ISSN 2077–9879

Eurasian Mathematical Journal

2013, Volume 4, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 4, Number 4 (2013), 64 – 87

LINEAR VOLTERRA INTEGRO-DIFFERENTIAL SECOND-ORDER EQUATIONS UNRESOLVED WITH RESPECT TO THE HIGHEST DERIVATIVE

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Communicated by V.I. Burenkov

Key words: Volterra integro-differential equation, C_0-semigroup, self-adjoint operator.

AMS Mathematics Subject Classification: 39B42, 39B99.

Abstract. We consider the Cauchy problem for Volterra integro-differential second-order linear equations which describe an evolution of dynamical systems with infinite numbers of degrees of freedom taking into account relaxation effects. Existence theorems for strong solutions for three classes of complete integro-differential second-order equations are obtained.

1 Introduction

In this paper we study the Cauchy problem for the Volterra integro-differential second-order equation in Hilbert space $\mathcal{H}$ of the following form

$$A\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu + \sum_{k=1}^{m} \int_0^t G_k(t,s)C_k u(s)ds = f(t), \quad u(0) = u^0,\ u'(0) = u^1. \tag{1.1}$$

Such equations describe evolution of dynamical systems with infinite numbers of degrees of freedom taking into account relaxation effects. The unknown function $u = u(t)$ with values in $\mathcal{H}$ describes the field of system displacements relative to the equilibrium state. The physical meanings of the operator coefficients in (1.1) are the following. A is a kinetic energy operator and therefore $A = A^* > 0$. Next, B is a potential energy operator; if the equilibrium state of the system is statically stable, then $B = B^* \geqslant 0$. The operator $F = F^* \geqslant 0$ takes into account energy dissipation while the operator $G = G^*$ describes Coriolis (gyroscopic) forces action. Finally, the integral terms take into account relaxation effects.

In this paper, A is supposed to be a bounded operator ($A \in \mathcal{L}(\mathcal{H})$) and the coefficients F, G, B, C_k are assumed to be unbounded noncommuting operators with the domains of definition dense in $\mathcal{H}$. These operators are compared by their domains of definition, that is we consider such classes of equations which have a unique so-called main operator; it has the narrowest domain of definition compared with the other operators.

The investigation of problem (1.1) is based on the methods stated in [2] for the case $A = I$, where I is the identity operator. We specify also monograph [6] where Cauchy problems are investigated for integro-differential and functional equations as well as corresponding spectral problems for the case when one of coefficients is the main operator while others are its powers.

The following facts will be essentially used in proving the basic statements of our paper.

Theorem 1.1. *Suppose we are given the following Volterra integral equation of the second kind*

$$u(t) - \int_0^t V(t,s)u(s)ds = f(t), \quad 0 \leqslant t \leqslant T. \tag{1.2}$$

Let the following conditions be satisfied:

1° *the function f with values in Banach space $\mathcal{E}$ is continuous in t, i.e.*

$$f \in C([0,T];\mathcal{E}). \tag{1.3}$$

2° *the operator-function $V(t,s)$ on the triangle $\Delta_T := \{(t,s) : \ 0 \leqslant s \leqslant t \leqslant T\}$ is strongly continuous in both variables and takes values in $\mathcal{L}(\mathcal{E})$, briefly*

$$V(t,s) \in SC(\Delta_T;\mathcal{L}(\mathcal{E})). \tag{1.4}$$

Then problem (1.2) *has a unique solution $u \in C([0,T];\mathcal{E})$, and it is possible to get this solution by the method of successive approximations.*

Theorem 1.2. *(see [2, p.16-25]). Let us consider the following Cauchy problem for Volterra integro-differential first-order equation*

$$\frac{du}{dt} + Fu + \sum_{k=1}^{m} \int_0^t G_k(t,s)C_k u(s)ds = f(t), \quad u(0) = u^0. \tag{1.5}$$

Let the following conditions be satisfied:

1° *operator $(-F)$ is a generator of C_0-semigroup;*

2° $f \in C^1([0,T];\mathcal{E})$;

3° $u^0 \in \mathcal{D}(F)$;

4° $\mathcal{D}(C_k) \supset \mathcal{D}(F), \quad k = \overline{1,m}$,

5° $G_k,\ \partial G_k/\partial t \in C(\Delta_T;\mathcal{L}(\mathcal{E})),\ k = \overline{1,m}$.

Then problem (1.5) *has a unique strong solution on the segment $[0,T]$, i.e. such function u, for which all the terms in* (1.5) *are elements of $C([0,T];\mathcal{E})$, and the initial condition $u(0) = u^0$ is satisfied.*

2 Incomplete linear Volterra integro-differential second-order equations unresolved with respect to the highest derivative

2.1 The Cauchy problem. The first approach

Let $\mathcal{H}$ be an arbitrary Hilbert space. We consider the case of the implicit integro-differential equation of form (1.1) when $F = G = 0$:

$$A\frac{d^2u}{dt^2} + Bu + \sum_{k=1}^{m} \int_0^t G_k(t,s)C_k u(s)ds = f(t), \quad u(0) = u^0, \quad u'(0) = u^1. \tag{2.1}$$

It should be added here that u is the unknown function, f is a given function, $A > 0$ is a positive bounded operator, $B = B^* \geq 0$ is a positive definite operator defined on the domain $\mathcal{D}(B) \subset \mathcal{H}$, $G_k(t,s)$ are bounded operator functions acting in $\mathcal{H}$, C_k, $k = \overline{1,m}$, are unbounded operators defined on the domains $\mathcal{D}(C_k) \subset \mathcal{H}$.

Remark 1. Let us assume that the operator A acts in the scale of spaces $\mathcal{E}^\alpha$, but not in $\mathcal{H}$. Let $\mathcal{E}^\alpha$, $\alpha > 0$, be the domain of definition of the power $(A^{-1})^\alpha$ of the operator A^{-1} defined on $\mathcal{D}(A^{-1}) = \mathcal{R}(A) \subset \mathcal{H}$. Then

$$\mathcal{H} = \mathcal{E}^0, \quad \mathcal{D}(A^{-1}) = \mathcal{E}^1, \quad \mathcal{D}(A^{-1/2}) = \mathcal{E}^{1/2}, \tag{2.2}$$

and $A^{-1/2} : \mathcal{E}^{\alpha/2} \to \mathcal{E}^{(\alpha-1)/2}$ is a bounded operator.

Let us define a strong solution of (2.1) with values in $\mathcal{E}^{1/2} = \mathcal{D}(A^{-1/2})$ taking into account Remark 1.

Definition 1. We call a function u on the segment $[0,T]$, with values in $\mathcal{E}^{1/2} = \mathcal{D}(A^{-1/2})$, a strong solution to Cauchy problem (2.1) if all the following conditions are satisfied:

1° $u \in C([0,T]; \mathcal{D}(A^{-1/2}B))$;
2° $u' \in C([0,T]; \mathcal{D}(B^{1/2}))$, $\quad u'' \in C([0,T]; \mathcal{E}^{-1/2})$;
3° all terms in (2.1) belong to $C([0,T]; \mathcal{E}^{1/2})$;
4° for all $t \in [0,T]$ equation (2.1) is true;
5° the initial conditions $u(0) = u^0$, $u'(0) = u^1$ are satisfied.

We note that the conditions

$$u^0 \in \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(B^{1/2}), \quad f \in C([0,T]; \mathcal{D}(A^{-1/2})) \tag{2.3}$$

are necessary for the existence of a strong solution with values in $\mathcal{D}(A^{-1/2})$ for problem (2.1) on the segment $[0,T]$.

If equation (2.1) has a strong solution with values in $\mathcal{E}^{1/2} = \mathcal{D}(A^{-1/2})$, then the first term in (2.1) can be rearranged in the equivalent forms taking into account Remark 1:

$$A\frac{d^2u}{dt^2} = \frac{d^2}{dt^2}(Au) = A^{1/2}\frac{d^2}{dt^2}(A^{1/2}u) \in C([0,T]; \mathcal{D}(A^{-1/2})). \tag{2.4}$$

Our aim is to find restrictions on the operators B, C_k and the operator-functions $G_k(t,s)$, $k = \overline{1,m}$, which ensure the existence and uniqueness theorem for a strong solution with values in $\mathcal{D}(A^{-1/2})$ of problem (2.1).

If we go over to the immediate consideration of problem (2.1), we note that it can be changed to the equivalent problem for the integro-differential first-order equation in the orthogonal sum of the spaces $\mathcal{H}^2 := \mathcal{H} \oplus \mathcal{H}$. On this way let problem (2.1) have a strong solution u with values in $\mathcal{D}(A^{-1/2})$. We replace in (2.1) the unknown function u by the new function v according to the rule $A^{1/2}u(t) =: v(t)$ and act on both parts of (2.1) by $A^{-1/2} \in \mathcal{L}(\mathcal{D}(A^{-1/2}), \mathcal{H})$. Then the following Cauchy problem arises

$$\frac{d^2v}{dt^2} + A^{-1/2}BA^{-1/2}v + \sum_{k=1}^{m}\int_0^t A^{-1/2}G_k(t,s)C_kA^{-1/2}v(s)ds = A^{-1/2}f(t), \tag{2.5}$$

$$v(0) = A^{1/2}u^0, \quad v'(0) = A^{1/2}u^1,$$

and all the terms in this equation belong to $C([0,T];\mathcal{H})$.

Let us introduce the new unknown function w by the relations:

$$-iB^{1/2}A^{-1/2}v(t) =: \frac{dw}{dt}, \quad w(0) = 0. \tag{2.6}$$

From condition 2° of Definition 1 it follows that $w \in C^2([0,T];\mathcal{H})$ and

$$\frac{d^2w}{dt^2} + iB^{1/2}A^{-1/2}\frac{dv}{dt} = 0, \quad w'(0) = -iB^{1/2}u^0. \tag{2.7}$$

Let us also transform the integral terms in (2.5) by using the formula

$$v(s) = \int_0^s v'(\xi)d\xi + v(0)$$

and change the order of integration. We obtain

$$\int_0^t A^{-1/2}G_k(t,s)C_kA^{-1/2}\left(\int_0^s v'(\xi)d\xi + v(0)\right)ds$$

$$= \int_0^t \left(\int_\xi^t A^{-1/2}G_k(t,s)C_kA^{-1/2}ds\right)v'(\xi)d\xi + \int_0^t A^{-1/2}G_k(t,s)C_kA^{-1/2}v(0)ds, \ k = \overline{1,m}.$$

Let us introduce the following notation:

$$\text{a)} \quad \begin{gathered}\hat{G}_k(t,s) := A^{-1/2}G_k(t,s)A^{1/2}, \quad \hat{C}_k := A^{-1/2}C_kA^{-1/2}, \\ \widehat{G}_k(t,s)\widehat{C}_k = A^{-1/2}G_k(t,s)C_kA^{-1/2},\end{gathered} \tag{2.8}$$

$$\text{b)} \quad \begin{gathered}\breve{G}_k(t,s) := A^{-1/2}G_k(t,s), \quad \breve{C}_k := C_kA^{-1/2}, \\ \breve{G}_k(t,s)\breve{C}_k = A^{-1/2}G_k(t,s)C_kA^{-1/2},\end{gathered} \tag{2.9}$$

and hereinafter we will consider the two cases corresponding to (2.8) and (2.9).

For case (2.8) with accounting (2.6), (2.7) problem (2.5) is equivalent to the Cauchy problem for the integro-differential first-order equation of the following form:

$$\frac{dz}{dt} + i\mathcal{B}z + \sum_{k=1}^m \int_0^t \widetilde{V}_k(t,\xi)\widetilde{C}_kz(\xi)d\xi = \widetilde{f}(t), \quad z(0) = z^0 := (A^{1/2}u^1; -iB^{1/2}u^0)^\tau, \tag{2.10}$$

where

$$\begin{gathered}z(t) := (v'(t); w'(t))^\tau \in \widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}, \\ \widetilde{f}(t) := (A^{-1/2}f(t) - \textstyle\sum_{k=1}^m \int_0^t A^{-1/2}G_k(t,s)C_ku^0ds; 0)^\tau,\end{gathered} \tag{2.11}$$

$$\begin{gathered}\mathcal{B} := \begin{pmatrix} 0 & A^{-1/2}B^{1/2} \\ B^{1/2}A^{-1/2} & 0 \end{pmatrix}, \\ \mathcal{D}(\mathcal{B}) := \mathcal{D}(B^{1/2}A^{-1/2}) \oplus \mathcal{D}(A^{-1/2}B^{1/2}) = \mathcal{R}(A^{1/2}B^{-1/2}) \oplus \mathcal{R}(B^{-1/2}A^{1/2}),\end{gathered} \tag{2.12}$$

and the operators $\widetilde{V}_k(t,\xi)$ and $\widetilde{C}_k$ are defined by the formulas

$$\widetilde{V}_k(t,\xi) := \operatorname{diag}(\widehat{V}_k(t,\xi); 0), \quad \widehat{V}_k(t,\xi) := \int_\xi^t \widehat{G}_k(t,s)ds = \int_\xi^t A^{-1/2}G_k(t,s)A^{1/2}ds, \tag{2.13}$$

$$\widetilde{C}_k := \mathrm{diag}(\widehat{C}_k; 0), \quad \mathcal{D}(\widetilde{C}_k) := \mathcal{D}(\widehat{C}_k) \oplus \mathcal{H}, \quad k = \overline{1,m}. \tag{2.14}$$

(Here the symbol $(\cdot;\cdot)^\tau$ means the transposition operation, in this case the transposition operation of a row vector.)

Note that in (2.10) the operator $\mathcal{B}$ is self-adjoint and therefore the operator $-i\mathcal{B}$ is a generator of a unitary operator group, and in particular, the generator of the C_0-semigroup. That is why for problem (2.10)-(2.14) the assertions of Theorem 1.2 are true under the following conditions:

1° $\widetilde{f} \in C^1([0,T];\mathcal{H}^2)$;
2° $z^0 \in \mathcal{D}(\mathcal{B})$;
3° $\mathcal{D}(\widetilde{C}_k) \supset \mathcal{D}(\mathcal{B})$, $\quad k = \overline{1,m}$;
4° $\widetilde{V}_k,\ \partial\widetilde{V}_k/\partial t \in C(\Delta_T;\mathcal{L}(\mathcal{H}^2))$.

It can be checked immediately that in order to realize these conditions it suffices to require that in the original problem (2.1) the following conditions are satisfied

$$u^0 \in \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(B^{1/2}), \quad f \in C^1([0,T];\mathcal{D}(A^{-1/2})), \tag{2.15}$$

$$\mathcal{D}(B^{1/2}A^{-1/2}) \subset \mathcal{D}(A^{-1/2}C_kA^{-1/2}), \tag{2.16}$$

$$G_k, \partial G_k/\partial t \in C(\Delta_T;\mathcal{L}(\mathcal{D}(A^{-1/2}))), \quad k = \overline{1,m}. \tag{2.17}$$

It should be checked only (see expression for $\widetilde{f}$ in (2.11) and condition (2.16)), that $A^{-1/2}C_ku^0 \in \mathcal{H}$ if $u^0 \in \mathcal{D}(A^{-1/2}B)$. But this fact follows from the relation

$$A^{-1/2}C_ku^0 = (A^{-1/2}C_kA^{-1/2})(A^{-1/2}BA^{-1/2})^{-1}(A^{-1/2}Bu^0),$$

if we note that

$$\mathcal{D}(A^{-1/2}BA^{-1/2}) \subset \mathcal{D}(B^{1/2}A^{-1/2}) \subset \mathcal{D}(A^{-1/2}C_kA^{-1/2}),$$

and therefore the operator $(A^{-1/2}C_kA^{-1/2})(A^{-1/2}BA^{-1/2})^{-1}$ is bounded.

Thus, if conditions (2.15)-(2.17) are satisfied, then problem (2.10)-(2.14) has a strong solution $z(t) \in C([0,T];\mathcal{H}^2)$, that is why problem (2.5) under the same conditions has a strong solution $v(t)$ too, i.e., it has such a solution when all the terms in (2.5) belong to $C([0,T];\mathcal{H})$.

In the view of the aforesaid let us state the following result.

Theorem 2.1. *Let conditions* (2.15)-(2.17) *be held. Then problem* (2.1) *has a unique strong solution* $u(t)$, $0 \leqslant t \leqslant T$, *with values in* $\mathcal{E}^{1/2} = \mathcal{D}(A^{-1/2})$.

Proof. In the view of the aforesaid it remains to note that the statement of the theorem will turn true after an inverse interchange $v(t) = A^{1/2}u(t)$ in (2.5) and applying $A^{1/2} \in \mathcal{L}(\mathcal{H},\mathcal{D}(A^{-1/2}))$ on both sides of the obtained equation. □

Now let us consider case (2.9). Then problem (2.5) is equivalent to problem (2.10)-(2.14), where now

$$\widetilde{V}_k(t,\xi) := \mathrm{diag}(\check{V}_k(t,\xi);0), \quad \check{V}_k(t,\xi) := \int_\xi^t \check{G}_k(t,s)ds = \int_\xi^t A^{-1/2}G_k(t,s)ds, \tag{2.18}$$

$$\widetilde{C}_k := \mathrm{diag}(\check{C}_k; 0), \quad \mathcal{D}(\widetilde{C}_k) := \mathcal{D}(\check{C}_k) \oplus \mathcal{H}, \quad k = \overline{1,m}. \tag{2.19}$$

Theorem 2.2. *Let conditions* (2.15) *and*

$$\mathcal{D}(B^{1/2}A^{-1/2}) \subset \mathcal{D}(C_k A^{-1/2}), \tag{2.20}$$

$$G_k, \partial G_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{H}; \mathcal{D}(A^{-1/2}))), \quad k = \overline{1,m}, \tag{2.21}$$

be held. Then problem (2.1) *has a unique strong solution u with values in $\mathcal{D}(A^{-1/2})$ on the segment* $[0,T]$.

Proof. Here, as above, we first prove the existence of a strong solution to the new problem (2.10)-(2.12), (2.18),(2.19). We obtain this proof by analogy with the proof of Theorem 1.2 (see [2]), but now taking into account relations (2.18), (2.19). Then we pass from (2.10)-(2.12), (2.18),(2.19) to problem (2.5), which has a strong solution $v \in C([0,T]; \mathcal{H})$, and return to problem (2.1). □

Remark 2. If the operator $B = B^* \geq 0$ is fixed, then conditions (2.20) for the operators C_k are more generic than (2.16); at the same time conditions (2.17) for the operator functions $G_k(t,s)$ are less generic than (2.21).

2.2 Cauchy problem. The second approach.

In studying problem (2.1) the second method of approach can be used, which is not based on the transition to the equivalent problem for two first-order equations, but on the usage of the theory of operator cosine- and sine-functions (see, for example, [5]). Here for $A = I$ the following assertion occurs (see [2, p. 67-71]).

Theorem 2.3. *Let in problem* (2.1) $A = I$ *and the following conditions be held*

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(B^{1/2}), \quad f \in C^1([0,T]; \mathcal{H}), \tag{2.22}$$

$$\mathcal{D}(C_k) \supset \mathcal{D}(B), \tag{2.23}$$

$$G_k, \ \partial G_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{H})), \quad k = \overline{1,m}. \tag{2.24}$$

Then this problem has a unique strong solution on the segment $[0,T]$, it means the existence of a function

$$u \in C([0,T]; \mathcal{D}(B)) \cap C^1([0,T]; \mathcal{D}(B^{1/2})) \cap C^2([0,T]; \mathcal{H}), \tag{2.25}$$

for which equation (2.1) *and the initial conditions are true (for $A = I$) for all $t \in [0,T]$.*

As above let us transform problem (2.1) to Cauchy problem (2.5) and use the conditions and assertions of Theorem 2.3 for this problem. Then conditions (2.22)-(2.24) lead us to the relations

$$u^0 \in \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(B^{1/2}), \quad f \in C^1([0,T]; \mathcal{D}(A^{-1/2})), \tag{2.26}$$

$$\mathcal{D}(A^{-1/2}BA^{-1/2}) \subset \mathcal{D}(A^{-1/2}C_k A^{-1/2}), \tag{2.27}$$

$$G_k, \ \partial G_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{D}(A^{-1/2}))), \quad k = \overline{1,m}. \tag{2.28}$$

Here to obtain (2.26) the following facts are used. Let us introduce a positive definite (as $B \geq 0$, $A^{-1} \geq 0$) self-adjoint operator $\widehat{B} := A^{-1/2}BA^{-1/2}$, defined on domain $\mathcal{D}(\widehat{B}) = \mathcal{R}(A^{1/2}B^{-1}A^{1/2}) \subset \mathcal{H}$. Then the operator $\widehat{B}^{1/2} = (\widehat{B}^{1/2})^* \geq 0$ exists, and the operator $B^{1/2}A^{-1/2}$ can be written in the polar representation (see [1, p. 280-285])

$$B^{1/2}A^{-1/2} = U\widehat{B}^{1/2} = U(A^{-1/2}BA^{-1/2})^{1/2}, \tag{2.29}$$

where the operator U according to the properties of $B^{1/2}$ and $A^{-1/2}$ is not only a partial isometry, but a unitary operator acting in $\mathcal{H}$. From this it follows that

$$\mathcal{D}(\widehat{B}^{1/2}) = \mathcal{D}(B^{1/2}A^{-1/2}), \tag{2.30}$$

and as in problem (2.5) it must be $v'(0) = A^{1/2}u^1 \in \mathcal{D}(\widehat{B}^{1/2})$, then by taking (2.30) into account we obtain condition $u^1 \in \mathcal{D}(B^{1/2})$.

Theorem 2.4. *Suppose the conditions* (2.26)-(2.28) *are satisfied. Then problem* (2.1) *has a unique strong solution u with values in $\mathcal{D}(A^{-1/2})$ on the segment* $[0, T]$.

Proof. It follows by the above arguments that if conditions (2.26)-(2.28) are satisfied, then Cauchy problem (2.5) has a unique strong solution v with values in $\mathcal{H}$ on the segment $[0, T]$. Therefore after the inverse change $v(t) = A^{1/2}u(t)$ in (2.5) and applying $A^{1/2}$ on the left we arrive at the assertion of the present theorem. □

Remark 3. Conditions (2.26)-(2.28), which imply the assertion of Theorem 2.4, are more generic that conditions (2.15)-(2.17), which imply Theorem 2.1. In fact, it is obvious that if condition (2.16) is satisfied then

$$\mathcal{D}(A^{-1/2}BA^{-1/2}) \subset \mathcal{D}(B^{1/2}A^{-1/2}) \subset \mathcal{D}(A^{-1/2}C_kA^{-1/2}), \tag{2.31}$$

and (2.27) follows by (2.16). At the same time (2.27) does not imply (2.16).

The next assertion is an analogue of Theorem 2.2 based on the usage of the theory of operator cosine- and sine-functions.

Theorem 2.5. *Let conditions* (2.26), (2.21) *and also the condition*

$$\mathcal{D}(A^{-1/2}BA^{-1/2}) \subset \mathcal{D}(C_kA^{-1/2}), \quad k = \overline{1, m}, \tag{2.32}$$

be satisfied. Then problem (2.1) *has a unique strong solution u with values in $\mathcal{D}(A^{-1/2})$ on the segment* $[0, T]$.

Proof. This proof is realized similarly to the proof of Theorem 2.3, with the consideration of the conditions of the present theorem. Namely, under these conditions there exists a unique strong solution v with values in $\mathcal{H}$ to problem (2.5), and hence a unique strong solution u to problem (2.1). □

Remark 4. The requirement $f \in C^1([0, T]; \mathcal{D}(A^{-1/2}))$ in Theorems 2.1-2.2, 2.4-2.5 can be weakened. It can be replaced by the condition

$$A^{-1/2}f \in W_p^1([0, T]; \mathcal{H}), \quad p > 1, \tag{2.33}$$

where

$$||f||_{W_p^1([0,T];\mathcal{H})} := \sum_{k=0}^{1} \left(\int_0^T ||f^{(k)}(t)||_{\mathcal{H}}^p dt \right)^{1/p}.$$

Indeed, as it is shown by S.Ya. Yakubov in [8] for $f \in W_p^1([0,T];\mathcal{H})$ Cauchy problem (1.5) for the differential (not integro-differential) equation has a strong solution u with values in $\mathcal{H}$. Just that property is used to prove the mentioned theorems.

3 Complete linear Volterra integro-differential second-order equations unresolved with respect to the highest derivative

3.1 Setting of the problem

Let us consider Cauchy problem (1.1) under the assumption of Section 1, i.e., let us assume that

$$0 < A = A^* \in \mathcal{L}(\mathcal{H}), \quad F = F^* \geq 0, \quad B = B^* \geq 0, \quad G = 0, \tag{3.1}$$

and we will formulate requirements on $G_k(t,s)$ and C_k below. The equation of form (1.1) is called complete since its main part consists of the terms, which depend on u and d^2u/dt^2 as well as on du/dt.

Definition 2. Function u with the values in $\mathcal{E}^{1/2} = \mathcal{D}(A^{-1/2})$ is said to be a strong solution to Cauchy problem (1.1) ($G = 0$) on the segment $[0,T]$ if the following conditions are satisfied:

1° $u \in C([0,T];\mathcal{D}(A^{-1/2})B)$;

2° $u' \in C([0,T];\mathcal{D}(B^{1/2})) \cap C([0,T];\mathcal{D}(A^{-1/2}F))$;

3° $u'' \in C([0,T];\mathcal{D}(A^{-1/2}))$;

4° all the terms in equation (1.1) are functions continuous in t belonging to $C([0,T];\mathcal{D}(A^{-1/2}))$;

5° for all $t \in [0,T]$ equation (1.1) is true;

6° the initial conditions $u(0) = u^0$, $u'(0) = u^1$ are satisfied.

Note that conditions

$$u^0 \in \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(B^{1/2}) \cap \mathcal{D}(A^{-1/2}F), \quad f \in C([0,T];\mathcal{D}(A^{-1/2}))$$

are necessary for the existence of a strong solution with values in $\mathcal{D}(A^{-1/2})$ for problem (1.1), (3.1) on the segment $[0,T]$.

Here again our goal is to find restrictions on the operators F, B, C_k and the operator-functions $G_k(t,s)$, $k = \overline{1,m}$, which ensure the existence and uniqueness theorem for a strong solution with values in $\mathcal{D}(A^{-1/2})$ to problem (1.1).

Let u be a strong solution to problem (1.1) in the sense of Definition 2. Let us pass (as well as in 2.1) from this problem to the Cauchy problem for the system of two first-order integro-differential equations. For that we replace in (1.1) the unknown function u with the new unknown function v according to the rule $A^{1/2}u(t) =: v(t)$

and act on both parts of (1.1) by $A^{-1/2} \in \mathcal{L}(\mathcal{D}(A^{-1/2}), \mathcal{H})$. Then the following Cauchy problem similar to problem (2.5) arises:

$$\frac{d^2v}{dt^2}+A^{-1/2}FA^{-1/2}\frac{dv}{dt}+A^{-1/2}BA^{-1/2}v+\sum_{k=1}^{m}\int_0^t A^{-1/2}G_k(t,s)C_kA^{-1/2}v(s)ds = A^{-1/2}f(t),$$
$$v(0) = A^{1/2}u^0, \quad v'(0) = A^{1/2}u^1. \tag{3.2}$$

Here all the terms belong to $C([0,T];\mathcal{H})$.

Let us introduce the new unknown function w according to the formula

$$-iB^{1/2}A^{-1/2}v =: \frac{dw}{dt}, \quad w(0) = 0. \tag{3.3}$$

As property 2° in Definition 2 is satisfied we obtain that $d^2w/dt^2 \in C([0,T];\mathcal{H})$ and therefore

$$\frac{d^2w}{dt^2} + iB^{1/2}A^{-1/2}\frac{dv}{dt} = 0, \quad w'(0) = -iB^{1/2}A^{-1/2}v(0) = -iB^{1/2}u^0. \tag{3.4}$$

Hence, as well as, in Subsection 2.1, we come to the conclusion that problem (1.1) is equivalent to the Cauchy problem for the first-order integro-differential equation

$$\frac{dz}{dt} + \mathcal{F}_0 z + \sum_{k=1}^{m}\int_0^t \widetilde{V}_k(t,\xi)\widetilde{C}_k z(\xi)d\xi = \widetilde{f}(t), \tag{3.5}$$

$$z(0) = z^0 := (A^{1/2}u^1; -iB^{1/2}u^0)^\tau, \tag{3.6}$$

$$z(t) := (v'(t); w'(t))^\tau \in \widetilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}, \tag{3.7}$$

$$\widetilde{f}(t) := (A^{-1/2}f(t) - \sum_{k=1}^{m}\int_0^t A^{-1/2}G_k(t,s)C_ku^0ds; 0)^\tau, \tag{3.8}$$

$$\mathcal{F}_0 := \begin{pmatrix} A^{-1/2}FA^{-1/2} & iA^{-1/2}B^{1/2} \\ iB^{1/2}A^{-1/2} & 0 \end{pmatrix}, \tag{3.9}$$

$$\mathcal{D}(\mathcal{F}_0) := (\mathcal{D}(A^{-1/2}FA^{-1/2}) \cap \mathcal{D}(B^{1/2}A^{-1/2})) \oplus \mathcal{D}(A^{-1/2}B^{1/2}). \tag{3.10}$$

Here the operators $\widetilde{V}_k(t,\xi)$ and $\widetilde{C}_k$ are defined by formulas (2.13), (2.14) when conditions (2.8) are satisfied and by formulas (2.18), (2.19) when conditions (2.9) are satisfied.

Further investigation of problem (3.5)-(3.10) is based on various of relations between the domains of definition of operators the $B^{1/2}A^{-1/2}$ and $A^{-1/2}FA^{-1/2}$. In the present paper the following three cases are studied:

1° low intensity of energy dissipation:

$$\mathcal{D}(B^{1/2}A^{-1/2}) \subset \mathcal{D}(A^{-1/2}FA^{-1/2}); \tag{3.11}$$

2° mean intensity of energy dissipation:

$$\mathcal{D}(A^{-1/2}BA^{-1/2}) \subset \mathcal{D}(A^{-1/2}FA^{-1/2}) \subset \mathcal{D}(B^{1/2}A^{-1/2}) \subset \mathcal{D}(F^{1/2}A^{-1/2}); \tag{3.12}$$

3° high intensity of energy dissipation:

$$\mathcal{D}(A^{-1/2}FA^{-1/2}) \subset \mathcal{D}(A^{-1/2}BA^{-1/2}) \subset \mathcal{D}(B^{1/2}A^{-1/2}). \tag{3.13}$$

We devote an individual section to each of these cases.

Remark 5. The right inclusion in (3.13) is obvious. To prove the right inclusion in (3.12) we use the well-known Heinz inequality (see, for example, [4, p. 254]) and the polar representation for unbounded operators (see [1, p. 280-285]).

Proof. Indeed, operators $\widetilde{B} := A^{-1/2}BA^{-1/2}$ and $\widetilde{F} := A^{-1/2}FA^{-1/2}$, which are defined on domains

$$\mathcal{D}(A^{-1/2}BA^{-1/2}) = \mathcal{R}(A^{1/2}B^{-1}A^{1/2}), \quad \mathcal{D}(A^{-1/2}FA^{-1/2}) = \mathcal{R}(A^{1/2}F^{-1}A^{1/2}), \tag{3.14}$$

are self-adjoint and positive definite operators. In addition by (3.12) then $\mathcal{D}(\widetilde{B}) \subset \mathcal{D}(\widetilde{F})$, and by Heinz inequality it follows that

$$\mathcal{D}(\widetilde{B}^{1/2}) \subset \mathcal{D}(\widetilde{F}^{1/2}). \tag{3.15}$$

Using the polar representations for these operators, i.e. formulas

$$\widetilde{B}^{1/2} = U_B B^{1/2}A^{-1/2}, \quad \widetilde{F}^{1/2} = U_F F^{1/2}A^{-1/2}, \tag{3.16}$$

where U_B and U_F are unitary operators, we make the conclusion that

$$\mathcal{D}(\widetilde{B}^{1/2}) = \mathcal{D}(B^{1/2}A^{-1/2}), \quad \mathcal{D}(\widetilde{F}^{1/2}) = \mathcal{D}(F^{1/2}A^{-1/2}), \tag{3.17}$$

and the right inclusion in (3.12) is proved. □

3.2 The case of low intensity of energy dissipation.

Under condition (3.11) the operator matrix $\mathcal{F}_0$ in (3.9) is well defined on the domain (see (3.10))

$$\mathcal{D}(\mathcal{F}_0) := \mathcal{D}(B^{1/2}A^{-1/2}) \oplus \mathcal{D}(A^{-1/2}B^{1/2}) = \mathcal{R}(A^{1/2}B^{-1/2}) \oplus \mathcal{R}(B^{-1/2}A^{1/2}). \tag{3.18}$$

Lemma 3.1. *Let condition* (3.11) *be held. Then the operator* $\mathcal{F}_0$ *defined on domain* (3.18) *is an unbounded maximal accretive one:*

$$\mathrm{Re}(\mathcal{F}_0 z, z)_{\mathcal{H}^2} = (A^{-1/2}FA^{-1/2}v', v')_{\mathcal{H}} = ||F^{1/2}A^{-1/2}v'||^2_{\mathcal{H}} \geqslant 0, \quad \forall z \in \mathcal{D}(\mathcal{F}_0). \tag{3.19}$$

There exists the following factorization of this operator

$$\begin{aligned}\mathcal{F}_0 &= \begin{pmatrix} A^{-1/2}FA^{-1/2} & iA^{-1/2}B^{1/2} \\ iB^{1/2}A^{-1/2} & 0 \end{pmatrix} \\ &= i\begin{pmatrix} I & -iA^{-1/2}FB^{-1/2} \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & A^{-1/2}B^{1/2} \\ B^{1/2}A^{-1/2} & 0 \end{pmatrix},\end{aligned} \tag{3.20}$$

where $A^{-1/2}FB^{-1/2}$ *is a bounded operator acting in* $\mathcal{H}$.

Proof. Property (3.20) can be checked immediately. It follows by the representation

$$A^{-1/2}FB^{-1/2} = (A^{-1/2}FA^{-1/2})(A^{1/2}B^{-1/2}) = (A^{-1/2}FA^{-1/2})(B^{1/2}A^{-1/2})^{-1} \quad (3.21)$$

and (3.11) that the operator $A^{-1/2}FB^{-1/2}$ is bounded. Further, the maximality property of $\mathcal{F}_0$ follows from the fact that the inverse operator

$$\mathcal{F}_0^{-1} = -i \begin{pmatrix} 0 & A^{1/2}B^{-1/2} \\ B^{-1/2}A^{1/2} & 0 \end{pmatrix} \begin{pmatrix} I & iA^{-1/2}FB^{1/2} \\ 0 & I \end{pmatrix} \quad (3.22)$$

is defined on the whole space $\widetilde{\mathcal{H}} = \mathcal{H}^2$. □

Corollary 3.1. *The operator $(-\mathcal{F}_0)$ is a generator of C_0-semigroup.*

The proved facts allow us to use the assertion of Theorem 1.2 to problem (3.5)-(3.9), (3.18). By this theorem we obtain that if the following conditions are satisfied:

1° $\mathcal{D}(\widetilde{C}_k) \supset \mathcal{D}(\mathcal{F}_0)$, $k = \overline{1, m}$;
2° $\widetilde{V}_k$, $\partial\widetilde{V}_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{H}^2))$;
3° $z^0 \in \mathcal{D}(\mathcal{F}_0)$;
4° $\widetilde{f} \in C^1([0,T]; \mathcal{H}^2)$;

then problem (3.5)-(3.9), (3.18) has a unique strong solution z on the segment $[0,T]$.

Let us find such conditions that make properties 1°-4° true. First let the operators $\widetilde{C}_k$ and the operator-functions $\widetilde{V}_k$ be defined by formulas (2.13), (2.14), (2.8).

Condition 1° leads us to the property

$$\mathcal{D}(B^{1/2}A^{-1/2}) \oplus \mathcal{D}(A^{-1/2}B^{1/2}) \subset \mathcal{D}(A^{-1/2}C_kA^{-1/2}) \oplus \mathcal{H}, \quad k = \overline{1, m},$$

which holds if

$$\mathcal{D}(B^{1/2}A^{-1/2}) \subset \mathcal{D}(A^{-1/2}C_kA^{-1/2}), \quad k = \overline{1, m}. \quad (3.23)$$

Further, from requirement 2° and formulas (2.13) the following properties follow

$$\widehat{V}_k(t,\xi) = \int_\xi^t A^{-1/2}G_k(t,s)A^{1/2}ds, \quad \partial\widehat{V}_k(t,\xi)/\partial t \in C(\Delta; \mathcal{L}(\mathcal{H})), \quad k = \overline{1, m}.$$

These properties are true if the following conditions are satisfied

$$G_k(t,s),\ \partial G_k(t,s)/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{D}(A^{-1/2}))), \quad k = \overline{1, m}. \quad (3.24)$$

It is easy to check that requirement 3° is equivalent to the conditions

$$u^0 \in \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(B^{1/2}). \quad (3.25)$$

Finally, it can be ascertained that condition 4° holds if

$$f \in C^1([0,T]; \mathcal{D}(A^{-1/2})). \quad (3.26)$$

Indeed, in this case $A^{-1/2}f \in C^1([0,T]; \mathcal{H})$, and all we need is to check that other terms in (3.8) belong to $C^1([0,T]; \mathcal{H})$ too. To check these facts let us represent the integrands as

$$A^{-1/2}G_k(t,s)C_ku^0 = (A^{-1/2}G_k(t,s)A^{1/2})(A^{-1/2}C_ku^0).$$

Note that by (3.24) $A^{-1/2}G_k(t,s)A^{1/2} \in C^1(\Delta_T;\mathcal{L}(\mathcal{H}))$ and then it suffices to check that $A^{-1/2}C_k u^0 \in \mathcal{H}$. But by (3.25) $u^0 = B^{-1/2}A^{1/2}\eta^0$, $\eta^0 \in \mathcal{H}$, and then

$$\begin{aligned} A^{-1/2}C_k u^0 &= (A^{-1/2}C_k A^{-1/2})(A^{1/2}B^{-1}A^{1/2})\eta^0 \\ &= (A^{-1/2}C_k A^{-1/2})(B^{1/2}A^{-1/2})^{-1}(B^{-1/2}A^{1/2}\eta^0) \in \mathcal{H}, \end{aligned} \tag{3.27}$$

since by (3.23) the product of the first and the second factors is a bounded operator, and the operator $B^{-1/2}A^{1/2}$ is bounded too.

Theorem 3.1. *Suppose that in problem* (1.1) *conditions* (3.1) *and* (3.23)-(3.26) *hold. Then this problem has a unique strong solution u with values in $\mathcal{D}(A^{-1/2}) = \mathcal{E}^{1/2}$ on the segment* $[0,T]$.

Proof. As was proved above if the conditions of the theorem are satisfied then the problem (3.5)-(3.9), (3.18) has a unique strong solution z with values in $\widetilde{\mathcal{H}} = \mathcal{H}^2$ on segment $[0,T]$. Turning back by formulas (3.4), (3.3) from (3.5) to problem (3.2) we make the conclusion that this problem has a unique strong solution v with values in $\mathcal{H}$ on segment $[0,T]$. Carrying out in (3.2) an inverse interchange $v(t) =: A^{1/2}u(t)$ and acting by $A^{1/2}$ we obtain the assertion of the theorem. □

Now we similarly consider another case when the operators $\widetilde{C}_k$ and the operator-functions $\widetilde{V}_k$ are defined by formulas (2.18), (2.19), (2.9). Here instead of (3.23), (3.24) the following conditions arise

$$\mathcal{D}(B^{1/2}A^{-1/2}) \subset \mathcal{D}(C_k A^{-1/2}), \quad k = \overline{1,m}, \tag{3.28}$$

$$G_k,\ \partial G_k/\partial t \in C(\Delta_T;\mathcal{L}(\mathcal{H},\mathcal{D}(A^{-1/2}))), \quad k = \overline{1,m}. \tag{3.29}$$

Not repeating the transformations similar to the derivation of formulas (3.23)-(3.27) we mention only that here instead of (3.27) the following formula takes place

$$C_k u^0 = C_k B^{-1}A^{1/2}\eta^0 = [(C_k A^{-1/2})(B^{1/2}A^{-1/2})^{-1}](B^{-1/2}A^{1/2}\eta^0) \in \mathcal{H}, \quad k = \overline{1,m}, \tag{3.30}$$

since conditions (3.28), (3.25) are satisfied.

Theorem 3.2. *Suppose that in problem* (1.1) *conditions* (3.1) *and* (3.28), (3.29), (3.25), (3.26) *hold. Then this problem has a unique strong solution u with values in $\mathcal{D}(A^{-1/2}) = \mathcal{E}^{1/2}$ on the segment* $[0,T]$.

Remark 6. Theorem 3.1 is a generalization of Theorem 2.1, and Theorem 3.2 is a generalization of Theorem 2.2 when in the investigated integro-differential equation (1.1) $F \neq 0$, $G = 0$.

3.3 The case of high intensity of energy dissipation.

Suppose that in problem (3.5)-(3.10) conditions (3.13) hold, i.e. let us consider the case of high intensity of energy dissipation. Here the operator matrix $\mathcal{F}_0$ is again defined by formula (3.9) but now

$$\mathcal{D}(\mathcal{F}_0) = \mathcal{D}(A^{-1/2}FA^{-1/2}) \oplus \mathcal{D}(A^{-1/2}B^{1/2}). \tag{3.31}$$

Here, it is useful to replace the unknown function z with the new unknown function y according to the rule

$$z(t) = e^{\alpha t} y(t), \quad \alpha > 0. \tag{3.32}$$

Then for the unknown function y we obtain the Cauchy problem

$$\frac{dy}{dt} + \mathcal{F}_\alpha y + \sum_{k=1}^{m} \int_0^t \widetilde{W}_k(t,\xi)\widetilde{C}_k y(\xi)d\xi = \widetilde{f}_\alpha(t), \tag{3.33}$$

$$y(0) = z(0) = (A^{1/2}u^1; -B^{1/2}u^0)^\tau, \tag{3.34}$$

$$\mathcal{F}_\alpha := \mathcal{F}_0 + \alpha\mathcal{I} = \mathcal{F}_{\alpha,1} + \mathrm{diag}(\alpha I; 0), \quad \mathcal{F}_{\alpha,1} := \begin{pmatrix} A^{-1/2}FA^{-1/2} & iA^{-1/2}B^{1/2} \\ iB^{1/2}A^{-1/2} & \alpha I \end{pmatrix}, \tag{3.35}$$

$$\mathcal{D}(\mathcal{F}_\alpha) = \mathcal{D}(\mathcal{F}_0) = \mathcal{D}(\mathcal{F}_{\alpha,1}) = \mathcal{D}(A^{-1/2}FA^{-1/2}) \oplus \mathcal{D}(A^{-1/2}B^{1/2}), \tag{3.36}$$

$$\widetilde{f}_\alpha(t) := e^{-\alpha t}\widetilde{f}_0(t), \quad \widetilde{W}_k(t,\xi) := e^{-\alpha(t-\xi)}\widetilde{V}_k(t,\xi), \tag{3.37}$$

where the function $\widetilde{f}_0$ is defined by formula (3.8), and $\widetilde{V}_k$ and $\widetilde{C}_k$ are defined by formulas (2.13), (2.14) if (2.8) is true and by formulas (2.18), (2.19) if (2.9) is true.

Lemma 3.2. *The operator $\mathcal{F}_\alpha$ in* (3.35),(3.36) *is a uniformly accretive operator on $\mathcal{D}(\mathcal{F}_\alpha)$, i.e.,*

$$\mathrm{Re}(\mathcal{F}_\alpha y, y)_{\mathcal{H}^2} \geqslant \alpha ||y||^2_{\mathcal{H}^2}, \quad \forall y \in \mathcal{D}(\mathcal{F}_\alpha), \quad \alpha > 0. \tag{3.38}$$

Proof. This fact follows immediately by (3.19) and definition (3.35) of the operator $\mathcal{F}_\alpha$. □

Let us introduce the auxiliary operators:

$$V := B^{1/2}F^{-1/2}, \quad V^+ := F^{-1/2}B^{1/2}, \quad \mathcal{D}(V^+) := \mathcal{D}(B^{1/2}). \tag{3.39}$$

Lemma 3.3. *The operators V and V^+ have the following properties:*

$$V \in \mathcal{L}(\mathcal{H}), \quad V^+ = V^*|_{\mathcal{D}(B^{1/2})}, \quad \overline{V^+} = V^* \in \mathcal{L}(\mathcal{H}). \tag{3.40}$$

Proof. Let us check first that the operator V is bounded and therefore it is defined on the whole space $\mathcal{H}$. In fact,

$$V = B^{1/2}F^{-1/2} = (B^{1/2}A^{-1/2})(F^{1/2}A^{-1/2})^{-1},$$

and the following condition holds

$$\mathcal{D}(F^{1/2}A^{-1/2}) \subset \mathcal{D}(B^{1/2}A^{-1/2}). \tag{3.41}$$

This condition follows by the left inclusion in (3.13), the Heinz inequality and can be proved exactly in the same way as in Remark 5. Therefore the operator $(B^{1/2}A^{-1/2})(F^{1/2}A^{-1/2})^{-1}$ is bounded, i.e., $V \in \mathcal{L}(\mathcal{H})$.

Let us now take $u \in \mathcal{D}(B^{1/2})$, $v \in \mathcal{H}$. Then

$$(V^+u, v)_{\mathcal{H}} = (F^{-1/2}B^{1/2}u, v)_{\mathcal{H}} = (u, B^{1/2}F^{-1/2}v)_{\mathcal{H}} = (u, Vv)_{\mathcal{H}}. \tag{3.42}$$

Hence, the second property in (3.40) follows. Further, since the operator V is bounded, V^* is bounded too, and V^+ and V^* coincide on the dense in $\mathcal{H}$ set $\mathcal{D}(B^{1/2})$. So the closure by the continuity of the operator V^+ from $\mathcal{D}(B^{1/2})$ to the whole $\mathcal{H}$ coincides with V^*. □

The following theorem is a corollary of Lemmas 3.2 and 3.3.

Theorem 3.3. *There exist the following factorizations of the operator matrix $\mathcal{F}_\alpha$ in (3.35), defined on domain (3.36):*

1°. *in the Schur–Frobenius form,*

$$\mathcal{F}_\alpha = \begin{pmatrix} I & 0 \\ iVF^{-1/2}A^{1/2} & I \end{pmatrix} \begin{pmatrix} A^{-1/2}FA^{-1/2} & 0 \\ 0 & VV^+ + \alpha I \end{pmatrix} \begin{pmatrix} I & iA^{1/2}F^{-1/2}V^+ \\ 0 & I \end{pmatrix} + \begin{pmatrix} \alpha I & 0 \\ 0 & 0 \end{pmatrix}; \tag{3.43}$$

2°. *with the symmetric bordering,*

$$\mathcal{F}_\alpha = \begin{pmatrix} A^{-1/2}F^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & iV^+ \\ iV & \alpha I \end{pmatrix} \begin{pmatrix} F^{1/2}A^{-1/2} & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} \alpha I & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.44}$$

The operator $\mathcal{F}_\alpha$ can be closed to the maximal accretive operator

$$\mathcal{F} := \overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_1 + \operatorname{diag}(\alpha I; 0). \tag{3.45}$$

This closure of the operator $\mathcal{F}_\alpha$ can be represented:

1°. *in the Schur–Frobenius form,*

$$\mathcal{F} = \begin{pmatrix} I & 0 \\ iVF^{-1/2}A^{1/2} & I \end{pmatrix} \begin{pmatrix} A^{-1/2}FA^{-1/2} & 0 \\ 0 & VV^* + \alpha I \end{pmatrix} \begin{pmatrix} I & iA^{1/2}F^{-1/2}V^* \\ 0 & I \end{pmatrix} + \begin{pmatrix} \alpha I & 0 \\ 0 & 0 \end{pmatrix}; \tag{3.46}$$

2°. *with the symmetric bordering,*

$$\mathcal{F} = \begin{pmatrix} A^{-1/2}F^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & iV^* \\ iV & \alpha I \end{pmatrix} \begin{pmatrix} F^{1/2}A^{-1/2} & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} \alpha I & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.47}$$

The operator $\mathcal{F}$ is defined on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{F}) := \{y = (y_1; y_2)^\tau : y_1 \in \mathcal{D}(F^{1/2}A^{-1/2}), \\ F^{1/2}A^{-1/2}y_1 + iV^*y_2 \in \mathcal{D}(A^{-1/2}F^{1/2})\}, \end{aligned} \tag{3.48}$$

by the formula

$$\mathcal{F}y = \begin{pmatrix} A^{-1/2}F^{1/2}(F^{1/2}A^{-1/2}y_1 + iV^*y_2) + \alpha y_1 \\ iB^{1/2}A^{-1/2}y_1 + \alpha y_2 \end{pmatrix}, \quad y \in \mathcal{D}(\mathcal{F}). \tag{3.49}$$

Proof. Let us first note that if $y = (y_1; y_2)^\tau \in \mathcal{D}(\mathcal{F})$, then $y_1 \in \mathcal{D}(F^{1/2}A^{-1/2})$, and taking into account (3.41), $y_1 \in \mathcal{D}(B^{1/2}A^{-1/2})$, i.e. formula (3.49) is well defined.

Further formulas (3.43), (3.44) can be checked immediately on the elements in $\mathcal{D}(\mathcal{F}_\alpha)$. The second and the third factors in (3.43) (the first term in the right part) can be closed by replacing the operator V^+ by $V^* \in \mathcal{L}(\mathcal{H})$. Consequently the operator $\mathcal{F}$ in (3.46) arises. The first term of $\mathcal{F}$ is a product of closed operators having a bounded inverse, and the second term is obviously a bounded operator. Therefore the range of the operator $\mathcal{F}$ in (3.46) is the whole space $\mathcal{H}^2$, i.e. $\mathcal{F}$ is a maximal uniformly accretive operator which keeps property (3.38) (see, for example, [3, p.109]).

Similarly we can state that the operator $\mathcal{F}$ in (3.47) is also a maximal uniformly accretive operator. Here (in the first term) the bordering factors are unbounded operators having a bounded inverse and the middle factor after replacing of operator V^+ by V^*, has the following property

$$\mathrm{Re}\left(\begin{pmatrix} I & iV^* \\ iV & \alpha I \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)_{\mathcal{H}^2} \geqslant \min\{1; \alpha^2\} ||y||^2, \quad \forall y \in \mathcal{H}^2,$$

i.e., it is a uniformly accretive operator, and so it has a bounded inverse too.

Let us finally note that formula (3.49) for $\mathcal{F}$ follows by representation (3.46) as well as by (3.47), and can be checked immediately. □

Taking into account the properties of the operator $\mathcal{F}$ let us consider the following Cauchy problem

$$\frac{dy}{dt} + \mathcal{F}y + \sum_{k=1}^{m} \int_0^t \widetilde{W}_k(t,\xi)\widetilde{C}_k y(\xi) d\xi = \widetilde{f}_\alpha(t), \tag{3.50}$$

$$y(0) = (A^{1/2}u^1; -iB^{1/2}u^0)^\tau, \tag{3.51}$$

together with (3.33), (3.34).

Since $\mathcal{F}$ is a maximal uniformly accretive operator, the operator $(-\mathcal{F})$ generates a C_0-semigroup. Therefore, by Theorem 1.2, problem (3.50), (3.51) has a unique strong solution y on the segment $[0,T]$ if the following conditions are satisfied (in the case (2.8), (2.13), (2.14)):

$$1^\circ \quad \mathcal{D}(\widetilde{C}_k) \supset \mathcal{D}(\mathcal{F}), \quad k = \overline{1,m}; \tag{3.52}$$

$$2^\circ \quad \widetilde{W}_k,\ \partial\widetilde{W}_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{H}^2)); \tag{3.53}$$

$$3^\circ \quad y(0) \in \mathcal{D}(\mathcal{F}_\alpha) \subset \mathcal{D}(\mathcal{F}); \quad 4^\circ \quad \widetilde{f}_\alpha \in C^1([0,T]; \mathcal{H}^2). \tag{3.54}$$

These facts allow us to obtain sufficient conditions of the solvability of problem (1.1), (3.1) in the case of high intensity of energy dissipation.

Theorem 3.4. *Suppose that in problem* (1.1) - (3.1) *condition* (3.13) *and the following conditions are satisfied*

$$u^0 \in \mathcal{D}(A^{-1/2}F) \subset \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(A^{-1/2}F), \quad f \in C^1([0,T]; \mathcal{D}(A^{-1/2})), \tag{3.55}$$

$$\mathcal{D}(A^{-1/2}C_k A^{-1/2}) \supset \mathcal{D}(F^{1/2}A^{-1/2}), \quad k = \overline{1,m}, \tag{3.56}$$

$$G_k, \partial G_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{D}(A^{-1/2}))), \quad k = \overline{1,m}. \tag{3.57}$$

Then this problem has a unique strong solution u on the segment $[0,T]$ with values in $\mathcal{D}(A^{-1/2})$.

Proof. 1) Let us check first that if conditions (3.55)-(3.57) are satisfied, then properties (3.52)-(3.54) are true.

Indeed, condition 3° here looks like

$$(A^{1/2}u^1; -iB^{1/2}u^0)^\tau \in \mathcal{D}(A^{-1/2}FA^{-1/2}) \oplus \mathcal{D}(A^{-1/2}B^{1/2}),$$

and to make it valid it suffices to make the first and the second conditions in (3.55) true. In particular, if $u^0 \in \mathcal{D}(A^{-1/2}F)$, then $u^0 = F^{-1}A^{1/2}\eta^0$, $\eta^0 \in \mathcal{H}$, and then

$$(A^{-1/2}B^{1/2})(B^{1/2}u^0) = (A^{-1/2}BA^{-1/2})(A^{-1/2}FA^{-1/2})^{-1}\eta^0 =: K\eta^0 \in \mathcal{H},$$

so according to the left condition of (3.13) the operator K is bounded.

To make condition 4° valid it suffices to suppose that $f \in C^1([0,T]; \mathcal{D}(A^{-1/2}))$ and besides (see (3.8), (3.27)),

$$A^{-1/2}G_k(t,s)C_k u^0 = (A^{-1/2}G_k(t,s)A^{1/2})(A^{-1/2}C_k u^0) \in C(\Delta_T; \mathcal{L}(\mathcal{H})),$$

$$A^{-1/2}(\partial G_k(t,s)/\partial t)C_k u^0 = (A^{-1/2}\partial G_k(t,s)A^{1/2})(A^{-1/2}C_k u^0) \in C(\Delta_T; \mathcal{L}(\mathcal{H})).$$

Since here according to (3.57) the first factors have this property and $u^0 = F^{-1}A^{1/2}\eta^0$, $\eta^0 \in \mathcal{H}$, then

$$A^{-1/2}C_k u^0 = \left[(A^{-1/2}C_k A^{-1/2})(F^{1/2}A^{-1/2})^{-1}\right] F^{-1/2}A^{1/2}\eta^0 \in \mathcal{H},$$

because according to (3.56) the square bracket is a bounded operator and the operator $F^{-1/2}A^{1/2}$ is also bounded .

It can also be checked that formulas (3.37) for $\widetilde{W}_k(t,\xi)$ and formulas (2.13), (2.14), (3.48) allow us to state that conditions 1° and 2° hold if properties (3.56), (3.57) are true.

Thus, if conditions (3.55)-(3.57) hold then problem (3.50), (3.51) has a unique strong solution y with values in $\mathcal{H}^2$ on the segment $[0,T]$.

2) Relying on this fact, let us prove the assertion of the theorem. Let us rewrite (3.50), (3.51) as the Cauchy problem for the system of two equations:

$$\frac{dy_1}{dt} + A^{-1/2}F^{1/2}(F^{1/2}A^{-1/2}y_1 + iV^*y_2) + \alpha y_1 + \sum_{k=1}^{m}\int_0^t e^{-\alpha(t-\xi)}\widehat{V}_k(t,\xi)\widehat{C}_k y_1(\xi)d\xi$$

$$= e^{-\alpha t}(A^{-1/2}f(t) - \sum_{k=1}^{m}\int_0^t \widehat{G}_k(t,s)\widehat{C}_k A^{1/2}u^0 ds), \quad y_1(0) = A^{1/2}u^1; \tag{3.58}$$

$$\frac{dy_2}{dt} + iB^{1/2}A^{-1/2}y_1 + \alpha y_2 = 0, \quad y_2(0) = -iB^{1/2}u^0. \tag{3.59}$$

Here, as has been proved, each term in the equations is a continuous function in t with values in $\mathcal{H}$.

Note now that problem (3.33), (3.34) can be rewritten as the Cauchy problem for the system of two equations as well. The second equation is (3.59), and the first equation according to the definition of operator $\mathcal{F}_\alpha$, is the following:

$$\frac{dy_1}{dt} + A^{-1/2}FA^{-1/2}y_1 + iA^{-1/2}F^{1/2}V^+y_2 + \alpha y_1 + \sum_{k=1}^{m}\int_0^t e^{-\alpha(t-\xi)}\widehat{V}_k(t,\xi)\widehat{C}_k y_1(\xi)d\xi$$

$$= e^{-\alpha t}(A^{-1/2}f(t) - \sum_{k=1}^{m}\int_0^t \widehat{G}_k(t,s)\widehat{C}_k A^{1/2}u^0 ds), \quad y_1(0) = A^{1/2}u^1. \tag{3.60}$$

Let us prove that (under the assumptions of the theorem) if problem (3.58), (3.59) has a unique strong solution then problem (3.60), (3.59) has a unique strong solution $(y_1(t); y_2(t))^\tau$ on the segment $[0,T]$ with values in $\mathcal{H}^2$. In other words taking into account the property $V^*|_{\mathcal{D}(B^{1/2})} = V^+ = F^{-1/2}B^{1/2}$ (Lemma 3.3), in equation (3.58) the brackets in the second term from the left can be opened and then in (3.60), (3.59) each term is a continuous function in t with values in $\mathcal{H}$.

By (3.59) it follows that

$$y_2(t) = -i\int_0^t e^{-\alpha(t-s)}B^{1/2}A^{-1/2}y_1(s)ds + y_2(0). \tag{3.61}$$

Substituting this relation in the brackets in (3.58), we obtain the function

$$\begin{gathered}\varphi(t) := F^{1/2}A^{-1/2}y_1(t)\\ +\int_0^t e^{-\alpha(t-s)}V^*B^{1/2}A^{-1/2}y_1(s)ds + V^*B^{1/2}u^0 \in C([0,T];\mathcal{D}(A^{-1/2}F^{1/2})).\end{gathered} \tag{3.62}$$

Let φ and u^0 be known. Then we obtain that function y_1 is a solution of the Volterra integral equation

$$y_1(t) + \int_0^t e^{-\alpha(t-s)}Ky_1(s)ds = \varphi_1(t) \in C([0,T];\mathcal{D}(A^{-1/2}FA^{-1/2})), \tag{3.63}$$

$$K := A^{1/2}F^{-1/2}V^*B^{1/2}A^{-1/2}, \tag{3.64}$$

$$\varphi_1(t) := A^{-1/2}F^{-1/2}\varphi(t) - A^{1/2}F^{-1/2}V^*B^{1/2}u^0. \tag{3.65}$$

Actually, relying on the inclusions

$$\mathcal{D}(A^{-1/2}F) \subset \mathcal{D}(A^{-1/2}B) \subset \mathcal{D}(B) \tag{3.66}$$

(see (3.55)), and Lemma 3.3, we obtain that

$$\begin{aligned} A^{1/2}F^{-1/2}V^*B^{1/2}u^0 &= A^{1/2}F^{-1/2}V^+B^{1/2}u^0 = A^{1/2}F^{-1}Bu^0\\ &= (A^{1/2}F^{-1}A^{1/2})(A^{-1/2}Bu^0) \in \mathcal{D}(A^{-1/2}FA^{-1/2}),\end{aligned} \tag{3.67}$$

as $A^{-1/2}Bu^0 \in \mathcal{H}$. As for the first term in (3.65), the property

$$A^{1/2}F^{-1/2}\varphi(t) \in C([0,T];\mathcal{D}(A^{-1/2}FA^{-1/2})) \tag{3.68}$$

follows by (3.62).

Thus, relation (3.62) can be considered as a Volterra integral equation of the second kind in the space

$$C([0,T];\mathcal{H}(A^{-1/2}FA^{-1/2})), \quad \mathcal{H}(A^{-1/2}FA^{-1/2}) := \mathcal{D}(A^{-1/2}FA^{-1/2}), \tag{3.69}$$

with the norm equivalent to the norm of the graphic:

$$||y_1||_{\mathcal{H}(A^{-1/2}FA^{-1/2})} := ||A^{-1/2}FA^{-1/2}y_1||_{\mathcal{H}}. \tag{3.70}$$

Let us check that the operator K in (3.64) is a linear bounded operator acting in $\mathcal{H}(A^{-1/2}FA^{-1/2})$. Note for this that if $y_1 \in \mathcal{D}(A^{-1/2}FA^{-1/2}) \subset \mathcal{D}(A^{-1/2}BA^{-1/2})$, then $B^{-1/2}A^{-1/2}y_1 \in \mathcal{D}(A^{-1/2}B^{1/2}) \subset \mathcal{D}(B^{1/2})$ and therefore by Lemma 3.3

$$Ky_1 = A^{1/2}F^{-1/2}V^*B^{1/2}A^{-1/2}y_1 = A^{1/2}F^{-1/2}V^+B^{1/2}A^{-1/2}y_1$$

$$= A^{1/2}F^{-1}BA^{-1/2}y_1 = (A^{-1/2}FA^{-1/2})^{-1}(A^{-1/2}BA^{-1/2})y_1 \in \mathcal{D}(A^{-1/2}FA^{-1/2}), \quad (3.71)$$

as $A^{-1/2}BA^{-1/2}y_1 \in \mathcal{H}$.

It follows from the above that $K|_{\mathcal{D}(A^{-1/2}FA^{-1/2})}$ is a bounded operator acting in $\mathcal{H}(A^{-1/2}FA^{-1/2})$. Then the kernel function $K(t,s) := e^{-\alpha(t-s)}K$ of the integral operator in equation (3.63) is a continuous in t,s operator function with values in $\mathcal{L}(\mathcal{H}(A^{-1/2}FA^{-1/2}))$. Therefore by the known existence and uniqueness theorem for Volterra integral equations of the second kind (see, for example, [7]) problem (3.63) has a unique solution $y_1 \in C([0,T];\mathcal{H}(A^{-1/2}FA^{-1/2}))$. So in (3.58) in the expression $F^{1/2}A^{-1/2}y_1(t) + iV^*y_2(t)$ each term belongs to $C([0,T];\mathcal{D}(A^{-1/2}F^{1/2}))$, and therefore the brackets in the second term can be opened. It follows from the above that problem (3.60), (3.59) has a unique solution

$$(y_1; y_2)^\tau \in C([0,T];\mathcal{H}^2). \quad (3.72)$$

3) Let us turn back from (3.60), (3.59) to original problem (1.1)-(3.1). First, we pass from (3.33), (3.34) to problem (3.5)-(3.10), then to problem (3.2) and finally to problem (1.1), (3.1). We obtain that problem (3.2) has on the segment $[0,T]$ a unique strong solution v with values in $\mathcal{H}$, since the original problem (1.1), (3.1) has on this segment a unique strong solution u with values in $\mathcal{D}(A^{-1/2})$. □

Remark 7. To prove Theorem 3.4 we use the property $\mathcal{D}(A^{-1/2}F) \subset \mathcal{D}(A^{-1/2}B)$, which follows from the relations

$$\mathcal{R}(A^{1/2}F^{-1}A^{1/2}) = \mathcal{D}(A^{-1/2}FA^{-1/2}) \subset \mathcal{D}(A^{-1/2}BA^{-1/2})$$
$$= \mathcal{R}(A^{1/2}B^{-1}A^{1/2}) \subset \mathcal{D}(A^{-1/2})$$

after applying operator $A^{-1/2}$:

$$A^{-1/2}\mathcal{R}(A^{1/2}F^{-1}A^{1/2}) = \mathcal{R}(F^{-1}A^{1/2}) = \mathcal{D}(A^{-1/2}F) \subset A^{-1/2}\mathcal{R}(A^{1/2}B^{-1}A^{1/2})$$
$$= \mathcal{R}(B^{-1}A^{1/2}) = \mathcal{D}(A^{-1/2}B). \quad (3.73)$$

Now we consider the case when $\widetilde{W}_k$ and $\widetilde{C}_k$ in (3.33) are defined by formulas (3.37), (2.18), (2.19), (2.9). Let us formulate without a proof the following result similar to Theorem 3.4.

Theorem 3.5. *Suppose in that problem* (1.1) - (3.1) *conditions* (3.13) *and* (3.55) *are satisfied, and the following conditions.*

$$\mathcal{D}(C_kA^{-1/2}) \supset \mathcal{D}(F^{1/2}A^{-1/2}), \quad k = \overline{1,m}, \quad (3.74)$$

$$G_k, \partial G_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{H}, \mathcal{D}(A^{-1/2}))), \quad k = \overline{1,m}, \quad (3.75)$$

are also satisfied.

Then this problem has a unique strong solution u on the segment $[0,T]$ with values in $\mathcal{D}(A^{-1/2})$.

3.4 The case of mean intensity of energy dissipation.

Let us finally consider the case when in problem (3.5)-(3.10) conditions (3.12) hold, i.e.,

$$\mathcal{D}(A^{-1/2}BA^{-1/2}) \subset \mathcal{D}(A^{-1/2}FA^{-1/2}) \subset \mathcal{D}(B^{1/2}A^{-1/2}) \subset (F^{1/2}A^{-1/2}). \tag{3.76}$$

In this case in the Cauchy problem (3.33)-(3.37) the operator $\mathcal{F}_\alpha$ in (3.35) is defined on $\mathcal{D}(\mathcal{F}_\alpha)$ in (3.36) again, but it has properties different from the properties of the operators $\mathcal{F}_\alpha$ and $\overline{\mathcal{F}}_\alpha = \mathcal{F}$, arising in Subsection 3.3. Let us note that Lemma 3.2 is still true. Now instead of Lemma 3.3 we formulate a new assertion.

Let us introduce the operators

$$Q := B^{1/2}F^{-1}A^{1/2}, \quad Q^+ := A^{1/2}F^{-1}B^{1/2}, \quad \mathcal{D}(Q^+) := \mathcal{D}(B^{1/2}), \tag{3.77}$$

$$\begin{gathered} V := B^{1/2}F^{-1/2}, \quad V^{-1} := F^{1/2}B^{-1/2}, \\ \mathcal{D}(V) := \mathcal{R}(V^{-1}), \quad \mathcal{R}(V) := \mathcal{D}(V^{-1}) = \mathcal{H}, \end{gathered} \tag{3.78}$$

$$\begin{gathered} V^+ := F^{-1/2}B^{1/2}, \quad (V^+)^{-1} = B^{-1/2}F^{1/2}, \\ \mathcal{D}(V^+) := \mathcal{D}(B^{1/2}), \quad \mathcal{D}(V^+)^{-1} := \mathcal{D}(F^{1/2}). \end{gathered} \tag{3.79}$$

Lemma 3.4. *The operators Q, Q^+, V u V^+ have the following properties:*

$$Q \in \mathcal{L}(\mathcal{H}), \quad Q^+ = Q^*|_{\mathcal{D}(B^{1/2})}, \quad \overline{Q^+} = Q^* \in \mathcal{L}(\mathcal{H}), \tag{3.80}$$

$$V^{-1} \in \mathcal{L}(\mathcal{H}), \quad (V^+)^{-1} = (V^{-1})^*|_{\mathcal{D}(F^{1/2})}, \quad \overline{(V^+)^{-1}} = (V^*)^{-1} = (V^{-1})^* \in \mathcal{L}(\mathcal{H}). \tag{3.81}$$

Proof. The proof is similar to the proof of Lemma 3.3.

1) Since

$$Q = B^{1/2}F^{-1}A^{1/2} = (B^{1/2}A^{-1/2})(A^{1/2}F^{-1}A^{1/2}) = (B^{1/2}A^{-1/2})(A^{-1/2}FA^{-1/2})^{-1}$$

and the middle inclusion in (3.76) holds, then the operator Q is bounded in $\mathcal{H}$.

2) Let us now take $u \in \mathcal{D}(B^{1/2})$ and $v \in \mathcal{H}$. Then

$$(Q^+u, v)_{\mathcal{H}} = (A^{1/2}F^{-1}B^{1/2}u, v)_{\mathcal{H}} = (u, B^{1/2}F^{-1}A^{1/2}v)_{\mathcal{H}} = (u, Qv)_{\mathcal{H}},$$

and therefore properties (3.80) are satisfied.

3) Similarly for V^{-1}, according to the right inclusion in (3.76), we have

$$V^{-1} = F^{-1/2}B^{1/2} = (F^{1/2}A^{-1/2})(A^{1/2}B^{1/2}) = (F^{1/2}A^{-1/2})(B^{1/2}A^{-1/2})^{-1} \in \mathcal{L}(\mathcal{H}).$$

4) Further let us take $u \in \mathcal{D}(F^{1/2}) = \mathcal{D}((V^+)^{-1})$ and $v \in \mathcal{H}$. Then

$$((V^+)^{-1}u, v)_{\mathcal{H}} = (B^{-1/2}F^{1/2}u, v)_{\mathcal{H}} = (u, F^{1/2}B^{-1/2}v)_{\mathcal{H}} = (u, V^{-1}v)_{\mathcal{H}}.$$

From the above it follows that properties (3.81) are valid. □

By Lemma 3.4, the operator $V := (V^{-1})^{-1}$ is an unbounded operator defined on the domain $\mathcal{D}(V) := \mathcal{R}(V^{-1})$; respectively V^* is also unbounded and $\mathcal{D}(V^*) = \mathcal{R}((V^*)^{-1}) = \mathcal{R}((V^{-1})^*)$. Moreover $\mathcal{R}(V) = \mathcal{R}(V^*) = \mathcal{H}$.

Theorem 3.6. *Taking into account* (3.76) *there exists the following factorization of the operator matrix* $\mathcal{F}_\alpha$ *in* (3.35), (3.36)

$$\mathcal{F}_\alpha = \begin{pmatrix} I & 0 \\ iQ & I \end{pmatrix} \begin{pmatrix} A^{-1/2}FA^{-1/2} & 0 \\ 0 & VV^+ + \alpha I \end{pmatrix} \begin{pmatrix} I & iQ^+ \\ 0 & I \end{pmatrix} + \begin{pmatrix} \alpha I & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.82)$$

where Q, Q^+, V *u* V^+ *are the operators defined by formulas* (3.77)-(3.79) *and having the properties described in Lemma 3.4.*

The operator $\mathcal{F}_\alpha$ *can be closed to the maximal accretive operator* $\mathcal{F}$ *which has the representation*

$$\mathcal{F} = \overline{\mathcal{F}}_\alpha = \begin{pmatrix} I & 0 \\ iQ & I \end{pmatrix} \begin{pmatrix} A^{-1/2}FA^{-1/2} & 0 \\ 0 & VV^* + \alpha I \end{pmatrix} \begin{pmatrix} I & iQ^* \\ 0 & I \end{pmatrix} + \begin{pmatrix} \alpha I & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.83)$$

The operator $\mathcal{F}$ *is defined on the domain*

$$\mathcal{D}(\mathcal{F}) := \{y = (y_1; y_2)^\tau \in \mathcal{H}^2 : y_1 \in \mathcal{D}(B^{1/2}A^{-1/2}),\ y_1 + iQ^*y_2 \in \mathcal{D}(A^{-1/2}FA^{-1/2})\} \quad (3.84)$$

acting by the formula

$$\mathcal{F}y = \begin{pmatrix} A^{-1/2}FA^{-1/2}(y_1 + iQ^*y_2) + \alpha y_1 \\ iB^{1/2}A^{-1/2}y_1 + \alpha y_2 \end{pmatrix}. \quad (3.85)$$

Proof. Taking into account(3.77)-(3.79) formula (3.82) can be checked immediately on the elements in $\mathcal{D}(\mathcal{F}_\alpha)$. The second and the third factors in (3.82) allow the closures by replacing operators V^+ and Q^+ by V^* and Q^* respectively. Thus operator (3.83) appears. The first term here is a product of closed operators having bounded inverses, and the second term is a bounded operator. Then the range of operator $\mathcal{F}$ is the whole space $\mathcal{H}^2$, therefore it is a closed operator. Since after closing $\mathcal{F}_\alpha$ inequality (3.38) is valid for $\mathcal{F} = \overline{\mathcal{F}}_\alpha$, then $\mathcal{F}$ is a maximal uniformly accretive operator.

Let us check that formulas (3.84), (3.85) are true. If $y \in \mathcal{D}(\mathcal{F})$, then by (3.83) it follows that

$$\mathcal{F}y = \begin{pmatrix} A^{-1/2}FA^{-1/2}(y_1 + iQ^*y_2) + \alpha y_1 \\ iQA^{-1/2}FA^{-1/2}(y_1 + iQ^*y_2) + (VV^* + \alpha I)y_2 \end{pmatrix}. \quad (3.86)$$

From the above we obtain that

$$y_1 + iQ^*y_2 \in \mathcal{D}(A^{-1/2}FA^{-1/2}), \quad y_2 \in \mathcal{D}(VV^*).$$

Let us take $y_2 \in \mathcal{D}(A^{-1/2}B^{1/2})$. Then

$$y_2 \in \mathcal{D}(VV^+) = \mathcal{D}(B^{1/2}F^{-1}B^{1/2}) = \mathcal{D}(QA^{-1/2}B^{1/2}) \subset \mathcal{D}(B^{1/2}), \quad y_2 \in \mathcal{D}(VV^*)$$

and therefore (according to the middle inclusion in (3.76))

$$\begin{aligned} Q^*y_2 &= Q^+y_2 = A^{1/2}F^{-1}B^{1/2}y_2 \\ &= (A^{-1/2}FA^{-1/2})^{-1}(A^{-1/2}B^{1/2}y_2) \in \mathcal{D}(A^{-1/2}FA^{-1/2}) \subset \mathcal{D}(B^{1/2}A^{-1/2}). \end{aligned}$$

Taking into account the relations $QA^{-1/2}FA^{-1/2} = B^{1/2}A^{-1/2}$ in the second row in (3.86) we obtain that $y_1 \in \mathcal{D}(B^{1/2}A^{-1/2})$, and this row equals

$$\begin{aligned} iB^{1/2}A^{-1/2}y_1 - B^{1/2}A^{-1/2}\left(A^{1/2}F^{-1}B^{1/2}y_2 + VV^{+}y_2 + \alpha y_2\right) \\ = iB^{1/2}A^{-1/2}y_1 - B^{1/2}F^{-1}B^{1/2}y_2 + B^{1/2}F^{-1}B^{1/2}y_2 + \alpha y_2 \\ = iB^{1/2}A^{-1/2}y_1 + \alpha y_2, \quad y_2 \in \mathcal{D}(A^{-1/2}B^{1/2}). \end{aligned} \tag{3.87}$$

Since $\mathcal{D}(A^{-1/2}B^{1/2}) = \mathcal{R}(B^{-1/2}A^{1/2})$ is dense in $\mathcal{H}$, after closing on elements $y_2 \in \mathcal{D}(A^{-1/2}B^{1/2})$ we obtain that (3.87) for all $y_2 \in \mathcal{H}$. □

Taking into account the properties of the operator $\mathcal{F}$, let us consider a more generic Cauchy problem than (3.33)-(3.34) under the assumption (2.13), (2.14), (2.8),

$$\frac{dy}{dt} + \mathcal{F}y + \sum_{k=1}^{m}\int_0^t \widetilde{W}_k(t,\xi)\widetilde{C}_k y(\xi)d\xi = \widetilde{f}_\alpha(t), \tag{3.88}$$

$$y(0) = (A^{1/2}u^1; -iB^{1/2}u^0)^\tau.$$

Since by Theorem 3.6, the operator $(-\mathcal{F})$ is a generator of the contractive C_0-semigroup, by Theorem 1.2 this problem has a strong solution on the segment $[0,T]$ if the following conditions are satisfied:

1° $\mathcal{D}(\widetilde{C}_k) \supset \mathcal{D}(\mathcal{F}), \quad k = \overline{1,m}$;

2° $\widetilde{W}_k$, $\partial\widetilde{W}_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{H}^2))$;

3° $y^0 \in \mathcal{D}(\mathcal{F})$;

4° $\widetilde{f}_\alpha \in C^1([0,T]; \mathcal{H}^2)$.

Let us connect the Cauchy problem for the second-order integro-differential equation in the space $\mathcal{H}$ with (3.88). We call the Cauchy problem

$$\begin{gathered} A\tfrac{d^2u}{dt^2} + FA^{-1/2}(A^{1/2}\tfrac{du}{dt} + Q^*B^{1/2}u) + \textstyle\sum_{k=1}^m \int_0^t G_k(t,s)C_k u(s)ds = f(t), \\ u(0) = u^0, \quad u'(0) = u^1, \end{gathered} \tag{3.89}$$

by the problem associated with the original problem (1.1) under conditions (3.1).

Definition 3. We call a function u on the segment $[0,T]$ with values in $\mathcal{D}(A^{-1/2}) \subset \mathcal{H}$ a strong solution of the associated problem (3.89), if all the following conditions are satisfied:

1° $u \in \mathcal{D}(B^{1/2})$ и $B^{1/2}u \in C([0,T]; \mathcal{H})$;

2° $A^{1/2}du/dt + Q^*B^{1/2}u \in \mathcal{D}(A^{-1/2}FA^{-1/2})$,
$FA^{-1/2}(A^{1/2}(du/dt) + Q^*B^{1/2}u) \in C([0,T]; \mathcal{D}(A^{-1/2}))$;

3° $Au \in C^2([0,T]; \mathcal{D}(A^{-1/2}))$;

4° for all $t \in [0,T]$ equation (3.89) and the initial conditions are true, and all the terms belong to $C([0,T]; \mathcal{D}(A^{-1/2}))$.

The following fact explains the term associated.

Lemma 3.5. *If the strong solution u to the associated problem* (3.89) *has the additional smoothness properties*

$$u \in \mathcal{D}(B), \quad Bu \in C([0,T];\mathcal{D}(A^{-1/2})), \tag{3.90}$$

then it is a strong solution to problem (1.1), (3.1) *in the sense of Definition 2, i.e. on the segment* $[0,T]$ *and with values in* $\mathcal{D}(A^{-1/2})$.

Proof. Indeed, if properties (3.90) are satisfied, then

$$Q^* B^{1/2} u(t) = Q^+ B^{1/2} u(t) = A^{1/2} F^{-1} B u(t) \in C([0,T];\mathcal{D}(A^{-1/2} F A^{-1/2})).$$

Therefore in (3.89) the brackets in the second term from the left can be opened:

$$FA^{-1/2}(A^{1/2}\frac{du}{dt} + A^{1/2}F^{-1}Bu(t)) = F\frac{du}{dt} + Bu,$$

and here every term belongs to $C([0,T];\mathcal{D}(A^{-1/2}))$. □

Thus, problem (3.89) is a generalization of problem (1.1), (3.1) in the case when properties (3.90) are not valid.

Theorem 3.7. *Suppose the following conditions are satisfied*

$$u^0 \in \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(A^{-1/2}F), \quad f \in C^1([0,T];\mathcal{D}(A^{-1/2})); \tag{3.91}$$

$$\mathcal{D}(A^{-1/2}C_k A^{-1/2}) \supset \mathcal{D}(B^{1/2}A^{-1/2}), \quad k = \overline{1,m}; \tag{3.92}$$

$$G_k,\ \partial G_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{D}(A^{-1/2}))), \quad k = \overline{1,m}. \tag{3.93}$$

Then the associated problem (3.89) *has on the segment* $[0,T]$ *a unique strong solution (in the sense of Definition 3) with values in* $\mathcal{D}(A^{-1/2})$.

Proof. The proof of this theorem is similar to the proof of Theorem 3.4.

1) Let us check that under conditions (3.91)- (3.93) properties $1^\circ - 4^\circ$, which provide the solvability of Cauchy problem (3.88), are valid

Indeed, it can be checked that under the first and the second conditions (3.91) property $y^0 \in \mathcal{D}(\mathcal{F}_\alpha) \subset \mathcal{D}(\mathcal{F})$ is true. Further, if f has property (3.91), then $\widetilde{f}_\alpha \in C^1([0,T];\mathcal{H}^2)$, under the condition (see the proof of Theorem 3.4)

$$A^{-1/2}G_k(t,s)C_k u^0,\ A^{-1/2}(\partial G_k(t,s)/\partial t)C_k u^0 \in C(\Delta_T;\mathcal{L}(\mathcal{H})). \tag{3.94}$$

However if $u^0 \in \mathcal{D}(A^{-1/2}B)$ then $u^0 = B^{-1}A^{1/2}\eta^0, \quad \eta^0 \in \mathcal{H}$, and therefore

$$A^{-1/2}G_k(t,s)C_k u^0$$

$$=(A^{-1/2}G_k(t,s)A^{1/2})\big[(A^{-1/2}C_k A^{-1/2})(B^{1/2}A^{-1/2})^{-1}\big](B^{-1/2}A^{1/2}\eta^0) \in C(\Delta_T;\mathcal{L}(\mathcal{H})),$$

as the first factor belongs to $C(\Delta_T;\mathcal{L}(\mathcal{H}))$, and the second factor according to (3.92) is bounded, and the operator $B^{-1/2}A^{1/2}$ is bounded too. The second property (3.94) can be proved similarly.

Thus, under conditions (3.91)-(3.93) properties 3° and 4° mentioned above are valid. It can be checked immediately, using formulas (3.37), (2.13), (2.14), (3.84), that conditions 1° и 2° hold if properties (3.92), (3.93) are true .

Thus, under conditions (3.91)-(3.93) Cauchy problem (3.88) has on the segment $[0, T]$ a unique strong solution y with values in $\mathcal{H}^2$.

2) Let us prove the assertion of the theorem. Let us rewrite (3.88) in a vector-matrix form

$$\frac{dy_1}{dt} + A^{-1/2}FA^{-1/2}(y_1 + iQ^*y_2) + \alpha y_1 + \sum_{k=1}^{m} \int_0^t e^{-\alpha(t-\xi)}\widehat{V}_k(t,\xi)\widehat{C}_k y_1(\xi)d\xi$$

$$= e^{-\alpha t}(A^{-1/2}f(t) - \sum_{k=1}^{m} \int_0^t \widehat{G}_k(t,s)\widehat{C}_k A^{1/2}u^0 ds), \quad y_1(0) = A^{1/2}u^1, \tag{3.95}$$

$$\frac{dy_2}{dt} + iB^{1/2}A^{-1/2}y_1 + \alpha y_2 = 0, \quad y_2(0) = -iB^{1/2}u^0. \tag{3.96}$$

Here in the equations all the terms belong to $C([0, T]; \mathcal{H})$.

Let us multiply both parts of the equations (3.95), (3.96) by $e^{\alpha t}$, and then express $e^{\alpha t}y_2(t)$ by (3.96) in terms of $e^{\alpha t}y_1(t)$. Further let us use relations (2.8), (2.13), (2.14), (3.3), (3.6), (3.8), (3.32) and $A^{1/2}u = v$. Then after acting on the left by the operator $A^{1/2}$ on the modified equation (3.95), we obtain equation (3.89), where all the terms belong to $C([0, T]; \mathcal{D}(A^{-1/2}))$. □

Now without proof we formulate an assertion about the correct solvability of problem (3.89) in the case (2.9), (2.18),(2.19), (3.37).

Theorem 3.8. *Suppose that the following conditions are satisfied:*

$$u^0 \in \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(A^{-1/2}F), \quad f \in C^1([0, T]; \mathcal{D}(A^{-1/2})); \tag{3.97}$$

$$\mathcal{D}(C_k A^{-1/2}) \supset \mathcal{D}(B^{1/2}A^{-1/2}), \quad k = \overline{1, m}; \tag{3.98}$$

$$G_k, \partial G_k/\partial t \in C(\Delta_T; \mathcal{L}(\mathcal{H}, \mathcal{D}(A^{-1/2}))), \quad k = \overline{1, m}. \tag{3.99}$$

Then the associated problem (3.89) *has on the segment* $[0, T]$ *a unique strong solution (in the sense of Definition 3)* u *with values in* $\mathcal{D}(A^{-1/2})$.

Note that Remark 4 is true for Theorems 3.1-3.2, 3.4-3.5, 3.7-3.8. In these theorems the requirement $f \in C^1([0, T]; \mathcal{D}(A^{-1/2}))$ can be weakened by replacing it by the condition (see Remark 4)

$$A^{-1/2}f \in W_p^1([0, T]; \mathcal{H}), \quad p > 1.$$

Acknowledgments

The authors are grateful to Professor D.A. Zakora for attention to this work and useful discussions, and to Professor V.I. Burenkov for editing the paper.

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Received: 25.10.2013