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PROPERTIES OF SINGULAR INTEGRALS IN TERMS OF MAXIMAL FUNCTIONS MEASURING SMOOTHNESS

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Abstract. This paper is devoted to the study of certain maximal functions measuring smoothness, related function spaces, and properties of multidimensional singular integrals. In this work we essentially use the relation between maximal functions measuring smoothness and local oscillation of functions.

1 Introduction

Maximal functions measuring smoothness play an important role in the study of structural properties of singular integrals, potential type integrals and other objects of Harmonic Analysis. It is well known that maximal functions of this type are useful in the study of smoothness of functions and of the mapping properties of various operators on smoothness spaces. The main theme of this paper is the study of certain maximal functions measuring smoothness, related function spaces and corresponding properties of singular integrals. In this work we essentially use the relation between maximal functions measuring smoothness and local oscillation of functions.

The paper is organized as follows.

Section 2 has auxiliary character and presents the basic definitions, some notation and well-known facts. In Section 3 we prove certain embedding theorems for the spaces $C_p^{k,\varphi}$ and $B_{p,q}^{k,\varphi}$ (for definitions see Section 3). In Section 4 the properties of a multidimensional singular integral in terms of maximal functions measuring smoothness are investigated. The main results are given in Theorems 3.1, 3.2, 4.2, and 4.3.

2 Preliminaries

Let Φ be the class of all positive monotonically increasing on $(0, +\infty)$ functions such that $\varphi(+0) = 0$, and Φ_k $(k \in N)$ (N is the set of all natural numbers) be the class of all functions $\varphi \in \Phi$ such that $\varphi(t) \cdot t^{-k}$ almost decreases[1]. By definition we assume that the function $\varphi(t) \equiv 1$ is an element of the classΦ.

[1] A function h is called almost decreasing on $(0, +\infty)$, if there exists a constant $c > 0$ such that for any $x_1 < x_2$ in $(0, +\infty)$ $h(x_2) \le ch(x_1)$.

Let R^n be the n-dimensional Euclidean space of points $x = (x_1, x_2, ..., x_n)$, $B(a,r) := \{x \in R^n : |x-a| \le r\}$ be the closed ball in R^n of radius $r > 0$ with the center at the point $a \in R^n$. Denote the class of all locally p-power summable functions defined on R^n by $L^p_{loc}(R^n)$, $(1 \le p < \infty)$, the class of all locally bounded functions defined on R^n by $L^\infty_{loc}(R^n)$.

Denote by P_k the totality of all polynomials on R^n whose degrees are equal to or less than $k \in N \bigcup \{0\}$. If $E \subset R^n$ is (Lebesgue) measurable , let $|E|$ denote the measure of E.

Let $k \in N$, $\varphi \in \Phi_k$, $1 \le p \le \infty$. Introduce the following notation

$$f^{\#}_{k,\varphi,p}(x) := \sup_{r>0} \frac{1}{\varphi(r)} \inf_{\pi \in P_{k-1}} \left\{ |B(x,r)|^{-\frac{1}{p}} \cdot \|f - \pi\|_{L^p(B(x,r))} \right\}, x \in R^n,$$

where $f \in L^p_{loc}(R^n)$.

For $f \in L^p_{loc}(R^n)$, $1 \le p \le \infty$, $1 \le q \le \infty$, $k \in N$ define the following functions

$$\mu^k_f(x;r)_p := \inf_{\pi \in P_{k-1}} \|f - \pi\|_{L^p(B(x,r))}, \quad r > 0, \ x \in R^n,$$

$$\mu^k_f(r)_{pq} := \begin{cases} \left\| \mu^k_f(\cdot;r)_p \right\|_{L^q(R^n)} & \text{if } 1 \le q < \infty, \\ \sup_{x \in R^n} \mu^k_f(x;r)_p & \text{if } q = \infty. \end{cases}$$

Let $x = (x_1, x_2, ..., x_n) \in R^n$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, ν_j $(i = 1, 2, ..., n)$ be non-negative integers, $|\nu| = \nu_1 + \nu_2 + ... + \nu_n$, $x^\nu = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots x_n^{\nu_n}$. Apply the orthogonalization process with respect to the inner product

$$(f,g) = \frac{1}{|B(0,1)|} \int_{B(0,1)} f(t) g(t) \, dt$$

to the system of the power functions $\{x^\nu\}$, $|\nu| \le k$, $(k \in N \bigcup \{0\})$ arranged in partially lexicographic order[2] [21]. Denote by $\{\varphi_\nu\}$, $|\nu| \le k$ the obtained orthonormal system.

Let $L^1_{loc}(R^n)$. Suppose that ([6], [19]):

$$P_{k,B(a,r)} f(x) = \sum_{|\nu| \le k} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} f(t) \varphi_\nu \left(\frac{t-a}{r} \right) dt \right) \varphi_\nu \left(\frac{x-a}{r} \right).$$

It is obvious that $P_{k,B(a,r)} f$ is a polynomial whose degree is less than or equal to k.

Denote

$$O_k(f, B(a,r))_p := \left\| f - P_{k-1,B(a,r)} f \right\|_{L^p(B(a,r))}$$

for $f \in L^p_{loc}(R^n)$ $(1 \le p \le \infty)$. Let us call $O_k(f, B(a,r))_p$ the local oscillation of k-th order of the function f on the ball $B(a,r)$ in the metric L^p.

Note that if $k = 0$, then

$$P_{k,B(a,r)} f(x) \equiv \frac{1}{|B(a,r)|} \int_{B(a,r)} f(t) \, dt =: f_{B(a,r)},$$

[2]This means that x^ν precedes x^μ if either $|\nu| < |\mu|$, or $|\nu| = |\mu|$ and the first nonzero difference $\nu_i - \mu_i$ is negative.

and therefore

$$O_1\left(f, B\left(a, r\right)\right)_1 = \int_{B(a,r)} \left|f\left(t\right) - f_{B(a,r)}\right| dt.$$

It is known [17] that for each polynomial $\pi \in P_{k-1}$ and each ball $B\left(x, r\right) \subset R^n$ the inequality

$$\left\|f - P_{k-1,B(x,r)}f\right\|_{L^p(B(x,r))} \leq C \left\|f - \pi\right\|_{L^p(B(x,r))}$$

is true, where a positive constant C does not depend on p, f, B and π. Hence it follows that

$$\exists C > 0,\ \ \forall x \in R^n,\ \ \forall r > 0:$$

$$\mu_f^k\left(x; r\right)_p \leq O_k\left(f, B\left(x, r\right)\right)_p \leq C \cdot \mu_f^k\left(x; r\right)_p. \tag{2.1}$$

It should be mentioned that the theory of spaces defined by local oscillation has been developed by several authors, for instance by F. John and L. Nirenberg [10], S. Campanato [5], N.G. Meyers [14], S. Spanne [28], J. Peetre [16], D. Sarason [26] etc. (see also [1], [9], [24], [30], [13]).

For $k = 1$, $\varphi\left(t\right) \equiv 1$, $p = 1$ we get

$$\exists C_1 > 0\ \ \exists C_2 > 0\ \ \forall x \in R^n:\ \ C_1 f^{\#}_{k,\varphi,p}\left(x\right) \leq f^{\#}\left(x\right) \leq C_2 f^{\#}_{k,\varphi,p}\left(x\right),$$

where

$$f^{\#}\left(x\right) := \sup_{r>0} \frac{1}{\left|B\left(x, r\right)\right|} \int_{B(x,r)} \left|f\left(t\right) - f_{B(x,r)}\right| dt.$$

$f^{\#}\left(x\right)$ is Fefferman-Stein's maximal function (introduced in [7]).

It is obvious that

$$f^{\#}_{k,\varphi,p}\left(x\right) = \gamma_n^{-\frac{1}{p}} \cdot \sup_{r>0} \frac{\mu_f^k\left(x; r\right)_p}{r^{n/p} \cdot \varphi\left(r\right)}, \quad x \in R^n, \tag{2.2}$$

where $\gamma_n := \left|B\left(0, 1\right)\right|$.

For $\varphi\left(t\right) = t^{\alpha}$ $\left(\alpha > 0\right)$, $k = [\alpha] + 1$ or $k = (\alpha) + 1$ the function $f^{\#}_{k,\varphi,p}\left(x\right)$, was considered in the papers of A. Calderon and R. Scott [4], R. DeVore and R. Sharpley [6]. In the latter paper the following notation was used:

$$f^{\#}_{\alpha,p}\left(x\right) := f^{\#}_{[\alpha]+1,\delta^{\alpha},p}\left(x\right); f^{b}_{\alpha,p}\left(x\right) := f^{\#}_{(\alpha)+1,\delta^{\alpha},p}\left(x\right)$$

(here $[\alpha]$ is an integer part of the number α; (α) is the largest integer less than α).

For $k = 1$, $\varphi \in \varPhi_1$ the function $f^{\#}_{k,\varphi,p}$ was used in the papers of V.I. Kolyada [11], E. Nakai and H. Sumitomo [15], and others.

Notice that if α is not an integer, then $f^{\#}_{\alpha,p}\left(x\right) \equiv f^{b}_{\alpha,p}\left(x\right)$. For any $\alpha > 0$ the inequality

$$f^{\#}_{\alpha,p}\left(x\right) \leq const \cdot f^{b}_{\alpha,p}\left(x\right), \quad x \in R^n$$

is valid.

Modulus of continuity of order $k \in N$ of the function f in the metric L^p $\left(1 \leq p \leq \infty\right)$ is defined by the equality

$$\omega_f^k\left(\delta\right)_p := \sup_{|h| \leq \delta} \left\|\Delta_h^k f\right\|_{L^p(R^n)}\ \ \left(\delta > 0\right),$$

where

$$\Delta_h^1 f(x) := f(x+h) - f(x), \quad \Delta_h^k f = \Delta_h^1 \left(\Delta_h^{k-1} f\right).$$

For $f \in L_{loc}^p(R^n)$, $1 \le p \le \infty$, $k \in N$ we consider also the following functions

$$m_f^k(x;\delta)_p := \sup_{r \le \delta} \left\{ |B(x,r)|^{-\frac{1}{p}} \cdot O_k(f, B(x,r))_p \right\}, \; x \in R^n, \; \delta > 0;$$

$$M_f^k(\delta)_p := \sup_{x \in R^n} m_f^k(x;\delta)_p, \; \delta > 0.$$

Let us notice that the functions

$$\mu_f^k(x;r)_p, \quad \mu_f^k(r)_{pq}, \quad \omega_f^k(\delta)_p, \quad m_f^k(x;r)_p \quad \text{and} \quad M_f^k(r)_p$$

monotonically increase on the interval $(0,+\infty)$ with respect to the argument r.

Further, for positive functions F and G we will use the notation $F(u) \approx G(u)$, $u \in U$ if there exist positive constants c_1 and c_2 such that

$$\forall u \in U : c_1 F(u) \le G(u) \le c_2 F(u).$$

Let us mention some known facts which we will use further in this work.

Theorem A [22]. *If $f \in L^q(R^n)$, $1 \le p \le q \le \infty$ (for $q = \infty$ it is supposed that f is equivalent to a continuous function) then the inequality*

$$\mu_f^k(\delta)_{pq} \le c \cdot \delta^{\frac{n}{p}} \omega_f^k(\delta)_q \;\; (\delta > 0)$$

holds, where $c > 0$ is independent of f and δ.

Theorem B [22]. *Let $f \in L_{loc}^q(R^n)$, $1 \le p \le \infty$, $1 \le q < \infty$,*

$$\int_0^1 t^{-\frac{n}{q}-1} \mu_f^k(t)_{qp}\, dt < +\infty.$$

Then the inequality

$$\omega_f^k(\delta)_p \le c \cdot \int_0^\delta t^{-\frac{n}{q}-1} \mu_f^k(t)_{qp}\, dt \;\; (\delta > 0)$$

is true, where the constant $c > 0$ is independent of f and δ.

Theorem C [22]. *Let $f \in L_{loc}^\infty(R^n)$. Then the inequality*

$$\omega_f^k(\delta)_\infty \le c \cdot \mu_f^k(\delta)_{\infty\infty} \;\; (\delta > 0)$$

is true, where the constant $c > 0$, is independent of f and δ.

Lemma 2.1. *Let $1 \le p \le \infty$, $k \in N$. Then the following relation*

$$f_{k,\varphi,p}^{\#}(x) \approx \sup_{r>0} \frac{m_f^k(x;r)_p}{\varphi(x)}, \; x \in R^n \tag{2.3}$$

hold, where the constants in the relation "≈" do not depend on $f \in L_{loc}^p(R^n)$.

Proof. Taking into account relation (2.1), by the definitions of functions $f^{\#}_{k,\varphi,p}(x)$, $\mu^k_f(x;r)_p$, $m^k_f(x;\delta)_p$ we obtain that

$$f^{\#}_{k,\varphi,p}(x) \le \sup_{r>0} \frac{O_k(f, B(x,r))_p}{|B(x,r)|^{-1/p} \cdot \varphi(r)} \le \sup_{r>0} \frac{m^k_f(x;r)_p}{\varphi(r)}, \ x \in R^n. \tag{2.4}$$

By the definition of $f^{\#}_{k,\varphi,p}(x)$, with the help of (2.1), it follows also that for all $r > 0$ the inequality

$$f^{\#}_{k,\varphi,p}(x)\,\varphi(r) \ge |B(x,r)|^{-\frac{1}{p}} \cdot \mu^k_f(x;r)_p \ge c\,|B(x,r)|^{-\frac{1}{p}} \cdot O_k(f, B(x,r))_p$$

holds. If to take supremum for $r \le \delta$ in both parts of this inequality we will get

$$f^{\#}_{k,\varphi,p}(x)\,\varphi(\delta) \ge c \cdot m^k_f(x;\delta)_p, \ x \in R^n, \ \delta > 0,$$

and hence

$$f^{\#}_{k,\varphi,p}(x) \ge c \cdot \frac{m^k_f(x;\delta)_p}{\varphi(\delta)}, \ x \in R^n, \ \delta > 0, \tag{2.5}$$

where the constant $c > 0$ is independent of x, δ and f.

By relations (2.4) and (2.5) we obtain the required relation (2.3). □

3 Spaces $C^{k,\varphi}_p$, $B^{k,\varphi}_{p,q}$ and some embedding theorems

Let $k \in N$, $\varphi \in \Phi_k$, $1 \le p \le \infty$. By $C^{k,\varphi}_p(R^n)$ we denote the totality of all the functions $f \in L^p(R^n)$ for which $f^{\#}_{k,\varphi,p} \in L^p(R^n)$, i.e.

$$C^{k,\varphi}_p = C^{k,\varphi}_p(R^n) := \left\{ f \in L^p(R^n) : \quad f^{\#}_{k,\varphi,p} \in L^p(R^n) \right\}.$$

Introduce the norm in $C^{k,\varphi}_p$ by means of the equality

$$\|f\|_{C^{k,\varphi}_p} := \|f\|_{L^p(R^n)} + |f|_{C^{k,\varphi}_p},$$

where

$$|f|_{C^{k,\varphi}_p} := \left\| f^{\#}_{k,\varphi,p} \right\|_{L^p(R^n)}.$$

With the introduced norm the space $C^{k,\varphi}_p$ is a Banach space.

In the paper of R. DeVore and R. Sharpley [6] for the space $C^{k,\varphi}_p$ in the case $\varphi(t) = t^\alpha$ $(\alpha > 0)$, the following notation was used:

$$C^{k,\delta^\alpha}_p =: \begin{cases} C^\alpha_p, & if \quad k = [\alpha] + 1, \\ \dot{C}^\alpha_p, & if \quad k = (\alpha) + 1 \end{cases}.$$

Some weighted analogues of the spaces C^α_p and $\dot{C}^\alpha_p$ were considered in the paper of D.C. Yang and S.B. Yang [31].

Theorem ([3]). If k is a positive integer and $1 < p \leq \infty$, then

$$\dot{C}_p^k(R^n) = W_p^k(R^n),$$

where $W_p^k(R^n)$ is the Sobolev space, and the norms in these spaces are equivalent.

Theorem ([6]). Let $\alpha > 0$. Then

$$\dot{C}_\infty^\alpha(R^n) = Lip\,\alpha,$$

where

$$Lip\,\alpha := \left\{ f \in L^\infty(R^n) : \quad \omega_f^{(\alpha)+1}(\delta)_\infty = O(\delta^\alpha), \;\; \delta > 0 \right\},$$

$$\|f\|_{Lip\,\alpha} := \|f\|_{L^\infty(R^n)} + \sup_{\delta>0} \frac{\omega_f^{(\alpha)+1}(\delta)_\infty}{\delta^\alpha},$$

and the norms in these spaces are equivalent.

For a discussion of the Lipschitz spaces $Lip\,\alpha$ see, for instance, [29].

Let $k \in N$, $\varphi \in \Phi_k$, $1 \leq p, q \leq \infty$. By $B_{p,q}^{k,\varphi}$ we denote the totality of all the functions $f \in L^p(R^n)$ (for $p = \infty$ it is assumed that f is equivalent to a continuous function) for which the following semi-norm is finite

$$|f|_{B_{p,q}^{k,\varphi}} := \begin{cases} \left(\int\limits_0^\infty \left(\frac{\omega_f^k(t)_p}{\varphi(t)} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if} \quad 1 \leq q < \infty, \\ \sup\limits_{t>0} \frac{\omega_f^k(t)_p}{\varphi(t)} & \text{if} \quad q = \infty. \end{cases}$$

In the space $B_{p,q}^{k,\varphi}$ we introduce the norm by the equality

$$\|f\|_{B_{p,q}^{k,\varphi}} := \|f\|_{L^p(R^n)} + |f|_{B_{p,q}^{k,\varphi}}.$$

Note that spaces of type $B_{p,q}^{k,\varphi}$ have been considered in works of several authors (see, e.g., [8] and the literature quoted there).

Lemma 3.1. *Let $1 \leq p \leq \infty$, $\varphi \in \Phi_k$, $k \in N$. Then the embedding*

$$B_{p,p}^{k,\varphi} \subset C_p^{k,\varphi} \tag{3.1}$$

holds and

$$\exists c > 0 \;\; \forall f \in B_{p,p}^{k,\varphi} : \;\; \|f\|_{C_p^{k,\varphi}} \leq c \|f\|_{B_{p,p}^{k,\varphi}}.$$

Proof. First we notice that if $\varphi \in \Phi_k$, then $\varphi(2r) \approx \varphi(r)$, $r > 0$.

Let $1 \leq p < \infty$. Then for all $r > 0$ and $x \in R^n$ we have

$$\int_0^\infty \left(\frac{\mu_f^k(x;t)_p}{t^{n/p}\varphi(t)} \right)^p \frac{dt}{t} \geq \int_r^{2r} \left(\frac{\mu_f^k(x;t)_p}{t^{n/p}\varphi(t)} \right)^p \frac{dt}{t} \geq c \cdot \left(\frac{\mu_f^k(x;r)_p}{r^{n/p}\varphi(r)} \right)^p,$$

where the constant $c > 0$ is independent of x, r and f. By equality (2.2) we obtain that for any $x \in R^n$

$$\left(f^{\#}_{k,\varphi,p}(x)\right)^p = \gamma_n^{-1} \cdot \left(\sup_{r>0} \frac{\mu_f^k(x;r)_p}{r^{n/p} \cdot \varphi(r)}\right)^p \le (c\gamma_n)^{-1} \cdot \int_0^\infty \left(\frac{\mu_f^k(x;t)_p}{t^{n/p}\varphi(t)}\right)^p \frac{dt}{t}.$$

Hence, with the help of Theorem A [22] we get

$$\left\| f^{\#}_{k,\varphi,p} \right\|^p_{L^p(R^n)} \le (c\gamma_n)^{-1} \cdot \int_0^\infty \left(\frac{\mu_f^k(t)_{pp}}{t^{n/p}\varphi(t)}\right)^p \frac{dt}{t} \le const \cdot \int_0^\infty \left(\frac{\omega_f^k(t)_p}{\varphi(t)}\right)^p \frac{dt}{t}.$$

The last inequality implies that if $1 \le p < \infty$, then $B^{k,\varphi}_{p,p} \subset C^{k,\varphi}_p$ and

$$\exists c > 0 \ \ \forall f \in B^{k,\varphi}_{p,p}: \ \ \|f\|_{C^{k,\varphi}_p} \le c\, \|f\|_{B^{k,\varphi}_{p,p}}.$$

The case $p = \infty$ is considered similarly. □

Lemma 3.2. *Let $1 \le p < \infty$, $\varphi \in \Phi_k$, $k \in N$ and*

$$\int_0^\delta \frac{\varphi(t)}{t} dt = O(\varphi(\delta))\,(\delta > 0). \tag{3.2}$$

Then the embedding

$$C^{k,\varphi}_p \subset B^{k,\varphi}_{p,\infty} \tag{3.3}$$

holds and

$$\exists c > 0 \ \ \forall f \in C^{k,\varphi}_p: \ \ \|f\|_{B^{k,\varphi}_{p,\infty}} \le c\, \|f\|_{C^{k,\varphi}_p}.$$

Proof. Let $1 \le p < \infty$ and $f \in C^{k,\varphi}_p$. Then with the help of Theorem B [22] taking into account (2.2) and (3.2), we obtain

$$\omega_f^k(\delta)_p \le c \cdot \int_0^\delta t^{-\frac{n}{p}-1} \mu_f^k(t)_{pp}\, dt = c \cdot \int_0^\delta \frac{\mu_f^k(t)_{pp}}{t^{n/p}\varphi(t)} \cdot \frac{\varphi(t)}{t} dt$$

$$= c \cdot \int_0^\delta \left\| \frac{\mu_f^k(\cdot\,;t)_p}{t^{n/p}\varphi(t)} \right\|_{L^p(R^n)} \cdot \frac{\varphi(t)}{t} dt \le c_1 \cdot \left\| f^{\#}_{k,\varphi,p} \right\|_{L^p(R^n)} \cdot \int_0^\delta \frac{\varphi(t)}{t} dt$$

$$\le c_2 \cdot \left\| f^{\#}_{k,\varphi,p} \right\|_{L^p(R^n)} \cdot \varphi(\delta) = c_2 \cdot |f|_{C^{k,\varphi}_p} \cdot \varphi(\delta),\ \delta > 0,$$

where the positive constants c_1, c_2 are independent of f and δ. Hence, it is easy to obtain the statements of the lemma. □

The following statement can be easily proved with the help of Theorem C [22].

Lemma 3.3. *Let $\varphi \in \Phi_k$, $\varphi(+0) = 0$, $k \in N$. Then the embedding*

$$C^{k,\varphi}_\infty \subset B^{k,\varphi}_{\infty,\infty} \tag{3.4}$$

holds and

$$\exists c > 0 \ \ \forall f \in C^{k,\varphi}_\infty: \ \ \|f\|_{B^{k,\varphi}_{\infty,\infty}} \le c\, \|f\|_{C^{k,\varphi}_\infty}.$$

Remark 4. *In the case* $\varphi(t) = t^{\alpha}$ $(\alpha > 0)$, $k = [\alpha] + 1$ *Lemmas 3.1, 3.2 and 3.3 were proved in the paper by R. DeVore and R. Sharpley [6]. In the same paper it is proved that embeddings* (3.1), (3.3) *and* (3.4) *are the best in the scale of Besov spaces.*

Combining Lemma 3.1 and 3.2, we get the following theorem.

Theorem 3.1. *Let* $1 \le p < \infty$, $\varphi \in \Phi_k$, $k \in N$ *and condition* (3.2) *be fulfilled. Then*

$$B_{p,p}^{k,\varphi} \subset C_p^{k,\varphi} \subset B_{p,\infty}^{k,\varphi},$$

and the embeddings are continuous.

Lemma 3.1 and 3.3 yield

Theorem 3.2. *Let* $\varphi \in \Phi_k$, $\varphi(+0) = 0$, $k \in N$. *Then*

$$C_{\infty}^{k,\varphi} = B_{\infty,\infty}^{k,\varphi},$$

and the norms in these spaces are equivalent.

4 Properties of singular integrals

Consider the singular integral operator

$$Af(x) = A_k f(x)$$

$$= \lim_{\varepsilon \to +0} \int_{R^n} \left\{ K_{\varepsilon}(x-y) - \left(\sum_{|\nu| \le k-1} \frac{x^{\nu}}{\nu!} D^{\nu} K(-y) \right) X_{\{|t|>1\}}(y) \right\} f(y)\, dy,$$

where

$$K(x) = \Omega\left(\frac{x}{|x|}\right) \cdot |x|^{-n}, \quad \int_{S^{n-1}} \Omega(x)\, ds = 0, \;\; K_{\varepsilon}(x) = K(x)\, X_{\{|t|>\varepsilon\}}(x),$$

$X_{\{|t|>\varepsilon\}}(x)$ is the characteristic function of the set $\{t \in R^n : \; |t| > \varepsilon\}$, S^{n-1} is the unit sphere in R^n, $k \in N$; for $k = 1$ it is assumed that $K(x)$ is differentiable and has bounded first order partial derivatives, for $k > 1$ the function $K(x)$ is k-times continuously differentiable on S^{n-1}; $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $\nu_1, \nu_2, ..., \nu_n$ are non-negative integers, $x = (x_1, x_2, ..., x_n) \in R^n$, $x^{\nu} = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdot ... \cdot x_n^{\nu_n}$, $|\nu| = \nu_1 + \nu_2 + ... + \nu_n$, $\nu! = \nu_1! \nu_2! \cdots \nu_n!$,

$$D^{\nu} f(x) := \frac{\partial^{|\nu|} f}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} ... \partial x_n^{\nu_n}}.$$

We can verify that if $f \in L^p(R^n)$ $(1 \le p < \infty)$, then the singular integral $Af = A_k f$ differs from the integral

$$Tf(x) = \lim_{\varepsilon \to +0} \int_{R^n} K_{\varepsilon}(x-y) f(y)\, dy$$

by a polynomial of degree at most $k - 1$.

Theorem D [20]. Let $x \in R^n$, $f \in L^p_{loc}(R^n)$, $1 < p < \infty$, $k \in N$ and

$$\int_1^\infty t^{-k-1} m_f^k (x;t)_p \, dt < +\infty.$$

Then the inequality

$$m^k_{A_k f}(x;r)_p \leq C \cdot r^k \int_r^\infty t^{-k-1} m_f^k (x;t)_p \, dt$$

is true for any $r > 0$, where the constant $C > 0$ is independent of f, x and r. (The statement of the theorem includes the existence of the singular integral $A_k f(x)$ almost everywhere.)

By Z we denote a class of all functions $\varphi \in \Phi$ satisfying condition (3.2). The class of all functions $\varphi \in \Phi_k$ satisfying the condition

$$\delta^k \int_\delta^\infty \frac{\varphi(t)}{t^{k+1}} dt = O(\varphi(\delta)) \, (\delta > 0) \tag{4.1}$$

is denoted by Z_k.

Let, for instance, $\varphi(t) = t^\alpha$, $t \in (0, +\infty)$. If $\alpha > 0$, then $\varphi \in Z$, and if $0 \leq \alpha < k$, then $\varphi \in Z_k$.

Theorem 4.1. *Let $f \in L^p_{loc}(R^n)$, $1 < p < \infty$, $\varphi \in Z_k$, $k \in N$. If at the point $x \in R^n$ the quantity $f^\#_{k,\varphi,p}(x)$ is finite, then the following inequality is valid*

$$(A_k f)^\#_{k,\varphi,p}(x) \leq c \cdot f^\#_{k,\varphi,p}(x), \tag{4.2}$$

where the constant $c > 0$ is independent of f and x.

Proof. By applying Theorem D [20] and taking into account relations (2.3) and (4.1), it follows that

$$\frac{m^k_{A_k f}(x;r)_p}{\varphi(r)} \leq C \cdot \frac{1}{\varphi(r)} \cdot r^k \int_r^\infty \frac{\varphi(t)}{t^{k+1}} \cdot \frac{m_f^k(x;t)_p}{\varphi(t)} dt$$

$$\leq C_1 \cdot f^\#_{k,\varphi,p}(x) \cdot \frac{1}{\varphi(r)} \cdot r^k \int_r^\infty \frac{\varphi(t)}{t^{k+1}} dt \leq C_2 \cdot f^\#_{k,\varphi,p}(x), \; r > 0,$$

where the constants $C_1 > 0$, $C_2 > 0$ are independent of f, x and r. Hence, inequality (4.2) follows. □

Note that Theorem 4.1 for $k = 1$, $\varphi(\delta) = \delta^\alpha$ $(\delta > 0)$, $0 < \alpha < 1$, was proved in the paper by R. Sharpley and Y.-S. Shim [27].

By Theorem 4.1 the following theorem immediately turns out.

Theorem 4.2. *Let $1 < p < \infty$, $\varphi \in Z_k$, $k \in N$. If $f \in L^p_{loc}(R^n)$ and $|f|_{C_p^{k,\varphi}} < +\infty$, then the following inequality is valid*

$$|A_k f|_{C_p^{k,\varphi}} \leq c \cdot |f|_{C_p^{k,\varphi}},$$

where the constant $c > 0$ is independent of f.

Theorem 4.3. *Let $\varphi \in Z \bigcap Z_k$, $k \in N$, $|f|_{C_\infty^{k,\varphi}} < +\infty$. Then the following inequality is true*

$$|A_k f|_{C_\infty^{k,\varphi}} \leq c \cdot |f|_{C_\infty^{k,\varphi}},$$

with the constant $c > 0$ independent of f.

Proof. By virtue of the results of [23] there is a constant $C_0 > 0$ such that if $|f|_{C_\infty^{k,\varphi}} < +\infty$, then for all $x \in R^n$ and $r > 0$

$$f^{\#}_{k,\varphi,\infty}(x) \leq C_0 \, |f|_{C_\infty^{k,\varphi}} \, \varphi(r).$$

Taking into account relation (2.3), we obtain that

$$M_f^k(r)_\infty \leq C_1 \, |f|_{C_\infty^{k,\varphi}} \, \varphi(r), r > 0, \tag{4.3}$$

where the constant $C_1 > 0$ is independent of f and r.

On the other hand the following estimate

$$M^k_{A_k f}(r)_\infty \leq C_2 \left(\int_0^r \frac{M_f^k(t)_\infty}{t} dt + r^k \int_r^\infty \frac{M_f^k(t)_\infty}{t^{k+1}} dt \right), \ r > 0 \tag{4.4}$$

is well known (see [18]). Taking into account the condition $\varphi \in Z \bigcap Z_k$, by inequalities (4.3) and (4.4) we obtain that

$$M^k_{A_k f}(r)_\infty \leq C_3 \, |f|_{C_\infty^{k,\varphi}} \, \varphi(r), \quad r > 0.$$

Further, for all $x \in R^n$ we have

$$(A_k f)^{\#}_{k,\varphi,\infty}(x) \approx \sup_{r>0} \frac{m^k_{A_k f}(x;r)_\infty}{\varphi(r)} \leq \sup_{r>0} \frac{M^k_{A_k f}(r)_\infty}{\varphi(r)} \leq C_3 \, |f|_{C_\infty^{k,\varphi}}.$$

Therefore

$$|A_k f|_{C_\infty^{k,\varphi}} := \left\| (A_k f)^{\#}_{k,\varphi,\infty} \right\|_{L^\infty(R^n)} \leq c \, |f|_{C_\infty^{k,\varphi}},$$

where the constant $c > 0$ is independent of f. □

The last theorem is an analogue of Plemelj-Privalov's theorem for multi-dimensional singular integrals (see, for instance, [25]).

In the case $\varphi(t) = t^\alpha$, where $\alpha > 0$ and α is non-integer, Theorems 4.2 and 4.3 for the integral $A_1 f$ (including the case $p = 1$) were proved in the paper by R. Sharpley, Y.-S. Shim [27].

By the from definition of the space $C_p^{k,\varphi}$ it follows that $C_p^{k,\varphi}$ continuously embedded in the space L^p.

From Theorem 4.2 we get the following

Theorem 4.4. *Let $1 < p < \infty$, $\varphi \in Z_k$, $k \in N$ and*

$$Tf(x) = v.p. \int_{R^n} K(x-y) f(y) \, dy = \lim_{\varepsilon \to +0} \int_{R^n} K_\varepsilon(x-y) f(y) \, dy.$$

Then the operator Tf boundedly acts in the space $C_p^{k,\varphi}$.

For the case $\varphi(t) = t^{\alpha}$, where $\alpha > 0$, this theorem was proved in the paper [20]. A theorem on the boundedness of a singular integral operator on the interval $(0,\ 1)$ in the space C_p^{α} for the case $0 < \alpha < 1$ was proved in the paper by A. Korenovskii [12].

The boundedness of singular integral operators in general Morrey-type spaces was studied in [2].

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