the next subsection.

4.4 Commuting relations

The mentioned problem reads as follows. Two operators were studied above: the Hausdorff operator and the Hilbert transform, for what $\varphi \in L^1(\mathbb{R})$ the two operators commute. The next theorem answers this question.

Theorem 4.4. *Assume $\varphi \in L^1(\mathbb{R})$.*

(i) *We have*

$$(\mathcal{H}_\varphi f)^{\sim} = \mathcal{H}_\varphi \widetilde{f} \quad \text{for all} \quad f \in H^1(\mathbb{R}) \tag{4.10}$$

if and only if

$$\varphi(x) = 0 \quad \text{for almost all} \quad x < 0. \tag{4.11}$$

(ii) *We have*

$$(\mathcal{H}_\varphi f)^\sim = -\mathcal{H}_\varphi \widetilde{f} \quad \text{for all} \quad f \in H^1(\mathbb{R})$$

if and only if

$$\varphi(x) = 0 \quad \text{for almost all} \quad x > 0. \tag{4.12}$$

Lemma 4.2 is used in the proof; in addition we need one more auxiliary result.

Lemma 4.4. *Let* $0 < \delta < a/2$ *and let*

$$g_{\delta,a}(t) := \begin{cases} t/\delta & \text{for} \quad 0 \le t < \delta, \\ 1 & \text{for} \quad \delta \le t \le a-\delta, \\ (a-t)/\delta & \text{for} \quad a-\delta < t \le a, \\ 0 & \text{for} \quad t < 0 \quad \text{or} \quad t > a. \end{cases}$$

Then $g_{\delta,a} \in \widehat{L}^1(\mathbb{R})$*, that is,* $g_{\delta,a}$ *is the Fourier transform of an integrable function.*

This technical lemma might be useful in many problems of harmonic analysis.

4.5 The case $p < 1$

Before proceeding to the case of the Hardy spaces H^p with $p < 1$, which in many respects is rather similar to H^1, let us make certain observations. There is a rather simple result for the Hausdorff operator in L^p, $1 \le p \le \infty$. For these p, Minkowski's inequality in the integral form gives

$$\left\| \int_0^\infty |t^{-1} f(t^{-1}x)\varphi(t)|\,dt \right\|_{L^p_x} \le \int_0^\infty t^{-1} \|f(t^{-1}x)\|_{L^p_x} |\varphi(t)|\,dt$$
$$= \int_0^\infty t^{-1+1/p} \|f\|_{L^p} |\varphi(t)|\,dt = A_{\varphi,p} \|f\|_{L^p},$$

where

$$A_{\varphi,p} = \int_0^\infty t^{-1+1/p} |\varphi(t)|\,dt. \tag{4.13}$$

From this inequality, we see that, if $1 \le p \le \infty$ and $A_{\varphi,p} < \infty$, then (4.8) gives a well-defined bounded linear operator $\mathcal{H}_\varphi$ in L^p.

If $f \in H^p$ with $0 < p < 1$, then $\widehat{f}$ is a continuous function satisfying

$$|\widehat{f}(\xi)| \le C_p \|f\|_{H^p} |\xi|^{1/p-1}$$

(see, e.g., [77, Chapt. III, § 5.4, p. 128]), and hence

$$\int_0^\infty |\widehat{f}(t\xi)\varphi(t)|\, dt \le \int_0^\infty C_p \|f\|_{H^p} |t\xi|^{1/p-1} |\varphi(t)|\, dt = C_p\, A_{\varphi,p} \|f\|_{H^p} |\xi|^{1/p-1}, \quad (4.14)$$

where $A_{\varphi,p}$ for $0 < p < 1$ is given by (4.13) as well. Hence, if $0 < p < 1$, $A_{\varphi,p} < \infty$, and $f \in H^p$, then the right-hand side of (4.3) gives a continuous function of $\xi \in \mathbb{R}$ that is uniformly of $O(|\xi|^{1/p-1})$ and, hence, the tempered distribution $\mathcal{H}_\varphi f$ is well-defined through (4.3). Thus, including also the case $p = 1$ as mentioned above, we give the following definition.

Definition 4.1. *If $0 < p \le 1$ and φ is a measurable function on $(0, \infty)$ with $A_{\varphi,p} < \infty$, then we define the continuous linear mapping $\mathcal{H}_\varphi : H^p \to \mathcal{S}'$ by (4.3).*

Kanjin [50] proved the following theorem.

Theorem 4.5. *Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. Suppose that $A_{\varphi,1} < \infty$, $A_{\varphi,2} < \infty$, and suppose that $\widehat{\varphi}$ is a function of class C^{2M} on $\mathbb{R}$ with $\sup_{\xi\in\mathbb{R}} |\xi|^M |\widehat{\varphi}^{(M)}(\xi)| < \infty$ and $\sup_{\xi\in\mathbb{R}} |\xi|^M |\widehat{\varphi}^{(2M)}(\xi)| < \infty$. Then the Hausdorff operator $\mathcal{H}_\varphi$ is a bounded operator in H^p.*

This theorem contains assumptions on $\widehat{\varphi}$. The proof is based on a more or less regular atomic decomposition.

In the case where $\varphi(t) = \alpha(1-t)^{\alpha-1}$ for $0 < t < 1$ and $\varphi(t) = 0$ otherwise, the operator $\mathcal{H}_\varphi = \mathcal{C}_\alpha$ is called the *Cesàro operator* of order α. Giang and Móricz [28] proved that the Cesàro operator $\mathcal{C}_1$ is bounded in the Hardy space H^1 (of course, this also follows by Theorem 4.3). Kanjin [50] proved that the Cesáro operator $\mathcal{C}_\alpha$ is a bounded operator in H^p provided α is a positive integer and $2/(2\alpha+1) < p < 1$. Kanjin proved this result by using Theorem 4.5. Later on, it was proved in [65] that the Cesáro operator $\mathcal{C}_\alpha$ is a bounded operator in H^p for every $\alpha > 0$ and every $0 < p < 1$. The proof is based on the ideas elaborated in [65] and uses the one-dimensional version of the modified atomic decomposition for H^p given in [64].

Definition 4.2. *Let $0 < p \le 1$ and let M be a positive integer. For $0 < s < \infty$, we define $\mathcal{A}_{p,M}(s)$ as the set of all those $f \in L^2$ for which $\widehat{f}(\xi) = 0$ for $|\xi| \le s^{-1}$ and $\|\widehat{f}^{(k)}\|_{L^2} \le s^{k-1/p+1/2}$ for $k = 0, 1, \cdots, M$. We define $\mathcal{A}_{p,M}$ as the union of $\mathcal{A}_{p,M}(s)$ over all $0 < s < \infty$.*

Lemma 4.5. *Let $0 < p \le 1$ and M be a positive integer satisfying $M > 1/p - 1/2$. Then there exists a constant $c_{p,M}$, depending only on p and M, such that the following statements hold.*

(1) $\|f(\cdot - x_0)\|_{H^p} \le c_{p,M}$ *for all $f \in \mathcal{A}_{p,M}$ and all $x_0 \in \mathbb{R}$;*

(2) *Every $f \in H^p$ can be decomposed as*

$$f = \sum_{j=1}^{\infty} \lambda_j f_j(\cdot - x_j), \tag{4.15}$$

where $f_j \in \mathcal{A}_{p,M}$, $x_j \in \mathbb{R}$, $0 \le \lambda_j < \infty$, *and*

$$\left(\sum_{j=1}^{\infty} \lambda_j^p\right)^{1/p} \le c_{p,M}\|f\|_{H^p},$$

and the series in (4.15) *converges in* H^p. *If* $f \in H^p \cap L^2$, *then this decomposition can be made so that the series in* (4.15) *converges in* L^2 *as well.*

This lemma is given in [64, Lemma 2] except for the assertion on the L^2 convergence. A complete proof of part (2) of the lemma can be found in [65, § 3].

In a recent paper [59] the following generalizations of [65] are obtained.

Theorem 4.6. *Let* $0 < p < 1$, $M = [1/p - 1/2] + 1$, *and let* ϵ *be a positive real number. Suppose* φ *is a function of class* C^M *on* $(0, \infty)$ *such that*

$$|\varphi^{(k)}(t)| \le \min\{t^{\epsilon}, t^{-\epsilon}\} t^{-1/p-k} \quad \textit{for} \quad k = 0, 1, \ldots, M.$$

Then $\mathcal{H}_\varphi$ *is a bounded linear operator in* H^p.

Theorem 4.7. *Let* $0 < p < 1$, $M = [1/p - 1/2] + 1$, *and let* ϵ *and* a *be positive real numbers. Suppose* φ *is a function on* $(0, \infty)$ *such that* $\operatorname{supp}\varphi$ *is a compact subset of* $(0, \infty)$, φ *is of class* C^M *on* $(0, a) \cup (a, \infty)$, *and*

$$|\varphi^{(k)}(t)| \le |t - a|^{\epsilon - 1 - k} \quad \textit{for} \quad k = 0, 1, \ldots, M.$$

Then $\mathcal{H}_\varphi$ *is a bounded linear operator in* H^p.

In this section we have given not all possible results and arguments. On the contrary, we present here only those specific for dimension one and not taking place or unclear for higher dimensions. One of the reasons is that more general approaches in the next section lead to results which, being taken in dimension one, clearly supplement those from the present section. No doubt that any interested reader can easily recognize such results.

5 Hausdorff operators on Euclidean spaces

In the multidimensional case the situation is, as usual, more complicated. The Cesàro means in [30] and the Hausdorff means in [61] were considered in dimension two only for the so-called product Hardy space $H^{11}(\mathbb{R} \times \mathbb{R})$ (the simplest partial case, see the corresponding subsection below):

$$(\mathcal{H}_\varphi f)(x) = \int_{\mathbb{R}^2} \frac{\varphi(u)}{|u_1 u_2|} f\left(\frac{x_1}{u_1}, \frac{x_2}{u_2}\right) du.$$

In [82] these and related results were slightly extended. Necessary and sufficient conditions for fulfillment of the commuting relations were also obtained in [62] for this simple situation. The case of usual Hardy space $H^1(\mathbb{R}^2)$ and moreover $H^1(\mathbb{R}^n)$ seemed to be unsolvable by the used method.

In [7] the problem was solved but for a “strange" Hausdorff type operator

$$(\mathcal{H}_\mu f)(x) = \int_{\mathbb{R}} |u|^{-n} f\left(\frac{x}{u}\right) d\mu(u), \tag{5.1}$$

where $x \in \mathbb{R}^n$, defined by *one-dimensional* averaging. This looks either not quite natural for the multivariate case or at least not giving most general results.

Below, we present a more natural generalization of (4.8) and conditions sufficient for the boundedness of such a naturally defined Hausdorff type operator taking $H^1(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.

5.1 Definition and basic properties

We define the Hausdorff type operator by

$$(\mathcal{H}f)(x) = (\mathcal{H}_\Phi f)(x) = \int_{\mathbb{R}^n} \Phi(u) f\big(xA(u)\big) du,$$

where $A = A(u) = (a_{ij})_{i,j=1}^n = \big(a_{ij}(u)\big)_{i,j=1}^n$ is the $n \times n$ matrix with the coefficients $a_{ij}(u)$ being measurable functions of u. This matrix may be singular on at most a set of measure zero; $xA(u)$ is the row n-vector obtained by multiplying the row n-vector x by the matrix $A(u)$. As an example, we mention that the corresponding Cesàro operator is given by

$$\Phi(u)|\det A^{-1}(u)| = \chi_{\{|\det A^{-1}(u)| \le 1\}}(u).$$

A similar definition of the Hausdorff operator was given in [12] (the only difference is that in [12] as well as in [66] the transformation $\mathcal{H}$ is considered to be a row vector and thus $f(A(u)x)$ stands in place of $f(xA(u))$; moreover, in [66] only diagonal matrices A with the diagonal entries equal to one another are studied), along with the following basic properties. Comparing the introduced definition with (5.1), one sees that it is possible to take $u \in \mathbb{R}^m$ for any $1 \le m \le n$, with subsequent m-dimensional averaging.

Let Φ satisfy the condition

$$\|\Phi\|_{L_A} = \int_{\mathbb{R}^n} |\Phi(u)\ \det A^{-1}(u)|\, du < \infty,$$

or, for similarity with the one-dimensional case, $\varphi(u) = \Phi(u) \det A^{-1}(u) \in L^1(\mathbb{R}^n)$. As in the one-dimensional case, before proceeding to the Hardy space, one must be sure in the L^1 boundedness of the corresponding Hausdorff operator.

Lemma 5.1. *The operator $\mathcal{H}f$ is bounded taking $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ provided $\Phi \in L_A$. Furthermore, there holds*

$$\|\mathcal{H}f\|_{L^1(\mathbb{R}^n)} \le \|\Phi\|_{L_A}\|f\|_{L^1(\mathbb{R}^n)}.$$

The proof follows by applying Fubini's theorem and by substituting $xA(u) = v$ (or $x = vA^{-1}(u)$).

If we again recall the discussion around (4.4), we shall realize that it is quite important to have an explicit formula for the Fourier transform of $\mathcal{H}f$ via the Fourier transform of f. First, let us define the latter as

$$\widehat{f}(u) = \int_{\mathbb{R}^n} f(x)e^{-i<u,x>}\,dx,$$

where $< u, x >= u_1x_1 + ... + u_nx_n$. For an integrable function f its Fourier transform is well defined. Let B^T be transposed to the matrix B.

Lemma 5.2. *Let $f \in L^1(\mathbb{R}^n)$ and $\Phi \in L_A$. The Fourier transform of $\mathcal{H}f$ is represented by the formula*

$$(\mathcal{H}f)\widehat{\ }(y) = \int_{\mathbb{R}^n} \Phi(u)\,|\det A^{-1}(u)|\,\widehat{f}\big(y(A^{-1})^T(u)\big)\,du.$$

It is worth mentioning that in dimension one a sort of symmetry takes place for f and φ in various representations of the Hausdorff operator (see Lemma 4.1 from the previous section). It turns out that this is an accidental circumstance, meaningless for several dimensions.

We can easily find the adjoint operator $\mathcal{H}^*$ as the one satisfying, for appropriate ("good") functions f and g,

$$\int_{\mathbb{R}^n} (\mathcal{H}f)(x)g(x)\,dx = \int_{\mathbb{R}^n} (\mathcal{H}^*g)(x)f(x)\,dx. \tag{5.2}$$

It is defined (compare with [12] and [66]) as

$$(\mathcal{H}^*f)(x) = (\mathcal{H}^*_{\Phi,A}f)(x) = \int_{\mathbb{R}^n} \Phi(u)|\det A^{-1}(u)|\,f(xA^{-1}(u))\,du.$$

This operator is also of Hausdorff type. Indeed, it actually can be written as $\mathcal{H}_{\psi,B}f$, where $\psi(u) = \Phi(u)|\det A^{-1}(u)|$ and $B = A^{-1}(u)$. Therefore the conditions for its boundedness readily follow from those for the initial operator.

The key ingredient in one of the proofs is Lemma 5.4 (see below) on the behavior in u of the BMO-norm of $f(xA(u))$. This also allows us to get conditions for the boundedness of both operators in $BMO(\mathbb{R}^n)$.

A different approach directly yields the bound for the H^1 norm of $f(xA(u))$, it is given in Lemma 5.3 below.

5.2 Real Hardy spaces in several dimensions

There are various definitions of Hardy spaces. Since each may become the one that helps to ensure the sharpness, we will give few of them.

Let ψ be a real-valued differentiable function on $\mathbb{R}^n$ which satisfies:

$$\begin{aligned}&(i) \qquad |\psi(x)| \ll (1+|x|)^{-n-1},\ |\nabla\psi(x)| \ll (1+|x|)^{-n-1},\\ &(ii) \qquad \int_{\mathbb{R}^n} \psi(x)dx = 0.\end{aligned}$$

Write $\psi_t(x) = \psi(x/t)t^{-n}$, $t > 0$. Given a function f with

$$\int_{\mathbb{R}^n} |f(x)|(1+|x|)^{-n-1}dx < \infty,$$

define the Lusin area integral $S_\psi f$ by

$$S_\psi f(x) = \left(\int_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x)$ is the cone $\{(y,t) : |y-x| < t\}$.

Given any Schwartz function η with $\int \eta \neq 0$, define the non-tangential maximal function by

$$M_\eta f(x) = \sup_{t>0} |f * \eta_t(x)|.$$

Classical results due to Ch. Fefferman and Stein (see [21]) state that

$$\|f\|_{H^1} \asymp \|M_\eta f\|_{L^1} \asymp \|S_\psi f\|_{L^1}. \tag{5.3}$$

Let us outline how an eventual proof of the boundedness of the Hausdorff operator could make use of the above definitions. By Fubini's theorem,

$$\begin{aligned}&\int_{\mathbb{R}^n} \eta_t(\xi)d\xi \int_{\mathbb{R}^n} \Phi(u) f\big((y-\xi)A(u)\big)du\\ \le\ &\int_{\mathbb{R}^n} |\Phi(u)|du \Big|\int_{\mathbb{R}^n} f\big((y-\xi)A(u)\big)\eta_t(\xi)d\xi\Big|.\end{aligned}$$

Substituting $\xi A(u) = v$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} f\big((y-\xi)A(u)\big)\eta_t(\xi)d\xi\\ =\ &\int_{\mathbb{R}^n} f(yA(u)-v)\eta_{\frac{t}{|\det A^{-1}(u)|^{1/n}}}\left(\frac{vA^{-1}(u)}{|\det A^{-1}(u)|^{1/n}}\right)dv.\end{aligned}$$

Hence, if

$$\frac{vA^{-1}(u)}{|\det A^{-1}(u)|^{1/n}} = v$$

for all v, then

$$M_\eta(\mathcal{H}_\Phi f)(y) \le \mathcal{H}_{|\Phi|}(M_\eta f)(y).$$

All this looks straitforward and natural, nevertheless it is not clear how to continue these calculations to derive something essential. We only wish to mention that in dimension one we can definitely obtain a proof of the boundedness of the Hausdorff operator on the Hardy space in this way, as well as the bound for the norm.

To overcome the difficulties of the multivariate case, we will make use of the other three definitions.

1. A very natural is the one via the Riesz transforms, n singular integral operators, as an analogue of the one-dimensional definition based on the unique singular operator - the Hilbert transform. The jth, $j = 1, 2, ..., n$, Riesz transform can be defined either explicitly

$$R_j f(x) = \frac{\Gamma(n/2 + 1/2)}{\pi^{n/2+1/2}} \int_{\mathbb{R}^n} \frac{u_j}{|u|^{n+1}} f(x-u)\, du,$$

or via the Fourier transform

$$\widehat{R_j f}(x) = i\frac{x_j}{|x|}\widehat{f}(x).$$

For $n = 1$ this is just the Hilbert transform. It is well known (see, e.g., [75, Ch.VII, § 3]) that

$$\|f\|_{H^1(\mathbb{R}^n)} \asymp \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)} = \sum_{p=0}^{n} \int_{\mathbb{R}^n} |R_p f(x)|\, dx,$$

where $R_0 f \equiv f$. This definition could be perfect if clear conditions for commuting of the Hausdorff operator and Riesz transforms existed. Unfortunately, no such results exist unless A is a diagonal matrix. Nevertheless, we will use this definition along with the others.

2. The definition using an atomic decomposition of the Hardy space proved to be very effective (see, e.g., [18]). We will give and apply the simplest version of an atomic decomposition. Let a denote an atom (more precisely, $(1, \infty, 0)$-atom), a function with *compact support*:

$$\operatorname{supp} a \subset B(x_0, r); \tag{5.4}$$

which satisfies the following *size condition* (L^∞ normalization)

$$\|a\|_\infty \le \frac{1}{|B(x_0,r)|}; \tag{5.5}$$

and the *cancellation condition*

$$\int_{\mathbb{R}^n} a(x)\,dx = 0. \tag{5.6}$$

Here $B(x_0,r)$ denotes the ball of radius r centered at x_0.

It is well known that

$$\|f\|_{H^1} \sim \inf\left\{\sum_k |c_k| : f(x) = \sum_k c_k a_k(x)\right\}, \tag{5.7}$$

where a_k are the above described atoms.

3. The dual space approach (see, e.g., [21]) is also one of the most effective and important. It employs the standard [1] procedure of linearizing the norm

$$\|h\|_{H^1(\mathbb{R}^n)} = \sup_{\|g\|_* \le 1} \left| \int_{\mathbb{R}^n} h(x)g(x)\,dx \right|, \tag{5.8}$$

where g is taken to be infinitely smooth and with compact support [2] and the semi-norm $\|g\|_*$ is that in BMO :

$$\|g\|_* = \sup_Q \inf_c \frac{1}{|Q|} \int_Q |g(x) - c|\,dx,$$

where Q is a ball, $|Q|$ is its Lebesgue measure, and the supremum is taken over all such balls.

5.3 Main result

We denote

$$\|B\|_1 = \|B(u)\|_1 = \max_j(|b_{1j}(u)| + ... + |b_{nj}(u)|),$$

[1]Very often, like in [21], to prove that no other linear functionals exist except those "standard" becomes a challenging task

[2]Such family of functions endowed with the BMO norm is known as VMO; however some authors call VMO a different space while the present one they call CMO, see, e.g., [10]

where b_{nj} are the entries of the matrix B, to be the operator ℓ-norm. We will say that $\Phi \in L^1_B$ if

$$\|\Phi\|_{L^1_B} = \int_{\mathbb{R}^n} |\Phi(u)| \, \|B(u)\|_1^n du < \infty.$$

The following result in [56] ensures the boundedness of Hausdorff type operators in $H^1(\mathbb{R}^n)$ for general matrices A.

Theorem 5.1. *The Hausdorff operator $\mathcal{H}f$ is bounded on the real Hardy space $H^1(\mathbb{R}^n)$ provided that $\Phi \in L^1_{A^{-1}}$, with*

$$\|\mathcal{H}f\|_{H^1(\mathbb{R}^n)} \le \|\Phi\|_{L^1_{A^{-1}}} \|f\|_{H^1(\mathbb{R}^n)}. \tag{5.9}$$

The proof using duality arguments is based on (5.8). The difference in conditions $\Phi \in L_{A^{-1}}$ and $\Phi \in L^1_{A^{-1}}$ seems to be quite natural. The main case when they coincide is that where A is a diagonal matrix with equal entries on the diagonal - this is the subject of [66]. In [56] and then in [57] the problem of the sharpness of Theorem 5.1 was posed. The proof of the following slightly less restrictive condition is based on an atomic decomposition of $H^1(\mathbb{R}^n)$. We will discuss and compare both results afterwards.

Let $\|B\|_2 = \max_{|x|=1} |Bx^T|$, where $|\cdot|$ denotes the Euclidean norm. It is known (see, e.g., [46, Ch. 5, 5.6.35]) that this norm does not exceed any other matrix norm. We will say that $\Phi \in L^2_B$ if

$$\|\Phi\|_{L^2_B} = \int_{\mathbb{R}^n} |\Phi(u)| \, \|B(u)\|_2^n du < \infty.$$

The following result is true.

Theorem 5.2. *The Hausdorff operator $\mathcal{H}f$ is bounded on the real Hardy space $H^1(\mathbb{R}^n)$ provided that $\Phi \in L^2_{A^{-1}}$. Furthermore,*

$$\|\mathcal{H}f\|_{H^1(\mathbb{R}^n)} \ll \|\Phi\|_{L^2_{A^{-1}}} \|f\|_{H^1(\mathbb{R}^n)}. \tag{5.10}$$

The proof is based on estimates of the multiple of transformed atoms. As is mentioned above, the obtained condition (5.10) is weaker that (5.9) but of course still more restrictive than (5.1). It is weaker in the sense that the $\|\cdot\|_2$ matrix norm is smaller than any other matrix norm. On the other hand, all norms in the finite-dimensional space are equivalent. However, having the $\|\cdot\|_2$ matrix norm as a bound is not meaningless, since otherwise the problem of a sharp constant, more precisely, its dependence on the dimension, the so-called Goldberg's problem (see, e.g., [34]) may appear.

5.4 BMO estimates

Analyzing the proof in [58], one can see that the main point is the following result.

Lemma 5.3. *Let* $F(x,u) = f(xB(u))$. *Then*

$$\|F(\cdot,u)\|_{H^1} \ll \|B^{-1}(u)\|_2^n \, \|f\|_{H^1}, \quad u \in \mathbb{R}^n.$$

The obtained estimate might be of interest by itself. Similarly, One of the advantages of the duality proof in [56] is the following lemma.

Lemma 5.4. *Let* $F(x,u) = f(xB(u))$. *Then*

$$\|F(\cdot,u)\|_* \le \|B(u)\|^n |\det B^{-1}(u)| \|f\|_*, \quad u \in \mathbb{R}^n.$$

With this lemma as a tool in hand, we can obtain results on the boundedness of Hausdorff type operators in $BMO(\mathbb{R}^n)$, that is, both Hausdorff operator and its adjoint. The interested reader can easily formulate corresponding results or see it in [56].

5.5 Product Hardy spaces

We will now proceed to sufficient conditions for the boundedness of Hausdorff type operators in the product Hardy space $H^1_m(\mathbb{R}^n) = H^1(\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_m})$, $n_1 + n_2 + ... + n_m$. Trivially, $H^1_1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$. These spaces are of interest and importance in certain problems of Fourier Analysis (see, e.g., [22], [29]).

No conditions for the boundedness of Hausdorff operators on such spaces as general as those for $H^1(\mathbb{R}^n)$ are known. The main restriction is posed on matrices A. Let

$$A(u) = A_1(u) \oplus ... \oplus A_m(u)$$

be *block diagonal* with square matrices (blocks) A_j be (almost everywhere) non-singular $n_j \times n_j$ matrices and with zero entries off the blocks. Such matrices are by no means artificial and are of importance in various subjects (see, e.g., [46]).

All the information we need on H^1_m one can find in one source [81]. We give it immediately. For an appropriate function f, first of all for a tempered distribution f, the Riesz operators $R_{j_1,...,j_m}$, $j_p = 0, 1, ..., n_p$; $p = 1, 2, ..., m$, are defined at $x \in \mathbb{R}^n$ by

$$(R_{j_1,...,j_m} f)^\wedge(x) = \left(\prod_{p=1}^{m} \left(i \frac{x_{n_1+...+n_{p-1}+j_p}}{|x^{(p)}|} \right) \right) \widehat{f}(x),$$

where

$$x^{(p)} = (x_{n_1+\ldots+n_{p-1}+1}, \ldots, x_{n_p}) \in \mathbb{R}^{n_p}.$$

Of course, $R_{0,\ldots,0}f = f$. There hold

$$\|R_{j_1,\ldots,j_m}f\|_{H^1_m} \ll \|f\|_{H^1_m} \tag{5.11}$$

and

$$\|f\|_{H^1_m} \asymp \sum_{p=1}^{m} \sum_{j_p=0}^{n_p} \|R_{j_1,\ldots,j_m}f\|_{L^1}. \tag{5.12}$$

The next result can immediately be derived from Theorem 5.2.

Theorem 5.3. *If the matrix A is block diagonal as above, then the Hausdorff operator $\mathcal{H}f$ is bounded on the real Hardy space H^1_m provided that*

$$\|\mathcal{H}f\|_{H^1_m} \le C_{n,m} \int_{\mathbb{R}^n} |\Phi(u)| \prod_{p=1}^{m} \sum_{j_p=0}^{1} \Delta_p^{j_p}(u)\, du < \infty,$$

where

$$\Delta_p^{j_p}(u) = \begin{cases} |\det A_p^{-1}(u)|, & j_p = 0, \\ ||A_p^{-1}(u)||_2^{n_p}, & j_p = 1. \end{cases}$$

Of course, this theorem covers the main result in [61]; the latter is just the simplest partial case. Unfortunately, no condition exists for a general matrix A to ensure the boundedness of the Hausdorff operator on the product Hardy space. However, block diagonal matrices have the advantage that each block naturally transforms only the corresponding group of variables not touching the others.

In [16] Hausdorff operators are studied on the product Besov spaces and on the local product Hardy space.

6 Open problems

Like some people who start reading detective stories from the final pages, not being patient enough to wait for the end of investigation, some may start from this section in order to learn about the prospects of the topic and to get food for thought immediately. In this section we overview open problems on Hausdorff operators in various settings. Some of them have, to a certain extent, already been mentioned in the text. In fact, the paper [57] is devoted to open problems in the topic, however, certain progress has been made during the passed time.

6.1 Power series

We formulate certain open problems that naturally arise from the above scrutiny in Section 3.

a) *Multivariate versions of the results in* Section 3.

Till recently the only known direct generalization was given [14], where the above results were extended in the simplest, product-wise way to the case of the polydisk in $\mathbb{C}^n$. In [4] much more is obtained by new characterization of Hardy spaces on Reinhardt domains. However, much is still can be done. For instance, boundedness of the quasi-Hausdorff operators in H^∞ is proved in a much more sophisticated way in dimension one (see [24]) and is open in several dimensions. For the unit ball in $\mathbb{C}^n$ only sufficient conditions are known, find necessary ones.

We note that very recently the results of [4] have been extended to much wider classes of domains in $\mathbb{C}^n$ by Aizenberg, Vidras and the author in [5].

It is worth mentioning [8], where the Cesàro means are studied in the upper half-plane. As in the other settings, it may give rise to further study of more general Hausdorff operators.

b) *Study partial sums of (3.2) as well as of its generalization for the Hausdorff and maybe quasi-Hausdorff matrices.*

c) *Find an example of a function (power series) NOT in H^1 for which the Hausdorff means or even the Cesàro means are in H^1; the same, of course, for the quasi-Hausdorff means.*

6.2 Fourier transform setting

In this subsection we overview open problems on Hausdorff operators in the Fourier transform setting both in one and several dimensions.

a) The main one reads as follows:

Given a Hausdorff operator (or even the Cesàro means), construct a counterexample of a function in L^1 but not in H^1 whose value taken by this Hausdorff operator is in H^1 (compare with **c)** *from the previous subsection).*

b) *Prove (or disprove) the sharpness of the obtained condition for the boundedness of the Hausdorff operator on $H^1(\mathbb{R}^n)$.*

c) *The same - similarly or instead - for* BMO.

d) *Find other (based on different definitions of $H^1(\mathbb{R}^n)$) proofs of the boundedness of the Hausdorff operator on $H^1(\mathbb{R}^n)$; with the same condition or maybe a BETTER one.*

e) Find conditions for commuting relations in $H^1(\mathbb{R}^n)$ (see Subsection 4.4 and the beginning of Section 5).

f) As we have seen above, the scale of spaces very similar to H^1 is that of H^p for $p < 1$. However, they differ much both in results and methods. The boundedness of the Cesàro means on $H^p(\mathbb{R})$ for all $0 < p < 1$ was proved in [65].

Find conditions for the boundedness of the Hausdorff operators in H^p for $p<1$ in the multidimensional case.

6.3 Partial integrals

We know from Lemma 5.2 that

$$(\mathcal{H}f)\widehat{\ }(y)=\int_{\mathbb{R}^n}\Phi(u)\,|\det A^{-1}(u)|\,\widehat{f}\big(y(A^{-1})^T(u)\big)\,du,$$

where $(A^{-1})^T$ is the transpose of A^{-1}.

Taking

$$\int_{|x|\le N}(\mathcal{H}f)\widehat{\ }(y)e^{i\langle x,y\rangle}dy,$$

we arrive at

$$\mathcal{H}_Nf(x)=\int_{\mathbb{R}^n}\Phi(u)|\det A^{-1}|\,du\int_{\mathbb{R}^n}f(v)\,dv\int_{|y|\le N}e^{-i\langle x-vA^{-1}(u),y\rangle}dy.$$

The last integral is well known (see, e.g., [78, Ch.IV, § 3]), and we get

$$(2\pi N)^{-\frac{n}{2}}\mathcal{H}_Nf(x)$$

$$=\int\limits_{\mathbb{R}^{2n}}\Phi(u)|\det A^{-1}|\,f(v)\,|x-vA^{-1}(u)|^{-\frac{n}{2}}J_{\frac{n}{2}}(N|x-vA^{-1}(u)|)\,du\,dv,$$

where $J_{\frac{n}{2}}$ is the Bessel function of order $\frac{n}{2}$, with f either in L^1 or in H^1. In dimension one it looks extremely simple: with $\varphi\in L^1$

$$\mathcal{H}_N^{\varphi}f(x)=\int_{\mathbb{R}^2}\varphi(u)f(t)\frac{\sin N(x-ut)}{x-ut}\,du\,dt.$$

A group of problems related to this is as follows.

a) *Study $\mathcal{H}_Nf$ or maybe $\mathcal{H}_N^*f=\sup_N|\mathcal{H}_Nf|$.*

b) *Find the rate - in N - of approximation to f by $\mathcal{H}_Nf$, almost everywhere, or in L^1 or H^1 norm.*

6.4 More problems

Certain possible problems worth being studied become apparent in discussions during the conference in Istanbul, 2006, first of all with O. Martio.

One of them is to consider general singular operators rather than those defined by the Hilbert or Riesz transforms.

We mention also the question the author was asked after his talk at that conference about *compactness* of Hausdorff operators.

Back to the BMO proof of the boundedness of the Hausdorff operator on $H^1(\mathbb{R}^n)$, instead of studying relation between the BMO norms of $f(xA(u))$ and $f(x)$ (see Lemma 5.4) one can try the same for $f(F_x(u))$ where $F_x(u)$ is a general family of mappings. A natural assumption on this family is to preserve BMO. In this context $f(xA(u))$ is a partial case when the mapping is linear. The latter definitely preserves BMO, and the only point in the above study was to figure out the bounds.

But the problem of preserving BMO is already solved in general case: necessary and sufficient conditions are given by P. Jones [49] (see also [6]) in dimension one and by Gotoh [38] in the multivariate setting; see also [10].

This leads to the study of a very general Hausdorff operator

$$(\mathcal{H}_{\Phi,F}f)(x) = \int_{\mathbb{R}^n} \Phi(u) f\big(F_x(u)\big)\, du.$$

All the above problems for such operators not only were never studied but even simplest initial results for them, such as Lemma 5.2, face essential difficulties. Considerable amount of them come from the unavoidable necessity to make use of implicit function theorems.

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