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SOME EQUIVALENT CRITERIA FOR THE BOUNDEDNESS OF HARDY-TYPE OPERATORS ON THE CONE OF QUASIMONOTONE FUNCTIONS

M.L. Goldman

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Abstract. Two-sided estimates are established for two types of generalized Hardy operators on the cones of functions in weighted Lebesgue spaces with some properties of monotonicity. In this paper we continue the proofs given in [9] for the main results announced in our paper [7]. Also we present here some other equivalent descriptions, consider some particular cases and establish results in the case of a degenerate measure.

1 Introduction

This paper is organized in the following way. For convenience of the reader we reproduce in the introduction the setting of the problems, the main notation and results in [9], Section 1. In the next Sections of this paper we preserve all the notation given in [9]. Anyway, in order to understand the content of this paper the reader has to be familiar with the related notation and facts in [9].

1.1. Let β and γ be non-negative Borel measures on $R_+ = (0, \infty)\,; p, q \in R_+$, and Ω be a certain cone of non-negative Borel measurable functions on R_+, and A be a positive operator. Introduce

$$H_\Omega(A) = \sup_{f\in\Omega}\left[\left(\int\limits_{R_+}(Af)^q\,d\gamma\right)^{1/q}\left(\int\limits_{R_+}f^p d\beta\right)^{-1/p}\right]. \qquad (1.1)$$

Here, we consider the cones of functions that are monotone with respect to the prescribed positive continuous functions k and m :

$$\Omega_k = \{f \geqslant 0 : f(\tau)/k(\tau) \downarrow\}; \qquad \Omega^m = \{f \geqslant 0 : f(\tau)/m(\tau) \uparrow\}. \qquad (1.2)$$

As operator A, consider the generalized Hardy operators $A = A_\mu$, and $A = B_\mu$ where μ is a non-negative Borel measure on R_+ ;

$$(A_\mu f)(t) = \int\limits_{(0,t]} f d\mu; \quad (B_\mu f)(t) = \int\limits_{[t,\infty)} f d\mu. \qquad (1.3)$$

1.2. First, we formulate the result for $H_{\Omega_k}(B_\mu)$. For this purpose we need some notations:

$$\omega_p(t) = \left(\int\limits_{(0,t)} k^p d\beta \right)^{1/p}, \quad t > 0; \quad \Psi(t,\tau) = \int\limits_{[t,\tau)} k d\mu, \quad t < \tau; \tag{1.4}$$

$$V_p(t) = \sup_{\tau \in (t,\infty)} \left[\frac{\Psi(t,\tau)}{\omega_p(\tau)} \right], \quad p \in (0,1]; \tag{1.5}$$

$$V_p(t) = \left\{ \int\limits_{(t,\infty)} \Psi^{p'}(t,\tau) \left(-d \left[\frac{1}{\omega_p^{p'}(\tau)} \right] \right) \right\}^{1/p'}; \quad p > 1, \quad \frac{1}{p} + \frac{1}{p'} = 1; \tag{1.6}$$

$$W_q(\tau) = \left(\int\limits_{(0,\tau)} d\gamma \right)^{1/q}; \quad \xi_\alpha(\tau) = \omega_p^{-1}(\alpha \omega_p(\tau)), \quad \tau \in R_+. \tag{1.7}$$

Here $\alpha \in (0,1)$ is fixed; ω_p^{-1} is the right-continuous inverse function for the (increasing) continuous function ω_p. Obviously, $\xi_\alpha(\tau) < \tau$.

The criterion of the boundedness for $H_{\Omega_k}(B_\mu)$ is determined by the following quantities:

$$E_{pq} = \sup_{\tau \in R_+} \left[\left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau) \, d\gamma(t) \right)^{1/q} \frac{1}{\omega_p(\tau)} \right], \quad p \leqslant q; \tag{1.8}$$

$$E_{pq} = \left\{ \int\limits_{R_+} \left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau) \, d\gamma(t) \right)^{s/q} \left(-d \left[\frac{1}{\omega_p^s(\tau)} \right] \right) \right\}^{1/s}, \quad p > q, \tag{1.9}$$

$$F_{pq} = \sup_{t \in R_+} \left[V_p(t) W_q(t) \right], \quad p \leqslant q, \tag{1.10}$$

$$F_{pq} = \left\{ \int\limits_{R_+} V_p^s(t) \, d\left[W_q^s(t) \right] \right\}^{1/s}, \quad p > q, \tag{1.11}$$

where as always in this paper $s = pq/(p-q)$ for $p > q$. In addition, introduce the non-degeneracy condition for measure β :

$$\beta \in N_p(k) \quad \Leftrightarrow \quad \int\limits_{(0,1)} k^p d\beta = 1, \quad \int\limits_{[1,\infty)} k^p d\beta = \infty. \tag{1.12}$$

Theorem 1.1. *Let $\beta \in N_p(k)$ and functions ω_p and W_q be positive and continuous on R_+, $\omega_p(+0) = 0$. Then there exists $c_0 = c_0(p,q,\alpha) \in [1,\infty)$ such that*

$$c_0^{-1}(E_{pq} + F_{pq}) \leqslant H_{\Omega_k}(B_\mu) \leqslant c_0(E_{pq} + F_{pq}). \tag{1.13}$$

1.3. Now, we present the corresponding results concerning $H_{\Omega^m}(A_\mu)$ (see (1.1)-(1.3)). To this end we denote

$$\bar{\omega}_p(t) = \left(\int\limits_{(t,\infty)} m^p d\beta \right)^{1/p}, \quad t > 0; \quad \Phi(\tau, t) = \int\limits_{(\tau,t]} m d\mu, \quad \tau < t; \tag{1.14}$$

$$V_p^{(0)}(t) = \sup_{\tau\in(0,t)} \left[\Phi(\tau, t) \frac{1}{\bar{\omega}_p(\tau)} \right], \quad p \in (0, 1]; \tag{1.15}$$

$$V_p^{(0)}(t) = \left\{ \int\limits_{(0,t)} \Phi^{p'}(\tau, t) d\left[\frac{1}{\bar{\omega}_p^{p'}(\tau)} \right] \right\}^{1/p'}, \quad p > 1; \tag{1.16}$$

$$\bar{W}_q(\tau) = \left(\int\limits_{(\tau,\infty)} d\gamma \right)^{1/q}; \quad \varsigma_\alpha(\tau) = \bar{\omega}_p^{-1}(\alpha\bar{\omega}_p(\tau)), \quad \tau \in R_+. \tag{1.17}$$

Here $\alpha \in (0, 1)$ is fixed; $\bar{\omega}_p^{-1}$ is the right-continuous inverse function for the (decreasing) continuous function $\bar{\omega}_p$. Obviously, $\tau < \varsigma_\alpha(\tau)$. Now, we introduce

$$E_{pq}^{(0)} = \sup_{\tau\in R_+} \left[\left(\int\limits_{(\tau,\varsigma_\alpha(\tau)]} \Phi^q(\tau, t) d\gamma(t) \right)^{1/q} \frac{1}{\bar{\omega}_p(\tau)} \right], \quad p \leqslant q; \tag{1.18}$$

$$E_{pq}^{(0)} = \left\{ \int\limits_{R_+} \left(\int\limits_{(\tau,\varsigma_\alpha(\tau)]} \Phi^q(\tau, t) d\gamma(t) \right)^{s/q} d\left[\frac{1}{\bar{\omega}_p^s(\tau)} \right] \right\}^{1/s}, \quad p > q. \tag{1.19}$$

$$F_{pq}^{(0)} = \sup_{t\in R_+} \left[V_p^{(0)}(t) \bar{W}_q(t) \right], \quad p \leqslant q, \tag{1.20}$$

$$F_{pq}^{(0)} = \left\{ \int\limits_{R_+} \left(V_p^{(0)} \right)^s (t) \left(-d\left[\bar{W}_q^s(t) \right] \right) \right\}^{1/s}, \quad p > q. \tag{1.21}$$

Now, the non-degeneracy condition on measure β is the following:

$$\beta \in \bar{N}_p(m) \quad \Leftrightarrow \quad \int\limits_{(0,1]} m^p d\beta = \infty, \quad \int\limits_{(1,\infty)} m^p d\beta = 1. \tag{1.22}$$

Theorem 1.2. *Let $\beta \in \bar{N}_p(m)$ and functions $\bar{\omega}_p$ and $\bar{W}_q$ be positive and continuous on R_+, $\bar{\omega}_p(+\infty) = 0$. Then for $c_0 \in [1, \infty)$ from Theorem 1.1,*

$$c_0^{-1} \left(E_{pq}^{(0)} + F_{pq}^{(0)} \right) \leqslant H_{\Omega^m}(A_\mu) \leqslant c_0 \left(E_{pq}^{(0)} + F_{pq}^{(0)} \right). \tag{1.23}$$

Remark 1. Let $p \leqslant q$ in Theorems 1.1 and 1.2. Then, we can change definitions (1.8) and (1.18). Namely, the estimates (1.13) and (1.23) with a certain constants $c_0, c_1; \quad c_i = c_i(p,q) \in [1,\infty), \quad i = 0,1$, remain true if we replace E_{pq} in (1.13) or $E_{pq}^{(0)}$ in (1.23), respectively, by

$$\dot{E}_{pq} = \sup_{\tau \in R_+} \left[\left(\int\limits_{(0,\tau)} \Psi^q(t,\tau)\, d\gamma(t) \right)^{1/q} \frac{1}{\omega_p(\tau)} \right], \quad p \leqslant q; \tag{1.24}$$

or by

$$\dot{E}_{pq}^{(0)} = \sup_{\tau \in R_+} \left[\left(\int\limits_{(\tau,\infty)} \Phi^q(\tau,t)\, d\gamma(t) \right)^{1/q} \frac{1}{\bar{\omega}_p(\tau)} \right], \quad p \leqslant q. \tag{1.25}$$

2 Some equivalent criteria and particular cases

2.1 Some particular cases of Theorems 1.1 and 1.2 are of special interest

Proposition 2.1. *Let $\beta \in N_p(k)$ and function ω_p be positive and continuous on $R_+, \quad \omega_p(+0) = 0$.*

1. If $V_p \circ \omega_p^{-1} \in \Delta_2$. that is, for $a = \alpha^{-1/3}$ with $\alpha \in (0,1)$ from (1.7)–(1.9),

$$D_a = \sup_{t \in R_+} \left[V_p\left(\omega_p^{-1}(t)\right) / V_p\left(\omega_p^{-1}(at)\right) \right] < \infty, \tag{2.1}$$

then there exists $c = c(p,q,\alpha) \in [1,\infty)$ such that

$$c^{-1} F_{pq} \leqslant H_{\Omega_k}(B_\mu) \leqslant c F_{pq}. \tag{2.2}$$

2. *If for $a = \alpha^{-1/3}$ with $\alpha \in (0,1)$ from (1.7)–(1.9)*

$$\delta_a = \inf_{t \in R_+} \left[W_q\left(\omega_p^{-1}(at)\right) / W_q\left(\omega_p^{-1}(t)\right) \right] > 1, \tag{2.3}$$

then there exists $c_1 = c_1(p,q,\alpha) \in [1,\infty)$ such that

$$c_1^{-1} E_{pq} \leqslant H_{\Omega_k}(B_\mu) \leqslant c_1 E_{pq}. \tag{2.4}$$

Now, we formulate corresponding result for $H_{\Omega^m}(A_\mu)$.

Proposition 2.2. *Let $\beta \in \bar{N}_p(\ m)$ and function $\bar{\omega}_p$ be positive and continuous on $R_+, \quad \bar{\omega}_p(+\infty) = 0$.*

1. If $V_p^{(0)} \circ \bar{\omega}_p^{-1} \in \Delta_2$, that is, for $a = \alpha^{-1/3}$ with $\alpha \in (0,1)$ from (1.17)–(1.19),

$$\bar{D}_a = \sup_{t \in R_+} \left[V_p^{(0)}\left(\bar{\omega}_p^{-1}(t)\right) / V_p^{(0)}\left(\bar{\omega}_p^{-1}(at)\right) \right] < \infty, \tag{2.5}$$

then there exists $c = c(p, q, \alpha) \in [1, \infty)$ *such that*

$$c^{-1} F_{pq}^{(0)} \leqslant H_{\Omega^m}(A_\mu) \leqslant c F_{pq}^{(0)}. \tag{2.6}$$

2. *If for* $a = \alpha^{-1/3}$ *with* $\alpha \in (0,1)$ *from (1.17)-(1.19),*

$$\bar{\delta}_a = \inf_{t \in R_+} \left[\bar{W}_q \left(\bar{\omega}_p^{-1}(at) \right) / \bar{W}_q \left(\bar{\omega}_p^{-1}(t) \right) \right] > 1, \tag{2.7}$$

then there exists $c_1 = c_1(p, q, \alpha) \in [1, \infty)$ *such that*

$$c^{-1} E_{pq}^{(0)} \leqslant H_{\Omega^m}(A_\mu) \leqslant c E_{pq}^{(0)}. \tag{2.8}$$

Remark 2. It is easy to see that if (2.1) holds for a given $a = \alpha^{-1/3} > 1$ then it holds for each $a > 1$, and (2.2) remains true with $c = c(p, q, \alpha, a) \in [1, \infty)$. Similar remark is relative to (2.5), and (2.6).

2.2. Here we present several special cases where the criterion of the boundedness for $H_{\Omega_k}(B_\mu)$ is obtained by application of some other approaches. Corresponding proofs are given in Section 3. Also, we show there that these equivalent descriptions may be obtained from the approaches developed in this paper.

Proposition 2.3. *Under notation (1.1)-(1.7), (1.24) let* $0 < p \leqslant \min\{1, q\}$. *Then,*

$$H_{\Omega_k}(B_\mu) = \dot{E}_{pq} = \sup_{\tau \in R_+} \left[\left(\int\limits_{(0,\tau)} \Psi^q(t,\tau)\, d\gamma(t) \right)^{1/q} \frac{1}{\omega_p(\tau)} \right]. \tag{2.9}$$

Proposition 2.4. *Let the hypotheses of Theorem 1.1 be fulfilled with* $p > q = 1$ *. Then,*

$$H_{\Omega_k}(B_\mu) \cong \left\{ \int\limits_{R_+} \left(\int\limits_{(0,\tau)} \Psi(t,\tau)\, d\gamma(t) \right)^{p'} \left(-d \left[\frac{1}{\omega_p^{p'}(\tau)} \right] \right) \right\}^{1/p'}. \tag{2.10}$$

3 Proofs of Propositions 2.1–2.4

3.1 Proof of Proposition 2.1.

1. First, we prove Part 1 of Proposition 2.1.

All the hypotheses of Theorem 1.1 are fulfilled, and we have assertions [9; (5.6) – (5.8)]. For

$$H_{\Omega_1}(B_{\mu_k}; p, q, \beta_{kp}, \gamma)$$

we can apply [9; Theorem 2.1], so that

$$H_{\Omega_1}(B_{\mu_k}; p, q, \beta_{kp}, \gamma) \cong \tilde{E}_{pq}^0(1, \beta_{kp}, \gamma, \mu_k) + \tilde{F}_{pq}(1, \beta_{kp}, \gamma, \mu_k), \tag{3.1}$$

where

$$\tilde{E}^0_{pq}(1,\beta_{kp},\gamma,\mu_k)=\left\{\sum_n a^{-ns}\left(\int\limits_{\Delta_n}\Psi^q(1,\mu_k;t,\lambda_{n+1})\,d\gamma(t)\right)^{s/q}\right\}^{1/s},$$

$$\tilde{F}_{pq}(1,\beta_{kp},\gamma,\mu_k)=\sup_n\left[V_p(1,\beta_{kp},\mu_k;\lambda_n)\,W_q(\gamma;\lambda_n)\right],\quad p\leqslant q;$$

$$\tilde{F}_{pq}(1,\beta_{kp},\gamma,\mu_k)=\left\{\sum_n V_p^s(1,\beta_{kp},\mu_k;\lambda_n)\left[W_q^s(\gamma;\lambda_n)-W_q^s(\gamma;\lambda_{n-1})\right]\right\}^{1/s},$$

for $p>q$. Here λ_n is determined by [9; (2.2)] with $\omega_p(t)=\omega_p(1,\beta_{kp};t)=\omega_p(k,\beta;t)$, thus $\lambda_n=\omega_p^{-1}(k,\beta;a^n)$. Note that (2.1) coincides with [9; (3.16)], and we can apply [9; Proposition 3.7]. Then,

$$\tilde{E}^0_{pq}(1,\beta_{kp},\gamma,\mu_k)\leqslant aD_a\tilde{F}_{pq}(1,\beta_{kp},\gamma,\mu_k),$$

and we obtain

$$H_{\Omega_1}(B_{\mu_k};p,q,\beta_{kp},\gamma)\cong\tilde{F}_{pq}(1,\beta_{kp},\gamma,\mu_k)\cong F_{pq}(1,\beta_{kp},\gamma,\mu_k)$$

(the last assertion follows from [9; (4.12))]. Together with [9; (5.6) – (5.8)] it implies

$$H_{\Omega_k}(B_\mu;p,q,\beta,\gamma)\cong F_{pq}(k,\beta,\gamma,\mu),$$

thus proving the first part of Proposition 2.1.

2. Now, we prove Part 2 of Proposition 2.1.

As before, we have assertions [9; (5.6)] and (3.1). Thus, our aim is to estimate $H_{\Omega_1}(B_{\mu_k};p,q,\beta_{kp},\gamma)$. Note that (2.3) coincides with [9; (3.25)], so that [9; Proposition 3.9] is applicable here. Then, according to [9; (3.26)],

$$\tilde{F}_{pq}(1,\beta_{kp},\gamma,\mu_k)\leqslant c\tilde{E}_{pq}(1,\beta_{kp},\gamma,\mu_k). \tag{3.2}$$

Together with [9; (3.3)], it gives

$$H_{\Omega_1}(B_{\mu_k};p,q,\beta_{kp},\gamma)\cong\tilde{E}_{pq}(1,\beta_{kp},\gamma,\mu_k). \tag{3.3}$$

Now, we apply estimate [9; (4.5)] for $p\leqslant q$, or the first estimate from [9; (4.11)] for $p>q$ and obtain

$$\tilde{E}_{pq}(1,\beta_{kp},\gamma,\mu_k)\leqslant c_1E_{pq}(1,\beta_{kp},\gamma,\mu_k). \tag{3.4}$$

From the other side, [9; (4.4)] for $p\leqslant q$, or the second estimate from [9; (4.11)] for $p>q$ imply

$$E_{pq}(1,\beta_{kp},\gamma,\mu_k)\leqslant c_2\left[\tilde{E}^0_{pq}(1,\beta_{kp},\gamma,\mu_k)+\tilde{F}_{pq}(1,\beta_{kp},\gamma,\mu_k)\right].$$

It is obvious from definitions [9; (3.1)], and [9; (2.5)] that

$$\tilde{E}^0_{pq}(1,\beta_{kp},\gamma,\mu_k)\leqslant\tilde{E}_{pq}(1,\beta_{kp},\gamma,\mu_k).$$

This inequality together with (3.2), yield the reverse estimate in (3.4), so that

$$\tilde{E}_{pq}(1,\beta_{kp},\gamma,\mu_k)\cong E_{pq}(1,\beta_{kp},\gamma,\mu_k). \tag{3.5}$$

We insert this estimate in (3.3) and obtain

$$H_{\Omega_1}(B_{\mu_k};p,q,\beta_{kp},\gamma)\cong E_{pq}(1,\beta_{kp},\gamma,\mu_k). \tag{3.6}$$

Now, assertions [9; (5.6)] and [9; (5.7)] imply (2.4) □

3.2 Proof of Proposition 2.2.

Here, we use approach developed in [9; Section 5.2] for reduction of Proposition 2.2 to Proposition 2.1.

1. First, we show that (2.5) coincides with (2.1) for $k(t) = m(t^{-1})$, and measures $\tilde{\beta}$, and $\tilde{\mu}$. Indeed, we see from [9; (5.23)] that

$$\omega_p\left(k, \tilde{\beta}; \tau\right) = \bar{\omega}_p\left(\tau^{-1}\right) \quad \Rightarrow \quad \omega_p^{-1}\left(k, \tilde{\beta}; t\right) = 1/\bar{\omega}_p^{-1}(t). \tag{3.7}$$

Now, [9; (5.25)] implies,

$$V_p\left(\omega_p^{-1}\left(k, \tilde{\beta}; t\right)\right) = V_p\left(1/\bar{\omega}_p^{-1}(t)\right) = V_p^{(0)}\left(\bar{\omega}_p^{-1}(t)\right), \quad t \in R_+,$$

so that,

$$D_a = \sup_{t \in R_+}\left(\frac{V_p\left(\omega_p^{-1}\left(k, \tilde{\beta}; t\right)\right)}{V_p\left(\omega_p^{-1}\left(k, \tilde{\beta}; at\right)\right)}\right) = \sup_{t \in R_+}\left(\frac{V_p^{(0)}\left(\bar{\omega}_p^{-1}(t)\right)}{V_p^{(0)}\left(\bar{\omega}_p^{-1}(at)\right)}\right) = \bar{D}_a < \infty.$$

Thus, the first Part of Proposition 2.1 is applicable here, and

$$H_{\Omega_k}\left(B_{\tilde{\mu}}; p, q, \tilde{\beta}, \tilde{\gamma}\right) \cong F_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right).$$

Moreover, we have equalities [9; (5.21)] and [9; (5.27)], so that

$$H_{\Omega^m}\left(A_\mu; p, q, \beta, \gamma\right) = H_{\Omega_k}\left(B_{\tilde{\mu}}; p, q, \tilde{\beta}, \tilde{\gamma}\right), \quad F_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) = F_{pq}^{(0)},$$

and (2.6) follows.

2. Similarly, (3.7) and [9; (5.26)] imply

$$W_q\left(\omega_p^{-1}\left(k, \tilde{\beta}; t\right)\right) = W_q\left(1/\bar{\omega}_p^{-1}(t)\right) = \bar{W}_q\left(\bar{\omega}_p^{-1}(t)\right), \quad t \in R_+,$$

and

$$\delta_a = \inf_{t \in R_+}\left(\frac{W_q\left(\omega_p^{-1}\left(k, \tilde{\beta}; at\right)\right)}{W_q\left(\omega_p^{-1}\left(k, \tilde{\beta}; t\right)\right)}\right) = \inf_{t \in R_+}\left(\frac{\bar{W}_q\left(\bar{\omega}_p^{-1}(at)\right)}{\bar{W}_q\left(\bar{\omega}_p^{-1}(t)\right)}\right) = \bar{\delta}_a > 1.$$

Therefore, (2.7) coincides with (2.3), and by Part 2 of Proposition 2.1,

$$H_{\Omega_k}\left(B_{\tilde{\mu}}; p, q, \tilde{\beta}, \tilde{\gamma}\right) \cong E_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right).$$

Now, we apply (2.5) and [9; (5.31)] and obtain (2.8). Proposition 2.2 is proved $\square$

3.3 Proof of Proposition 2.3.

Under the hypotheses of Proposition 2.3, i.e., for $0 < p \leqslant \min\{1, q\}$, we can apply the results of paper [2; Theorem 3] showing that $\sup\limits_{f\in\Omega_k} \ldots$ in (1.1) is achieved on the family of functions $\{f_\tau\}_{\tau\in R_+} \subset \Omega_k$, where

$$f_\tau(t) = k(t)\,\chi_{(0,\tau)}(t), t \in R_+. \tag{3.8}$$

Therefore, see also (1.24),

$$H_{\Omega_k}(B_\mu) = \sup_{\tau>0}\left[\left(\int\limits_{R_+} (B_\mu f_\tau)^q\, d\gamma\right)^{1/q} \left(\int\limits_{R_+} f_\tau^p d\beta\right)^{-1/p}\right]$$
$$= \sup_{\tau>0}\left[\left(\int\limits_{R_+}\left(\int\limits_{[t,\infty)} k\chi_{(0,\tau)}d\mu\right)^q d\gamma(t)\right)^{1/q} \left(\int\limits_{R_+} \chi_{(0,\tau)}k^p d\beta\right)^{-1/p}\right] = \dot{E}_{pq}.$$

□

Remark 3. *Let us show that the equivalent description follows from the results of Section 1.*

Let the hypotheses of Theorem 1.1 be fulfilled. Then, for $0 < p \leqslant \min\{1, q\}$ we obtain from Remark 1,

$$H_{\Omega_k}(B_\mu) \cong \dot{E}_{pq}, \tag{3.9}$$

which corresponds to (2.9). Indeed, according to Remark 1,

$$H_{\Omega_k}(B_\mu) \cong \dot{E}_{pq} + F_{pq}, \tag{3.10}$$

where F_{pq} is determined by (1.10) with V_p from (1.5). It means that

$$F_{pq} = \sup_{t\in R_+}\sup_{\tau\in(t,\infty)} \frac{\Psi(t,\tau)}{\omega_p(\tau)} W_q(t) = \sup_{\tau\in R_+}\frac{1}{\omega_p(\tau)}\sup_{t\in(0,\tau)}\Psi(t,\tau)\left(\int\limits_{(0,t)} d\gamma(\xi)\right)^{1/q}$$
$$\leqslant \sup_{\tau\in R_+}\frac{1}{\omega_p(\tau)}\sup_{t\in(0,\tau)}\left(\int\limits_{(0,t)} \Psi^q(\xi,\tau)\,d\gamma(\xi)\right)^{1/q} \leqslant \dot{E}_{pq}.$$

Consequently, (3.10) implies (3.9).

3.4 Proof of Proposition 2.4.

We apply equality [9; (5.6)] with $q = 1$, and obtain by the change of the order of integration

$$H_{\Omega_k}(B_\mu) = \sup_{f\in\Omega_1}\left[\int\limits_{R_+}\left(\int\limits_{[t,\infty)} fkd\mu\right)d\gamma(t)\left(\int\limits_{R_+} f^p k^p d\beta\right)^{-1/p}\right] =$$

$$= \sup_{f\in\Omega_1}\left[\int\limits_{R_+} f(\xi)\,k(\xi)\left(\int\limits_{(0,\xi]} d\gamma\right)d\mu(\xi)\left(\int\limits_{R_+} f^p k^p d\beta\right)^{-1/p}\right] \equiv G_0. \quad (3.11)$$

Now, we denote

$$d\beta_0(\xi) = k^p(\xi)\,d\beta(\xi)\,, d\gamma_0(\xi) = k(\xi)\left(\int\limits_{(0,\xi]} d\gamma\right)d\mu(\xi)\,, \quad (3.12)$$

so that,

$$G_0 = \sup_{f\in\Omega_1}\left[\int\limits_{R_+} fd\gamma_0\left(\int\limits_{R_+} f^p d\beta_0\right)^{-1/p}\right].$$

To estimate G_0 we apply one result from [8; Theorem 1.1] in corresponding notations, and obtain

$$G_0 \cong \left\{\int\limits_{R_+}\left(\int\limits_{(0,\tau)} d\gamma_0\right)^{1/p'}\left(-d\left[\frac{1}{\omega^{p'}(\tau)}\right]\right)\right\}^{1/p'}. \quad (3.13)$$

Here,

$$\omega(\tau) := \left(\int\limits_{(0,\tau)} d\beta_0\right)^{1/p} = \left(\int\limits_{(0,\tau)} k^p d\beta\right)^{1/p} = \omega_p(\tau)\,,$$

$$\int\limits_{(0,\tau)} d\gamma_0 = \int\limits_{(0,\tau)} k(\xi)\left(\int\limits_{(0,\xi]} d\gamma\right)d\mu(\xi) = \int\limits_{(0,\tau)}\left(\int\limits_{[t,\tau)} kd\mu\right)d\gamma(t),$$

so that,

$$\int\limits_{(0,\tau)} d\gamma_0 = \int\limits_{(0,\tau)} \Psi(t,\tau)\,d\gamma(t).$$

These assertions show that,

$$H_{\Omega_k}(B_\mu) \cong A_0 := \left\{\int\limits_{R_+}\left(\int\limits_{(0,\tau)} \Psi(t,\tau)\,d\gamma(t)\right)^{1/p'}\left(-d\left[\frac{1}{\omega_p^{p'}(\tau)}\right]\right)\right\}^{1/p'}. \quad (3.14)$$

This completes the proof of Proposition 2.4 □

3.5 Direct proof of estimate (3.14).

Below, we present the direct proof of assertion (3.14) without application of the result from [8]. This illustrates the correspondence of the results given here to ones obtained by applying of some other approaches.

So, let the hypotheses of Theorem 1.1 be fulfilled with $p > q = 1$. Then, we apply the equality [9; (5.6)], assertions [9; (2.8)], and [9; (3.45)], and obtain

$$H_{\Omega_k}(B_\mu) = H_{\Omega_1}(B_{\mu_k}; p, 1, \beta_{kp}, \gamma) \cong \tilde{E}^0_{p1} + \tilde{F}_{p1} \cong \tilde{E}^0_{p1} + \tilde{f}_{p1}, \tag{3.15}$$

where $\tilde{E}^0_{p1}$, and $\tilde{f}_{p1}$ are determined by [9; (2.5)] and [9; (3.43)] with $p > q = 1, s = \sigma = p'$, so that

$$\tilde{E}^0_{p1} = \left\{ \sum_m \left[a^{-m} \int\limits_{\Delta_m} \Psi(t, \lambda_{m+1})\, d\gamma \right]^{p'} \right\}^{1/p'}, \tag{3.16}$$

$$\begin{aligned}\tilde{f}_{p1} &= \left\{ \sum_n \sum_{m\geqslant n} \left[\psi(\lambda_m) a^{-m}\right]^{p'} \left[W_1^{p'}(\lambda_n) - W_1^{p'}(\lambda_{n-1}) \right] \right\}^{1/p'} \\ &= \left\{ \sum_m \left[\psi(\lambda_m) a^{-m}\right]^{p'} \sum_{n\leqslant m} \left[W_1^{p'}(\lambda_n) - W_1^{p'}(\lambda_{n-1}) \right] \right\}^{1/p'} \\ &= \left\{ \sum_m \left[\psi(\lambda_m) a^{-m} W_1(\lambda_m)\right]^{p'} \right\}^{1/p'}. \end{aligned} \tag{3.17}$$

Now, our aim is to prove that

$$A_0 \cong \tilde{E}^0_{p1} + \tilde{f}_{p1}. \tag{3.18}$$

Let us take into account the increase of the non-negative function

$$f(\tau) = \int\limits_{(0,\tau)} \Psi(t, \tau)\, d\gamma(t).$$

Then, we can apply the discretisation procedure [9; (2.2)–(2.4)] for A_0 in (3.14), and obtain, by analogy with [9; Proposition 2.2],

$$A_0 \cong \left\{ \sum_m a^{-mp'} f^{p'}(\lambda_m) \right\}^{1/p'} = \left\{ \sum_m a^{-mp'} \left(\int\limits_{(0,\lambda_m)} \Psi(t, \lambda_m)\, d\gamma \right)^{p'} \right\}^{1/p'}. \tag{3.19}$$

Let us show that

$$\int\limits_{(0,\lambda_m)} \Psi(t, \lambda_m)\, d\gamma = I_m + J_m, \tag{3.20}$$

where

$$I_m = \sum_{l \leqslant m-1} \int_{\Delta_l} \Psi(t, \lambda_{l+1})\, d\gamma, \tag{3.21}$$

$$J_m = \sum_{l \leqslant m-2} \left(\int_{\Delta_l} d\gamma \right) \sum_{j=l+1}^{m-1} \psi(\lambda_j). \tag{3.22}$$

Indeed,

$$\int_{(0,\lambda_m)} \Psi(t, \lambda_m)\, d\gamma = \int_{\Delta_{m-1}} \Psi(t, \lambda_m)\, d\gamma + \sum_{l \leqslant m-2} \int_{\Delta_l} \Psi(t, \lambda_m)\, d\gamma$$

$$= \int_{\Delta_{m-1}} \Psi(t, \lambda_m)\, d\gamma + \sum_{l \leqslant m-2} \int_{\Delta_l} \Psi(t, \lambda_{l+1})\, d\gamma + \sum_{l \leqslant m-2} \Psi(\lambda_{l+1}, \lambda_m) \int_{\Delta_l} d\gamma$$

$$= I_m + \sum_{l \leqslant m-2} \left(\sum_{j=l+1}^{m-1} \Psi(\lambda_j, \lambda_{j+1}) \right) \int_{\Delta_l} d\gamma = I_m + J_m.$$

Now, we change the order of summation in (3.22) and obtain,

$$J_m = \sum_{j \leqslant m-1} \psi(\lambda_j) \sum_{l \leqslant j-1} \int_{\Delta_l} d\gamma = \sum_{j \leqslant m-1} \psi(\lambda_j) \int_{(0,\lambda_j)} d\gamma = \sum_{j \leqslant m-1} \psi(\lambda_j)\, W_1(\lambda_j). \tag{3.23}$$

From (3.20), (3.21), and (3.23), we see that

$$\int_{(0,\lambda_m)} \Psi(t, \lambda_m)\, d\gamma = \sum_{l \leqslant m-1} \beta_l, \quad \beta_l = \int_{\Delta_l} \Psi(t, \lambda_{l+1})\, d\gamma + \psi(\lambda_l)\, W_1(\lambda_l). \tag{3.24}$$

We substitute this equality in (3.19), and obtain by applying of [9; Proposition 2.3],

$$A_0 \cong \left\{ \sum_m a^{-mp'} \left(\sum_{l \leqslant m-1} \beta_l \right)^{p'} \right\}^{1/p'} \cong \left\{ \sum_m a^{-mp'} \beta_{m-1}^{p'} \right\}^{1/p'} =$$

$$= a^{-1} \left\{ \sum_m a^{-mp'} \beta_m^{p'} \right\}^{1/p'}.$$

From here, from (3.24), (3.16), and (3.17), it follows (3.18).

Assertions (3.18), and (3.15) give (3.14) □

4 The case of degeneracy for measure β

Here, we consider the case of degeneracy, when the condition $\beta \in N_p(k)$ (1.12) is violated. The essentially non-trivial situation appears in the following case (see (1.4))

$$\omega_p(1) = 1 \leqslant \omega_p(\infty) < \infty. \tag{4.1}$$

We preserve here the notations of Section 1 and introduce also

$$e_{pq}=\sup_{\tau\in(0,1]}\left[\left(\int\limits_{[\xi_\alpha(\tau),\tau)}\Psi^q(t,\tau)\,d\gamma(t)\right)^{1/q}\frac{1}{\omega_p(\tau)}\right],\quad p\leqslant q; \tag{4.2}$$

$$e_{pq}=\left\{\int\limits_{(0,1]}\left(\int\limits_{[\xi_\alpha(\tau),\tau)}\Psi^q(t,\tau)\,d\gamma(t)\right)^{s/q}\left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)\right\}^{1/s},\quad p>q, \tag{4.3}$$

$$f_{pq}=\sup_{t\in(0,1)}\left[\dot{V}_p(t)\,W_q(t)\right],\quad p\leqslant q, \tag{4.4}$$

$$f_{pq}=\left\{\int\limits_{(0,1)}\dot{V}_p^s(t)\,d\left[W_q^s(t)\right]\right\}^{1/s},\quad p>q, \tag{4.5}$$

where for $t\in(0,1)$,

$$\dot{V}_p(t)=\sup_{\tau\in(t,1]}\left[\frac{\Psi(t,\tau)}{\omega_p(\tau)}\right],\quad p\in(0,1]; \tag{4.6}$$

$$\dot{V}_p(t)=\left\{\int\limits_{(t,1]}\Psi^{p'}(t,\tau)\left(-d\left[\frac{1}{\omega_p^{p'}(\tau)}\right]\right)\right\}^{1/p'};\quad p>1,\quad \frac{1}{p}+\frac{1}{p'}=1. \tag{4.7}$$

For $t\geqslant 1$ we assume that $\dot{V}_p(t)=0$.

Theorem 4.1. *Under the hypotheses of Theorem 1.1, let the condition (1.12) be replaced by (4.1). Then, the following assertions hold*

$$H_{\Omega_k}(B_\mu)\cong e_{pq}+f_{pq}+D_{pq}, \tag{4.8}$$

$$H_{\Omega_k}(B_\mu)\cong E_{pq}+F_{pq}+D_{pq}, \tag{4.9}$$

where

$$D_{pq}=\frac{1}{\omega_p(\infty)}\left(\int\limits_{R_+}\Psi^q(t,\infty)\,d\gamma(t)\right)^{1/q}. \tag{4.10}$$

Remark 4. The structure of the answers in (4.8) and in (4.9) is the same, but in e_{pq} and in f_{pq} only the values of functional parameters on $(0,1]$ are involved.

Remark 5. *Analogously to Remark 1, if $p\leqslant q$ in Theorem 4.1, we can replace the answer (4.9) by the following equivalent one:*

$$H_{\Omega_k}(B_\mu)\cong\dot{E}_{pq}+F_{pq},$$

with $\dot{E}_{pq}$ from (1.24).

Indeed, inequality [9; (4.2)] holds without assumption $\beta \in N_p(k)$. Together with (4.9), it gives

$$H_{\Omega_k}(B_\mu) \cong \dot{E}_{pq} + H_{\Omega_k}(B_\mu) \cong \dot{E}_{pq} + F_{pq} + D_{pq} \cong \dot{E}_{pq} + F_{pq}.$$

Here, the last assertion follows from the inequality $0 \leqslant D_{pq} \leqslant \dot{E}_{pq}$.

Proof. **1.** Our first aim is to prove (4.8). Let us note that under assumptions (4.1)

$$\left(\int\limits_{(0,1)} f^p d\beta\right)^{1/p} \leqslant \left(\int\limits_{R_+} f^p d\beta\right)^{1/p} \leqslant \omega_p(\infty)\left(\int\limits_{(0,1)} f^p d\beta\right)^{1/p}, \quad f \in \Omega_k. \tag{4.11}$$

Indeed,

$$f \in \Omega_k \Rightarrow f(\xi) \geqslant \frac{f(1)}{k(1)} k(\xi), \xi \in (0,1); \quad f(\xi) \leqslant \frac{f(1)}{k(1)} k(\xi), \xi \geqslant 1. \tag{4.12}$$

Therefore

$$\int\limits_{(0,1)} f^p d\beta \geqslant (f(1)/k(1))^p \int\limits_{(0,1)} k^p d\beta = (f(1)/k(1))^p; \tag{4.13}$$

$$\int\limits_{[1,\infty)} f^p d\beta \leqslant (f(1)/k(1))^p \int\limits_{[1,\infty)} k^p d\beta = (f(1)/k(1))^p \left[\omega_p^p(\infty) - 1\right],$$

and

$$\int\limits_{[1,\infty)} f^p d\beta \leqslant \left[\omega_p^p(\infty) - 1\right] \int\limits_{(0,1)} f^p d\beta.$$

Consequently,

$$\int\limits_{R_+} f^p d\beta = \int\limits_{(0,1)} f^p d\beta + \int\limits_{[1,\infty)} f^p d\beta \leqslant \omega_p^p(\infty) \int\limits_{(0,1)} f^p d\beta,$$

which gives (4.11). Now, (4.11) implies

$$H_{\Omega_k}(B_\mu) \cong \dot{H} \equiv \sup_{f \in \Omega_k} \left[\left(\int\limits_{R_+} (B_\mu f)^q d\gamma\right)^{1/q} \left(\int\limits_{(0,1)} f^p d\beta\right)^{-1/p}\right]. \tag{4.14}$$

The denominator in $\dot{H}$ is independent of values of $f(\xi)$ for $\xi \geqslant 1$. It means that $\sup\limits_{f \in \Omega_k} \ldots$ is achieved on the most function $f_0 \in \Omega_k$ which has the same values as f for $\xi \in (0,1)$, namely on

$$f_0(\xi) = f(\xi), \quad \xi \in (0,1); \quad f_0(\xi) = \frac{f(1)}{k(1)} k(\xi), \xi \geqslant 1$$

(see (4.12)). Therefore,

$$\dot{H} \cong H_{(0)} + \dot{H}_{(0)}, \tag{4.15}$$

where

$$H_{(0)} = \sup_{f\in\Omega_k} \left[\left(\int\limits_{(0,1)} \left(\int\limits_{[t,1)} f d\mu \right)^q d\gamma \right)^{1/q} \left(\int\limits_{(0,1)} f^p d\beta \right)^{-1/p} \right], \tag{4.16}$$

$$\dot{H}_{(0)} = \left[\int\limits_{(0,1)} \left(\int\limits_{[1,\infty)} k d\mu \right)^q d\gamma + \int\limits_{[1,\infty)} \left(\int\limits_{[t,\infty)} k d\mu \right)^q d\gamma \right]^{1/q}. \tag{4.17}$$

In (4.17) it was taken into account that

$$\sup_{f\in\Omega_k} \left[\frac{f(1)}{k(1)} \left(\int\limits_{(0,1)} f^p d\beta \right)^{-1/p} \right] = \left[\frac{k(1)}{k(1)} \left(\int\limits_{(0,1)} k^p d\beta \right)^{-1/p} \right] = 1,$$

because of (4.13). Now, (4.17) implies

$$\dot{H}_{(0)} = \left[\int\limits_{(0,1)} \Psi^q(1,\infty)\, d\gamma + \int\limits_{[1,\infty)} \Psi^q(t,\infty)\, d\gamma \right]^{1/q} \leqslant \left(\int\limits_{R_+} \Psi^q(t,\infty)\, d\gamma \right)^{1/q}. \tag{4.18}$$

Estimates (4.14), (4.15), and (4.18) show that

$$H_{\Omega_k}(B_\mu) + D_{pq} \cong H_{(0)} + \dot{H}_{(0)} + D_{pq} \cong H_{(0)} + D_{pq}. \tag{4.19}$$

Let us note that $k \in \Omega_k$, so that

$$H_{\Omega_k}(B_\mu) \geqslant \frac{\left(\int\limits_{R_+} (B_\mu k)^q\, d\gamma \right)^{1/q}}{\left(\int\limits_{R_+} k^p d\beta \right)^{1/p}} = \frac{\left(\int\limits_{R_+} \Psi^q(t,\infty)\, d\gamma \right)^{1/q}}{\omega_p(\infty)} = D_{pq}. \tag{4.20}$$

Therefore, from (4.19) we obtain,

$$H_{\Omega_k}(B_\mu) \cong H_{(0)} + D_{pq}. \tag{4.21}$$

Now, our aim is to estimate $H_{(0)}$ (4.16). We define measure $\beta_1 \in N_p(k)$ (see (1.12)) such that $\beta_1(e) = \beta(e)$ for each Borel set $e \subset (0,1)$ and consider

$$H_{(1)} = \sup_{f\in\Omega_k} \left[\left(\int\limits_{(0,1)} \left(\int\limits_{[t,1)} f d\mu \right)^q d\gamma \right)^{1/q} \left(\int\limits_{R_+} f^p d\beta_1 \right)^{-1/p} \right]. \tag{4.22}$$

Note that the numerator in (4.22) is independent of values of $f(\xi)$ for $\xi \geqslant 1$. It means that $\sup\limits_{f\in\Omega_k} \ldots$ is achieved in (4.22) on the smallest function $f_1 \in \Omega_k$ which has the same values as f for $\xi \in (0,1)$, namely on

$$f_1(\xi) = f(\xi), \quad \xi \in (0,1); \quad f_1(\xi) = 0, \xi \geqslant 1. \tag{4.23}$$

Therefore, $H_{(1)} = H_{(0)}$. From the other side, we see from (4.22) that

$$H_{(1)} = \sup_{f\in\Omega_k} \left[\left(\int\limits_{R_+} \left(\int\limits_{[t,\infty)} f d\mu_1 \right)^q d\gamma \right)^{1/q} \left(\int\limits_{R_+} f^p d\beta_1 \right)^{-1/p} \right], \tag{4.24}$$

where measure μ_1 is extension by zero of measure μ from $(0,1)$ on R_+, i.e., $\mu_1(e) = \mu(e \cap (0,1))$ for each Borel set $e \subset R_+$. Thus, we have

$$H_{\Omega_k}(B_\mu) \cong H_{(1)} + D_{pq}, \tag{4.25}$$

with $H_{(1)}$ determined by (4.24), where measure $\beta_1 \in N_p(k)$. It means that Theorem 1.1 is applicable for $H_{(1)}$, and we have

$$H_{(1)} \cong E_{pq}^{(1)} + F_{pq}^{(1)},$$

where $E_{pq}^{(1)}$ and $F_{pq}^{(1)}$ are determined by formulas (1.4)-(1.11) with β and μ replaced by β_1 and μ_1. Thus,

$$H_{\Omega_k}(B_\mu) \cong E_{pq}^{(1)} + F_{pq}^{(1)} + D_{pq}, \tag{4.26}$$

Let us describe explicitly the quantities $E_{pq}^{(1)}$ and $F_{pq}^{(1)}$. We denote

$$\omega_{p1}(t) = \left(\int\limits_{(0,t)} k^p d\beta_1 \right)^{1/p}, \quad t \in R_+,$$

so that

$$\omega_{p1}(t) = \omega_p(t), \quad t \in (0,1].$$

Further,

$$\Psi_1(t,\tau) = \int\limits_{[t,\tau)} k d\mu_1 = \begin{cases} \Psi(t,\tau), & 0 < t < \tau \leqslant 1; \\ \Psi(t,1), & 0 < t < 1 < \tau; \\ 0, \quad t \geqslant 1. \end{cases} \tag{4.27}$$

Then, for $p \leqslant q$

$$E_{pq}^{(1)} = \sup_{\tau\in R_+} \left[\left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi_1^q(t,\tau)\, d\gamma(t) \right)^{1/q} \frac{1}{\omega_{p1}(\tau)} \right] \cong e_{pq} + e'_{pq}; \tag{4.28}$$

where e_{pq} was defined in (4.2), and

$$e'_{pq} = \sup_{\tau>1}\left[\left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi_1^q(t,\tau)\,d\gamma(t)\right)^{1/q}\frac{1}{\omega_{p1}(\tau)}\right] \leqslant \left(\int\limits_{R_+}\Psi^q(t,\infty)\,d\gamma(t)\right)^{1/q}.$$

Here we take into account that $\omega_{p1}(\tau) \geqslant \omega_{p1}(1) = 1$ for $\tau > 1$. Simultaneously, for $p > q$

$$E^{(1)}_{pq} = \left\{\int\limits_{R_+}\left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi_1^q(t,\tau)\,d\gamma(t)\right)^{s/q}\left(-d\left[\frac{1}{\omega^s_{p1}(\tau)}\right]\right)\right\}^{1/s} \cong e_{pq} + e'_{pq}, \quad (4.29)$$

where e_{pq} was defined in (4.3), and

$$e'_{pq} = \left\{\int\limits_{(1,\infty)}\left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi_1^q(t,\tau)\,d\gamma(t)\right)^{s/q}\left(-d\left[\frac{1}{\omega^s_{p1}(\tau)}\right]\right)\right\}^{1/s}$$

$$\leqslant \left(\int\limits_{R_+}\Psi^q(t,\infty)\,d\gamma(t)\right)^{1/q}\left\{\int\limits_{(1,\infty)}\left(-d\left[\frac{1}{\omega^s_{p1}(\tau)}\right]\right)\right\}^{1/s}$$

$$= \left(\int\limits_{R_+}\Psi^q(t,\infty)\,d\gamma(t)\right)^{1/q}.$$

(recall that $\omega_{p1}(1) = 1, \omega_{p1}(\infty) = \infty$). Consequently, for all $p, q > 0$,

$$E^{(1)}_{pq} \cong e_{pq} + e'_{pq}, \quad 0 \leqslant e'_{pq} \leqslant \omega_p(\infty)\,D_{pq}.$$

It means that,

$$E^{(1)}_{pq} + D_{pq} \cong e_{pq} + D_{pq}. \quad (4.30)$$

Now, we estimate

$$F^{(1)}_{pq} = \sup_{t\in R_+}\left[V_{p1}(t)\,W_q(t)\right], \quad p \leqslant q; \quad (4.31)$$

$$F^{(1)}_{pq} = \left\{\int\limits_{R_+} V^s_{p1}(t)\,d\left[W^s_q(t)\right]\right\}^{1/s}, \quad p > q. \quad (4.32)$$

Here,

$$V_{p1}(t) = \sup_{\tau\in(t,\infty)}\left(\Psi_1(t,\tau)/\omega_{p1}(\tau)\right), \quad p \in (0,1], \quad (4.33)$$

$$V_{p1}(t) = \left\{ \int\limits_{(t,\infty)} \Psi_1^{p'}(t,\tau) \left(-d \left[\frac{1}{\omega_{p1}^{p'}(\tau)} \right] \right) \right\}^{1/p'} ; \quad p > 1, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{4.34}$$

We see from (4.27) that $V_{p1}(t) = 0, t \geqslant 1$. It means that

$$V_{p1}(t) \cong \dot{V}_p(t) + \tilde{V}_p(t), \quad t \in R_+, \tag{4.35}$$

with $\dot{V}_p(t)$ from (4.6) or (4.7), and for $t \geqslant 1$ we have $\tilde{V}_p(t) = 0$; for $t \in (0,1)$ we have

$$\tilde{V}_p(t) = \sup_{\tau \in [1,\infty)} \frac{\Psi_1(t,\tau)}{\omega_{p1}(\tau)} \leqslant \Psi(t,\infty) \frac{1}{\omega_{p1}(1)} = \Psi(t,\infty), \quad p \in (0,1],$$

or

$$\tilde{V}_p(t) = \left\{ \int\limits_{[1,\infty)} \Psi_1^{p'}(t,\tau) \left(-d \left[\frac{1}{\omega_{p1}^{p'}(\tau)} \right] \right) \right\}^{1/p'}$$

$$\leqslant \Psi(t,\infty) \left\{ \int\limits_{[1,\infty)} \left(-d \left[\frac{1}{\omega_{p1}^{p'}(\tau)} \right] \right) \right\}^{1/p'} = \Psi(t,\infty); \quad p > 1.$$

We see that always

$$0 \leqslant \tilde{V}_p(t) \leqslant \Psi(t,\infty), \quad t \in R_+. \tag{4.36}$$

Now, we substitute (4.35) in (4.31) and in (4.32), and obtain

$$F_{pq}^{(1)} \cong f_{pq} + f'_{pq}, \tag{4.37}$$

with f_{pq} from (4.4)-(4.5), and

$$f'_{pq} = \sup_{t \in R_+} \left[\tilde{V}_p(t) W_q(t) \right], \quad p \leqslant q; \tag{4.38}$$

$$f'_{pq} = \left\{ \int\limits_{R_+} \tilde{V}_p^s(t) \, d\left[W_q^s(t) \right] \right\}^{1/s}, \quad p > q, \tag{4.39}$$

where $\tilde{V}_p$ satisfies inequality (4.36). Our nearest aim is to show that (4.36) implies the estimate

$$0 \leqslant f'_{pq} \leqslant c_{pq} \omega_p(\infty) D_{pq}. \tag{4.40}$$

For $p \leqslant q$ we substitute (4.36) in (4.38) and obtain taking into account the decrease of $\Psi(t,\infty)$ and equality (1.7):

$$f'_{pq} \leqslant \sup_{t \in R_+} \left(\Psi(t,\infty) \left(\int\limits_{(0,t)} d\gamma \right)^{1/q} \right) \leqslant \sup_{t \in R_+} \left(\int\limits_{(0,t)} \Psi^q(\xi,\infty) \, d\gamma(\xi) \right)^{1/q}$$

$$= \omega_p(\infty) D_{pq}.$$

This gives (4.40) for $p \leqslant q$ with $c_{pq} = 1$. If $p > q$, we substitute (4.36) in (4.39) and obtain

$$f'_{pq} \leqslant \left\{ \int\limits_{R_+} \Psi^s(t,\infty)\, d\left[W_q^s(t)\right] \right\}^{1/s}. \tag{4.41}$$

The decrease of $\Psi(t,\infty)$ yields the following estimate

$$\left\{ \int\limits_{R_+} \Psi^s(t,\infty)\, d\left[W_q^s(t)\right] \right\}^{1/s} \leqslant c(s,q) \left\{ \int\limits_{R_+} \Psi^q(t,\infty)\, d\left[W_q^q(t)\right] \right\}^{1/q}. \tag{4.42}$$

Here $s = pq/(p-q) > q$. Now, (4.41), (4.42), and (1.7) imply

$$f'_{pq} \leqslant c(s,q) \left\{ \int\limits_{R_+} \Psi^q(t,\infty)\, d\left[W_q^q(t)\right] \right\}^{1/q} = c(s,q) \left\{ \int\limits_{R_+} \Psi^q(t,\infty)\, d\gamma \right\}^{1/q}.$$

which gives (4.40) with $c_{pq} = c(s,q)$.

Further, (4.40) and (4.37) imply

$$F_{pq}^{(1)} + D_{pq} \cong f_{pq} + D_{pq}. \tag{4.43}$$

Finally, by (4.26), (4.30) and (4.43) we have (4.8)

2. Now, to prove (4.9) it suffices to show that

$$E_{pq} + F_{pq} + D_{pq} \cong e_{pq} + f_{pq} + D_{pq}. \tag{4.44}$$

The arguments are similar to ones given at the step1. For $p \leqslant q$ we have from (1.8) and (4.2)

$$E_{pq} = \max\left\{e_{pq}, e''_{pq}\right\}, \tag{4.45}$$

where

$$e''_{pq} = \sup_{\tau>1} \left[\left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau)\, d\gamma(t) \right)^{1/q} \frac{1}{\omega_p(\tau)} \right], \tag{4.46}$$

and

$$0 \leqslant e''_{pq} \leqslant \left(\int\limits_{R_+} \Psi^q(t,\infty)\, d\gamma(t) \right)^{1/q} \sup_{\tau>1} \left[\frac{1}{\omega_p(\tau)} \right] = \omega_p(\infty)\, D_{pq}, \tag{4.47}$$

because $\omega_p(\tau)\uparrow, \quad \omega_p(1) = 1$. For $p > q$ we have from (1.9) and (4.3)

$$E_{pq} = \left\{ e_{pq}^s + \left(e''_{pq}\right)^s \right\}^{1/s} \cong e_{pq} + e''_{pq}, \tag{4.48}$$

where

$$e''_{pq} = \left\{ \int\limits_{(1,\infty)} \left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau)\, d\gamma(t) \right)^{s/q} \left(-d\left[\frac{1}{\omega_p^s(\tau)} \right] \right) \right\}^{1/s}. \tag{4.49}$$

Therefore,

$$e''_{pq} \leqslant \left(\int\limits_{R_+} \Psi^q (t, \infty) \, d\gamma (t) \right)^{1/q} \left\{ \int\limits_{(1,\infty)} \left(-d \left[\frac{1}{\omega_p^s (\tau)} \right] \right) \right\}^{1/s} \leqslant \omega_p (\infty) \, D_{pq},$$

because

$$\left\{ \int\limits_{(1,\infty)} \left(-d \left[\frac{1}{\omega_p^s (\tau)} \right] \right) \right\}^{1/s} = \left\{ \frac{1}{\omega_p^s (1)} - \frac{1}{\omega_p^s (\infty)} \right\}^{1/s} \leqslant \frac{1}{\omega_p (1)} = 1.$$

These estimates show that for all $p, q > 0$

$$E_{pq} + D_{pq} \cong e_{pq} + D_{pq}. \tag{4.50}$$

Now, let us prove that

$$F_{pq} + D_{pq} \cong f_{pq} + D_{pq}. \tag{4.51}$$

From (1.5) and (4.6) we see that for $p \in (0, 1]$

$$V_p (t) = \max \left\{ \dot{V}_p (t) \, , \bar{V}_p (t) \right\} , \quad t \in R_+, \tag{4.52}$$

where

$$\bar{V}_p (t) = \sup_{\tau \in (\delta_t, \infty)} \left(\frac{\Psi (t, \tau)}{\omega_p (\tau)} \right) , \qquad \delta_t = \max \{1, t\} \, . \tag{4.53}$$

Also, from (1.6) and (4.7) it follows that for $p > 1$

$$V_p (t) \cong \dot{V}_p (t) + \bar{V}_p (t) \, , \quad t \in R_+, \tag{4.54}$$

where

$$\bar{V}_p (t) = \left\{ \int\limits_{(\delta_t,\infty)} \Psi^{p'} (t, \tau) \left(-d \left[\frac{1}{\omega_p^{p'} (\tau)} \right] \right) \right\}^{1/p'} \tag{4.55}$$

(recall that $\dot{V}_p (t) = 0, \quad t \geqslant 1$). For all $p > 0$ we have the estimate

$$0 \leqslant \bar{V}_p (t) \leqslant \Psi (t, \infty) \, , \quad t \in R_+,$$

which is established by analogy with (4.36). It means that for all $p > 0$

$$V_p (t) \cong \dot{V}_p (t) + \bar{V}_p (t) \, , \quad 0 \leqslant \bar{V}_p (t) \leqslant \Psi (t, \infty) \, . \tag{4.56}$$

We substitute this estimate in (1.10) for $p \leqslant q$ or in (1.11) for $p > q$, and obtain

$$F_{pq} \cong f_{pq} + f''_{pq}, \tag{4.57}$$

with f_{pq} from (4.4)-(4.5), and

$$f''_{pq} = \sup_{t \in R_+} \left[\bar{V}_p (t) \, W_q (t) \right] , \quad p \leqslant q; \tag{4.58}$$

$$f''_{pq} = \left\{ \int\limits_{R_+} \bar{V}_p^s(t)\, d\left[W_q^s(t)\right] \right\}^{1/s}, \quad p > q, \tag{4.59}$$

Now, inequality

$$0 \leqslant f''_{pq} \leqslant c_{pq}\, \omega_p(\infty)\, D_{pq} \tag{4.60}$$

follows from (4.58), (4.59), and (4.56) in just the same way as (4.40) follows from (4.38), (4.39) and from estimate (4.36).

Then, (4.57) and (4.60) imply (4.51). Finally, (4.50), and (4.51) establish (4.44). □

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Mikhail L'vovich Goldman
Department of nonlinear analysis and optimization
Peoples' Friendship University of Russia
Moscow
E-mail: seulydia@yandex.ru

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