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ON HARDY-TYPE INEQUALITIES IN WEIGHTED VARIABLE EXPONENT SPACES $L_{p(x),\omega}$ FOR $0 < p(x) < 1$

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Abstract. In this paper two-weighted inequalities for the Hardy operator and its dual operator acting from one weighted variable Lebesgue space to another weighted variable Lebesgue space are proved. In particular, sufficient conditions on the weights ensuring the validity of two-weighted inequalities of Hardy type are found. Also an embedding theorem for weighted variable Lebesgue spaces is proved.

1 Introduction

It is known that for constant exponent Lebesgue L_p-spaces with $0 < p < 1$ the Hardy inequality is not satisfied for arbitrary non-negative measurable functions, but it is satisfied for non-negative monotone functions. Moreover, in [6] and [7] the sharp constant in the Hardy-type inequality for non-negative non-increasing functions was found. Recently, in [8] the Hardy-type inequality for usual L_p-spaces with $0 < p < 1$ is proved for some spaces of hypodecreasing functions (see also [17]). Therefore the investigation of the Hardy inequality in variable exponent Lebesgue spaces $L_{p(x)}$ for $0 < p(x) < 1$ is actual. Note that many investigations are devoted to the problem of boundedness of the Hardy-type operator in variable exponent Lebesgue spaces $L_{p(x)}$ for $p(x) \geq 1$ (see, for example, [2], [3], [9]). But the investigation of the Hardy inequality in variable exponent Lebesgue space $L_{p(x)}$ for $0 < p(x) < 1$ is an open problem. It is well known that the variable exponent Lebesgue spaces $L_{p(x)}$ for $p(x) \geq 1$ appeared in the literature for the first time in [14]. Further development of this theory was connected with the theory of modular function space. Somewhat later, a more explicit version of these spaces, namely modular function spaces, were investigated by many mathematicians (see [13]). The next step in the investigation of variable exponent spaces was made in [18] and in [11]. But the variable exponent Lebesgue spaces for $0 < p(x) < 1$ are much less studied. Note that the space $L_{p(x)}$ for $0 < p(x) < 1$ is not a modular function space. The study of these spaces has been stimulated by problems in elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [15], [19], [20]). For detailed information about variable exponent Lebesgue space $L_{p(x)}$ for $p(x) \geq 1$ we refer to [10].

In this paper two-weighted inequalities for the Hardy operator and its dual operator acting from one weighted variable Lebesgue space to another weighted variable Lebesgue space are proved. In particular, sufficient conditions on the weights ensuring the validity of two-weighted inequalities of Hardy type are found.

2 Preliminaries

Let R^n be the n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$, Ω be a Lebesgue measurable subset in R^n, and $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. Suppose that p is a Lebesgue measurable function on Ω such that $0 < \underline{p} \le p(x) \le \overline{p} < \infty$, $\underline{p} = \text{ess inf}_{x\in\Omega}\, p(x)$, $\overline{p} = \text{ess sup}_{x\in\Omega}\, p(x)$, and ω is a weight function on Ω, i.e. ω is a non-negative, almost everywhere (a.e.) positive function on Ω. The Lebesgue measure of a set Ω will be denoted by $|\Omega|$.

Definition 2.1. *By $L_{p(x),\omega}(\Omega)$ we denote the set of all measurable functions f on Ω such that*

$$I_{p,\omega}(f) = \int\limits_{\Omega} \left(|f(x)|\,\omega(x)\right)^{p(x)}\, dx < \infty.$$

Note that the expression

$$\|f\|_{L_{p(\cdot),\omega}(\Omega)} = \|f\|_{p,\omega,\Omega} = \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left(\frac{|f(x)|\,\omega(x)}{\lambda}\right)^{p(x)} dx \le 1\right\}$$

defines a quasi-norm on $L_{p(x),\omega}(\Omega)$. $L_{p(x),\omega}(\Omega)$ is a quasi-Banach space equipped with this quasi-norm (see [16]).

We note one important property of the spaces $L_{p(x),\omega}(\Omega)$. We have (see [16])

$$\min\left\{\|f\|^{\underline{p}}_{p,\omega,\Omega},\, \|f\|^{\overline{p}}_{p,\omega,\Omega}\right) \le I_{p,\omega}(f) \le \max\left\{\|f\|^{\underline{p}}_{p,\omega,\Omega},\, \|f\|^{\overline{p}}_{p,\omega,\Omega}\right). \tag{2.1}$$

Theorem 2.1. *Let $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < \infty$ and $r(x) = \dfrac{p(x)\,q(x)}{q(x) - p(x)}$.*

Then the inequality

$$\|fg\|_{L_{p(\cdot)}(\Omega)} \le \left(A + B + \|\chi_{\Omega_2}\|_{L_\infty(\Omega)}\right)^{1/\underline{p}} \|f\|_{L_{q(\cdot)}(\Omega)}\, \|g\|_{L_{r(\cdot)}(\Omega)} \tag{2.2}$$

holds for every $f \in L_{q(x)}(\Omega)$, $g \in L_{r(x)}(\Omega)$, where $\Omega_1 = \{x\in\Omega : p(x) < q(x)\}$, $\Omega_2 = \{x\in\Omega : p(x) = q(x)\}$, $A = \sup\limits_{x\in\Omega_1} \dfrac{p(x)}{q(x)}$, $B = \sup\limits_{x\in\Omega_1} \dfrac{q(x)-p(x)}{q(x)}$, and $\|g\|_{L_{r(\cdot)}(\Omega)} = \max\{\|g\|_{L_{r(\cdot)}(\Omega_1)}, \|g\|_{L_\infty(\Omega_2)}\}$.

Proof. We have

$$\|fg\|_{L_{p(\cdot)}(\Omega_2)} \le \|f\|_{L_{p(\cdot)}(\Omega_2)}\, \|g\|_{L_\infty(\Omega_2)} = \|f\chi_{\Omega_2}\|_{L_{p(\cdot)}(\Omega)}\, \|g\|_{L_\infty(\Omega_2)}$$

$$\leq \|f\|_{L_{p(\cdot)}(\Omega)} \|\chi_{\Omega_2}\|_{L_\infty(\Omega)} \|g\|_{L_\infty(\Omega_2)} .$$

Therefore $\left\| \dfrac{fg}{\|f\|_{L_{p(\cdot)}(\Omega)} \|g\|_{L_\infty(\Omega_2)}} \right\|_{L_{p(\cdot)}(\Omega_2)} \leq \|\chi_{\Omega_2}\|_{L_\infty(\Omega)} \leq 1$. By virtue of inequality (2.1)

$$\int\limits_{\Omega_2} \left(\frac{|f(x)\, g(x)|}{\|f\|_{L_{p(\cdot)}(\Omega)} \|g\|_{L_\infty(\Omega_2)}} \right)^{p(x)} dx \leq \|\chi_{\Omega_2}\|^{p}_{L_\infty(\Omega)} = \|\chi_{\Omega_2}\|_{L_\infty(\Omega)} . \tag{2.3}$$

It is well known that for $s > 1$ the inequality

$$ab \leq \frac{a^s}{s} + \frac{b^{s'}}{s'}, \tag{2.4}$$

holds, where $s' = \dfrac{s}{s-1}$, $a,\, b > 0$. We take $s = s(x) = \dfrac{q(x)}{p(x)}$, $a = \left(\dfrac{|f(x)|}{\|f\|_{L_{q(\cdot)}(\Omega_1)}} \right)^{p(x)}$ and $b = \left(\dfrac{|g(x)|}{\|g\|_{L_{r(\cdot)}(\Omega_1)}} \right)^{p(x)}$. Thus $s' = s'(x) = \dfrac{q(x)}{q(x) - p(x)}$ and by inequality (2.4) we have

$$\left(\frac{|f(x)||g(x)|}{\|f\|_{L_{q(\cdot)}(\Omega_1)} \ \|g\|_{L_{r(\cdot)}(\Omega_1)}} \right)^{p(x)} \leq \frac{p(x)}{q(x)} \left(\frac{|f(x)|}{\|f\|_{L_{q(\cdot)}(\Omega_1)}} \right)^{q(x)} + \frac{q(x)-p(x)}{q(x)} \left(\frac{|g(x)|}{\|g\|_{L_{r(\cdot)}(\Omega_1)}} \right)^{r(x)}$$

$$\leq A \left(\frac{|f(x)|}{\|f\|_{L_{q(\cdot)}(\Omega_1)}} \right)^{q(x)} + B \left(\frac{|g(x)|}{\|g\|_{L_{r(\cdot)}(\Omega_1)}} \right)^{r(x)} .$$

Obviously, $1 \leq A + B \leq 2$. Integrating with respect to Ω_1 and Definition 2.1, we get

$$\int\limits_{\Omega_1} \left(\frac{|f(x)||g(x)|}{\|f\|_{L_{q(\cdot)}(\Omega_1)} \ \|g\|_{L_{r(\cdot)}(\Omega_1)}} \right)^{p(x)} dx$$

$$\leq A \int\limits_{\Omega_1} \left(\frac{|f(x)|}{\|f\|_{L_{q(\cdot)}(\Omega_1)}} \right)^{q(x)} dx + B \int\limits_{\Omega_1} \left(\frac{|g(x)|}{\|g\|_{L_{r(\cdot)}(\Omega_1)}} \right)^{r(x)} dx \leq A + B. \tag{2.5}$$

Inequalities (2.3) and (2.5) imply that

$$\int\limits_{\Omega} \left(\frac{|f(x)||g(x)|}{\|f\|_{L_{q(\cdot)}(\Omega)} \ \|g\|_{L_{r(\cdot)}(\Omega)}} \right)^{p(x)} dx = \int\limits_{\Omega_1} \left(\frac{|f(x)||g(x)|}{\|f\|_{L_{q(\cdot)}(\Omega)} \ \|g\|_{L_{r(\cdot)}(\Omega)}} \right)^{p(x)} dx$$

$$+ \int\limits_{\Omega_2} \left(\frac{|f(x)||g(x)|}{\|f\|_{L_{q(\cdot)}(\Omega)} \ \|g\|_{L_{r(\cdot)}(\Omega)}} \right)^{p(x)} dx \leq \int\limits_{\Omega_1} \left(\frac{|f(x)||g(x)|}{\|f\|_{L_{q(\cdot)}(\Omega_1)} \ \|g\|_{L_{r(\cdot)}(\Omega_1)}} \right)^{p(x)} dx$$

$$+ \int\limits_{\Omega_2} \left(\frac{|f(x)||g(x)|}{\|f\|_{L_{p(\cdot)}(\Omega)} \ \|g\|_{L_\infty(\Omega_2)}} \right)^{p(x)} dx \leq A + B + \|\chi_{\Omega_2}\|_{L_\infty(\Omega)} .$$

From the last inequality we have

$$1 \geq \int_{\Omega} \left(\frac{|f(x)|\,|g(x)|}{\left(A+B+\|\chi_{\Omega_2}\|_{L_\infty(\Omega)}\right)^{1/p(x)} \ \|f\|_{L_{q(\cdot)}(\Omega)}\ \|g\|_{L_{r(\cdot)}(\Omega)}} \right)^{p(x)} dx$$

$$\geq \int_{\Omega} \left(\frac{|f(x)|\,|g(x)|}{\left(A+B+\|\chi_{\Omega_2}\|_{L_\infty(\Omega)}\right)^{1/\underline{p}} \ \|f\|_{L_{q(\cdot)}(\Omega)}\ \|g\|_{L_{r(\cdot)}(\Omega)}} \right)^{p(x)} dx.$$

Hence (2.2) follows. □

Let ω_1 and ω_2 be weights functions defined on Ω. Replacing f by $f\omega_2$ and taking $g = \dfrac{\omega_1}{\omega_2}$ in Theorem 2.1 we obtain the following corollary.

Corollary 2.1. *Let $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < \infty$ and $r(x) = \dfrac{p(x)\,q(x)}{q(x) - p(x)}$. Suppose that ω_1 and ω_2 are weights functions defined in Ω satisfying the condition*

$$\left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} < \infty.$$

Then the inequality

$$\|f\|_{L_{p(\cdot),\,\omega_1}(\Omega)} \leq \left(A+B+\|\chi_{\Omega_2}\|_{L_\infty(\Omega)}\right)^{1/\underline{p}} \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} \|f\|_{L_{q(\cdot),\,\omega_2}(\Omega)},$$

holds for every $f \in L_{q(x),\omega_2}(\Omega)$.

Remark 1. *Note that Theorem 2.1 in the case $1 \leq \underline{p} \leq p(x) \leq q(x) \leq \overline{q} \leq \infty$ was proved in [10] (see [10], Lemma 3.2.20). If $|\Omega_2| = 0$, then the constant in [10] is equal to $A + B$. Since $(A + B)^{1/\underline{p}} \leq A + B$, then the constant in (2.2) is better than the constant in [10]. Note that Corollary 2.1 in the case $\omega_1 = \omega_2 = 1$ and $|\Omega| < \infty$ was proved in [16]. In the case $1 \leq \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < \infty$ for general measures Corollary 2.1 was proved in [4] (see, also [10]).*

The following Lemmas are known.

Lemma 2.1. *[1] Let $1 \leq \underline{p} \leq p(x) \leq q(y) \leq \overline{q} < \infty$ for all $x \in \Omega_1 \subset R^n$ and $y \in \Omega_2 \subset R^m$. If $p \in C(\Omega_1)$, then the inequality*

$$\Big\| \|f\|_{L_{p(\cdot)}(\Omega_1)} \Big\|_{L_{q(\cdot)}(\Omega_2)} \leq C_{p,q} \Big\| \|f\|_{L_{q(\cdot)}(\Omega_2)} \Big\|_{L_{p(\cdot)}(\Omega_1)}$$

is valid, where

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + \frac{\overline{p}}{\underline{q}} - \frac{\underline{p}}{\overline{\overline{q}}} \right) \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty\right),$$

$\underline{q} = \operatorname{ess}\inf_{\Omega_2} q(x)$, $\overline{q} = \operatorname{ess}\sup_{\Omega_2} q(x)$, $\Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2 :\ p(x) = q(y)\}$, $\Delta_2 = \Omega_1 \times \Omega_2 \setminus \Delta_1$ *and $C(\Omega_1)$ is the space of continuous functions in Ω_1 and $f : \Omega_1 \times \Omega_2 \to R$ is any measurable function such that $\| \|f\|_{L_{q(\cdot)}(\Omega_2)} \|_{L_{p(\cdot)}(\Omega_1)} < \infty$.*

Lemma 2.2. *[6] Let $0 < s < 1$, $-\infty < a < b \leq \infty$ and f be a non-negative and non-increasing function defined on (a, b). Then*

$$\left(\int_a^b f(x)\,dx\right)^s \leq s\int_a^b f^s(x)\,(x-a)^{s-1}\,dx.$$

Lemma 2.3. *[6] Let $0 < s < 1$, $-\infty \leq a < b < \infty$ and f be a non-negative and non-decreasing function defined on (a, b). Then*

$$\left(\int_a^b f(x)\,dx\right)^s \leq s\int_a^b f^s(x)\,(b-x)^{s-1}\,dx.$$

3 Main results

We consider the classical Hardy operator and its dual operator defined as

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad H^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt,$$

where f is a non-negative function on $(0, \infty)$.

Lemma 3.1. *Let $0 < \underline{p} \leq p_n \leq \overline{p} \leq 1$, $p_n \geq p_{n+1}$ and $\{x_n\}_{n\geq 1}$ be any non-negative sequence of real numbers such that $x_n^{p_n} \geq x_{n+1}^{p_{n+1}}$ for any $n \in \mathbb{N}$.*

Then

$$\left(\sum_{n=1}^\infty x_n^{\frac{p_n}{\underline{p}}}\right)^{\underline{p}} \leq \sum_{n=1}^\infty x_n^{p_n}\left[n^{p_n} - (n-1)^{p_n}\right] \leq \sum_{n=1}^\infty x_n^{p_n}. \tag{3.1}$$

Proof. First we prove that

$$\left(\sum_{n=1}^m x_n^{\frac{p_n}{p_m}}\right)^{p_m} \leq \sum_{n=1}^m x_n^{p_n}\left[n^{p_n} - (n-1)^{p_n}\right]. \tag{3.2}$$

We consider the function $h(t) = \dfrac{(1+t)^q - 1}{t^q}$, where $t \geq 0$ and $0 < q < 1$. It is obvious that $h'(t) = \dfrac{q\left[1-(1+t)^{q-1}\right]}{t^{q+1}} \geq 0$ for all $t \geq 0$. In particular, the function h monotonically increases on the segment $[0, B]$. Therefore $h(t) \leq h(B)$, i.e.,

$$(1+t)^q \leq 1 + t^q\left[\left(B^{-1}+1\right)^q - B^{-q}\right] \quad \text{for any} \quad 0 \leq t \leq B. \tag{3.3}$$

Since $x_1^{p_1} \geq x_2^{p_2}$, then $x_2 \leq x_1^{\frac{p_1}{p_2}}$. Therefore taking $t = \dfrac{x_2}{x_1^{\frac{p_1}{p_2}}}$, $B = 1$ and $q = p_2$ in (3.3), we have

$$\left(x_1^{\frac{p_1}{p_2}} + x_2\right)^{p_2} \leq x_1^{p_1} + x_2^{p_2}\left(2^{p_2} - 1\right). \tag{3.4}$$

It is obvious that inequality (3.4) is inequality (3.2) for $m=2$. By the assumptions of Lemma 2.1 $p_2 \geq p_3$, and so $2^{p_3} \leq 2^{p_2}$. Since $x_3 \leq \frac{x_1^{\frac{p_1}{p_3}}+x_2^{\frac{p_2}{p_3}}}{2}$ by (3.3) for $t = \frac{x_3}{x_1^{\frac{p_1}{p_3}}+x_2^{\frac{p_2}{p_3}}}$, $B=\frac{1}{2}$ and $q=p_3$ and (3.4), we get

$$\left(x_1^{\frac{p_1}{p_3}}+x_2^{\frac{p_2}{p_3}}+x_3\right)^{p_3} \leq \left(x_1^{\frac{p_1}{p_3}}+x_2^{\frac{p_2}{p_3}}\right)^{p_3} + x_3^{p_3}\left(3^{p_3}-2^{p_3}\right)$$

$$\leq x_1^{p_1}+x_2^{p_2}\left(2^{p_3}-1\right)+x_3^{p_3}\left(3^{p_3}-2^{p_3}\right) \leq x_1^{p_1}+x_2^{p_2}\left(2^{p_2}-1\right)+x_3^{p_3}\left(3^{p_3}-2^{p_3}\right).$$

The last inequality is (3.1) for $m=3$. Clearly

$$x_1^{\frac{p_1}{p_{m+1}}}+x_2^{\frac{p_2}{p_{m+1}}}+\ldots+x_m^{\frac{p_m}{p_{m+1}}}+x_{m+1} \geq (m+1)x_{m+1}.$$

Hence

$$x_{m+1} \leq \frac{x_1^{\frac{p_1}{p_{m+1}}}+x_2^{\frac{p_2}{p_{m+1}}}+\ldots+x_m^{\frac{p_m}{p_{m+1}}}}{m}.$$

Therefore taking

$$t=\frac{x_{m+1}}{x_1^{\frac{p_1}{p_{m+1}}}+x_2^{\frac{p_2}{p_{m+1}}}+\ldots+x_m^{\frac{p_m}{p_{m+1}}}},\ B=\frac{1}{m} \text{ and } q=p_{m+1}$$

in (3.3), we have

$$\left(\sum_{n=1}^{m+1} x_n^{\frac{p_n}{p_{m+1}}}\right)^{p_{m+1}} = \left(\sum_{n=1}^{m} x_n^{\frac{p_n}{p_{m+1}}}+x_{m+1}\right)^{p_{m+1}}$$

$$\leq \left(\sum_{n=1}^{m} x_n^{\frac{p_n}{p_{m+1}}}\right)^{p_{m+1}} + x_{m+1}^{p_{m+1}}\left[(m+1)^{p_{m+1}}-m^{p_{m+1}}\right]$$

$$\leq \sum_{n=1}^{m} x_n^{p_n}\left[n^{p_n}-(n-1)^{p_n}\right]+x_{m+1}^{p_{m+1}}\left[(m+1)^{p_{m+1}}-m^{p_{m+1}}\right]$$

$$=\sum_{n=1}^{m+1} x_n^{p_n}\left[n^{p_n}-(n-1)^{p_n}\right].$$

By the induction principle inequality (3.2) is proved for any $m \in \mathbb{N}$.

Since the sequence $\{p_n\}_{n\geq 1}$ is decreasing, then $\lim_{n\to\infty} p_n = \underline{p}$. Therefore passing to the limit as $m \to \infty$ in (3.2) we have the left part of inequality (3.1). By using the inequality $n^{p_n} \leq (n-1)^{p_n}+1$, we have the right part of inequality (3.1). □

Example 3.1. Let $x_n = \begin{cases} n^{-\frac{\underline{p}}{2\,p_n}}, & \text{for } n=k^2 \\ 0, & \text{for } n \neq k^2, \end{cases}$ and $\overline{p} < \frac{\underline{p}+1}{2}$.

It is obvious that the sequence $\{x_n^{p_n}\}_{n\geq 1}$ is not monotone and $\sum_{n=1}^{\infty} x_n^{\frac{p_n}{\underline{p}}} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$. On the other hand, $n^{p_n} - (n-1)^{p_n} \sim p_n\, n^{p_n - 1} \sim n^{p_n-1}$ as $n \to \infty$. Therefore

$$\sum_{n=1}^{\infty} x_n^{p_n}\, [n^{p_n} - (n-1)^{p_n}] \sim \sum_{n=1}^{\infty} x_n^{p_n}\, n^{p_n-1} = \sum_{k=1}^{\infty} k^{-\underline{p}+2\,p_k-2} \leq \sum_{k=1}^{\infty} k^{2\,\overline{p}-\underline{p}-2}.$$

It is well known that the series $\sum_{k=1}^{\infty} k^{2\,\overline{p}-\underline{p}-2}$ converges if and only if $\overline{p} < \dfrac{\underline{p}+1}{2}$. Thus for $\overline{p} < \dfrac{\underline{p}+1}{2}$ inequality (3.1) does not hold.

The example shows that the condition of monotonicity of the sequence $\{x_n^{p_n}\}_{n\geq 1}$ is essential.

Remark 2. *Note that Lemma 3.1 in the case $p_1 = p_2 = \ldots = p_n = \ldots = p = const$ was proved in [6]. The idea of proving Lemma 3.1 is taken from [6].*

Theorem 3.1. *Let $x \in (0,\infty)$, $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$, $r(x) = \dfrac{\underline{p}\,p(x)}{p(x) - \underline{p}}$ and f be a non-negative and non-increasing function defined on $(0,\infty)$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then for any $f \in L_{p(x),\omega_1}(0,\,\infty)$ the inequality

$$\|Hf\|_{L_{q(\cdot),\,\omega_2}(0,\infty)} \leq \underline{p}^{\frac{1}{\underline{p}}}\, c_{p,q}\, d_p \left\| \frac{t^{1/p'}\ \left\|\frac{\omega_2}{x}\right\|_{L_{q(\cdot)}(t,\infty)}}{\omega_1} \right\|_{L_{r(\cdot)}(0,\infty)} \|f\|_{L_{p(\cdot),\,\omega_1}(0,\infty)}$$

holds, where

$$c_{p,q} = \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \underline{p}\left(\frac{1}{\underline{q}} - \frac{1}{\overline{q}}\right)\right)\left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)}\right),$$

$$S_1 = \left\{x \in (0,\infty):\ p(x) = \underline{p}\right\},\ \ S_2 = (0,\infty)\backslash S_1, \text{and } d_p = \left(1 + \frac{\overline{p} - \underline{p}}{\overline{p}} + \|\chi_{S_1}\|_{L_\infty(0,\,\infty)}\right)^{1/\underline{p}}.$$

Proof. Taking $a = 0$, $b = x$, $s = \underline{p}$ and applying Lemma 2.2, we have

$$\|Hf\|_{L_{q(\cdot),\,\omega_2}(0,\infty)} = \|\omega_2 Hf\|_{L_{q(\cdot)}(0,\infty)} = \left\| \frac{\omega_2}{x} \int_0^x f(t)\,dt \right\|_{L_{q(\cdot)}(0,\infty)}$$

$$\leq \underline{p}^{\frac{1}{\underline{p}}} \left\| \frac{\omega_2(x)}{x} \left(\int_0^x f^{\underline{p}}(t)\, t^{\underline{p}-1}\,dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,\infty)}.$$

Now applying Lemma 2.1, we get

$$\left\| \frac{\omega_2(x)}{x} \left(\int_0^x f^{\underline{p}}(t)\, t^{\underline{p}-1}\, dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,\infty)}$$

$$= \left\| \left(\int_0^\infty f^{\underline{p}}(t)\, \chi_{(0,\,x)}(t) \left[\frac{\omega_2(x)}{x} \right]^{\underline{p}} t^{\underline{p}-1}\, dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,\infty)}$$

$$= \left\| \int_0^\infty f^{\underline{p}}(t)\, \chi_{(0,\,x)}(t) \left[\frac{\omega_2(x)}{x} \right]^{\underline{p}} t^{\underline{p}-1}\, dt \right\|^{1/\underline{p}}_{L_{\frac{q(\cdot)}{\underline{p}}}(0,\infty)}$$

$$\le c_{p,q} \left(\int_0^\infty \left\| f^{\underline{p}}(t)\, \chi_{(0,\,x)}(t) \left[\frac{\omega_2(x)}{x} \right]^{\underline{p}} t^{\underline{p}-1} \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,\infty)} dt \right)^{1/\underline{p}}$$

$$= c_{p,q} \left(\int_0^\infty f^{\underline{p}}(t)\, t^{\underline{p}-1} \left\| \chi_{(0,\,x)}(t) \left[\frac{\omega_2(x)}{x} \right]^{\underline{p}} \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,\infty)} dt \right)^{1/\underline{p}}$$

$$= c_{p,q} \left(\int_0^\infty f^{\underline{p}}(t)\, t^{\underline{p}-1} \left\| \frac{\omega_2}{x} \right\|^{\underline{p}}_{L_{q(\cdot)}(t,\infty)} dt \right)^{1/\underline{p}} = c_{p,q} \left\| f\, t^{1/\overline{p}'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)} \right\|_{L_{\underline{p}}(0,\infty)}.$$

Finally, applying Corollary 2.1, we get

$$\left\| f\, t^{1/\overline{p}'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)} \right\|_{L_{\underline{p}}(0,\infty)} \le d_p \left\| \frac{t^{1/\overline{p}'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)}}{\omega_1} \right\|_{L_{r(\cdot)}(0,\infty)} \|f\|_{L_{p(\cdot),\,\omega_1}(0,\infty)}.$$

Thus

$$\|Hf\|_{L_{q(\cdot),\,\omega_2}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}}\, c_{p,q}\, d_p \left\| \frac{t^{1/p'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)}}{\omega_1} \right\|_{L_r(\cdot)(0,\infty)} \|f\|_{L_{p(\cdot),\,\omega_1}(0,\infty)}.$$

□

Theorem 3.2. *Let* $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \dfrac{\underline{p}\, p(x)}{p(x) - \underline{p}}$ *and* f *be a non-negative and non-decreasing function defined on* $(0,1)$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,1)$.

Then for any $f \in L_{p(x),\omega_1}(0,1)$ *the inequality*

$$\|Hf\|_{L_{q(\cdot),\,\omega_2}(0,1)} \le \underline{p}^{\frac{1}{\underline{p}}}\, c_{p,q}\, d_p \left\| \left\| \frac{(x-t)^{1/\overline{p}'}\, \omega_2}{x} \right\|_{L_{q(\cdot)}(t,1)} \frac{1}{\omega_1} \right\|_{L_{r(\cdot)}(0,1)} \|f\|_{L_{p(\cdot),\,\omega_1}(0,1)} \qquad (3.5)$$

holds, where $c_{p,q}$ *and* d_p *are the constants in Theorem 3.1.*

Proof. Taking $a=0$, $b=x$, $s=\underline{p}$ and applying Lemma 2.3, we have

$$\|Hf\|_{L_{q(\cdot),\,\omega_2}(0,1)} = \|\omega_2 Hf\|_{L_{q(\cdot)}(0,1)} = \left\| \frac{\omega_2}{x} \int\limits_0^x f(t)\,dt \right\|_{L_{q(\cdot)}(0,1)}$$

$$\le \left(\underline{p}\right)^{1/\underline{p}} \left\| \frac{\omega_2(x)}{x} \left(\int\limits_0^x f^{\underline{p}}(t)\,(x-t)^{\underline{p}-1}\,dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,1)}.$$

Now applying Lemma 2.1, we get

$$\left\| \frac{\omega_2(x)}{x} \left(\int\limits_0^x f^{\underline{p}}(t)\,(x-t)^{\underline{p}-1}\,dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,1)}$$

$$= \left\| \left(\int\limits_0^1 f^{\underline{p}}(t)\,\chi_{(0,\,x)}(t) \left[\frac{\omega_2(x)}{x}\right]^{\underline{p}} (x-t)^{\underline{p}-1}\,dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,1)}$$

$$= \left\| \int\limits_0^1 f^{\underline{p}}(t)\,\chi_{(0,\,x)}(t) \left[\frac{\omega_2(x)}{x}\right]^{\underline{p}} (x-t)^{\underline{p}-1}\,dt \right\|^{1/\underline{p}}_{L_{\frac{q(\cdot)}{\underline{p}}}(0,1)}$$

$$\le c_p \left(\int\limits_0^1 \left\| f^{\underline{p}}(t)\,\chi_{(0,\,x)}(t) \left[\frac{\omega_2(x)}{x}\right]^{\underline{p}} (x-t)^{\underline{p}-1} \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,1)} dt \right)^{1/\underline{p}}$$

$$= c_p \left(\int\limits_0^1 f^{\underline{p}}(t) \left\| \chi_{(0,\,x)}(t) \left[\frac{(x-t)^{1/\overline{p}'}}{x}\,\omega_2(x)\right]^{\underline{p}} \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,1)} dt \right)^{1/\underline{p}}$$

$$= c_p \left(\int\limits_0^1 f^{\underline{p}}(t) \left\| \frac{(x-t)^{1/\overline{p}'}}{x}\,\omega_2 \right\|^{\underline{p}}_{L_{q(\cdot)}(t,1)} dt \right)^{1/\underline{p}}$$

$$= c_p \left\| f \left\| \frac{(x-t)^{1/\overline{p}'}}{x}\,\omega_2 \right\|_{L_{q(\cdot)}(t,1)} \right\|_{L_{\underline{p}}(0,1)}.$$

Finally, applying Corollary 2.1, we get

$$\left\| f \left\| \frac{(x-t)^{1/\overline{p}'}}{x}\,\omega_2 \right\|_{L_{q(\cdot)}(t,1)} \right\|_{L_{\underline{p}}(0,1)}$$

$$\le \left\| \left\| \frac{(x-t)^{1/\overline{p}'}\,\omega_2}{x} \right\|_{L_{q(\cdot)}(t,1)} \frac{1}{\omega_1} \right\|_{L_{r(\cdot)}(0,1)} \|f\|_{L_{p(\cdot),\,\omega_1}(0,1)}.$$

Hence inequality (3.5) follows. □

For the dual operator H^* the theorem below is proved analogously.

Theorem 3.3. *Let $x \in (0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \dfrac{\underline{p}\,p(x)}{p(x)-\underline{p}}$ and f be a non-negative and non-increasing function defined on $(0,\infty)$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then for any $f \in L_{p(x),\omega_1}(0,\,\infty)$ the inequality

$$\|H^* f\|_{L_{q(\cdot),\,\omega_2}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}}\, c_{p,q}\, d_p \left\| \left\| \frac{(t-x)^{1/\overline{p}'}\,\omega_2}{x} \right\|_{L_{q(\cdot)}(0,\,t)} \frac{1}{\omega_1} \right\|_{L_{r(\cdot)}(0,\infty)} \|f\|_{L_{p(\cdot),\,\omega_1}(0,\infty)}$$

holds, where $c_{p,q}$ and d_p are the constants in Theorem 3.1.

Remark 3. *Note that Theorem 3.1, Theorem 3.2 and Theorem 3.4 in the case $p(x) = q(x) = p = const$ and $\omega_1(x) = \omega_2(x) = x^\alpha$ were proved in [6] with sharp constant in Hardy inequality (see also [5]). In the case $1 \le p(x) \le q(x) \le \overline{q} < \infty$ Hardy inequality is well studied (see [2], [3], [9] and etc.). In the constant exponent case $1 \le p \le q \le \infty$ for detailed information we refer to [12].*

Example 3.2. *Let $x \in (0,\infty)$, $0 < p(x) = p = const < 1$, $q(x) = \begin{cases} \frac{1}{4}, & for\ 0 < x < 1 \\ \frac{1}{2}, & for\ x \ge 1, \end{cases}$ $0 < p \le q(x)$ and $p' = \dfrac{p}{p-1}$. Suppose $\omega_1(x) = x^\alpha$, $\omega_2(x) = x^{\beta+1}$, $\beta < -2$, $\beta \ne -4$ and $\beta + 2 + \dfrac{1}{p'} < \alpha < \min\left\{\dfrac{1}{p'};\ \beta + 4 + \dfrac{1}{p'}\right\}$, where $r(x) = \infty$.*

Then the pair (ω_1, ω_2) satisfies the assumptions of Theorem 3.1.

Example 3.3. *Let $x \in (0,\,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$ and $\overline{p}' = \dfrac{\underline{p}}{\underline{p}-1}$. Suppose $\omega_1(x) = x^{1/\overline{p}'} \left\| \dfrac{\omega_2}{x} \right\|_{L_{q(\cdot)}(x,\infty)}$. Then the condition $\|1\|_{L_{r(\cdot)}(0,\infty)} < \infty$ guarantees the validity of the assumptions of Theorem 3.1. Note that by Definition 2.1 the condition $\|1\|_{L_{r(\cdot)}(0,\infty)} < \infty$ is equivalent to*

$$\int_0^\infty \delta^{\frac{\underline{p}\,p(x)}{p(x)-\underline{p}}}\, dx < \infty,$$

where $\delta \in (0,1)$. Then the pair (ω_1, ω_2) satisfies the assumptions of Theorem 3.1.

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