EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 4, Number 3 (2013), 84 – 106

QUALITATIVE PROPERTIES OF STRUCTURALLY DAMPED WAVE MODELS

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Communicated by H. Begehr

Key words: structurally damped waves, energies of higher order, smoothing effect, WKB analysis, scattering states

AMS Mathematics Subject Classification: 35B40, 35M20, 74D99.

Abstract. Qualitative properties such as precise decay estimates for energies of higher order and smoothing effects for a special class of structurally damped wave models are investigated in this paper. The main tools are diagonalization techniques and the theory of zones in the extended phase space. Some special effects for visco-elastic damped wave models are explained. These models are limit cases of structurally damped wave models interpolating between classical damped waves and visco-elastic waves.

1 Introduction

The structurally damped wave model has been motivated in an abstract form

$$u_{tt} + Au + A^{\sigma} u_t = 0$$

in [3] for $\sigma \in [0,1]$ and is also of interest in the framework of control theory (see [1], [2] and [6]). Most of the applications of this abstract model are devoted to mixed problems in bounded domains. There are very few contributions to the Cauchy problem. Choosing $A = -\Delta$, where Δ is the Laplacian on $\mathbb{R}^n$, we consider for $\sigma \in [0,1]$ the Cauchy problem with a time-dependent coefficient in the damping term

$$u_{tt} - \Delta u + b(t)\big(-\Delta\big)^{\sigma} u_t = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{1.1}$$

This special damped wave model represents an interpolation between the external damped wave model ($\sigma = 0$) with the damping term $b(t)u_t$ and the visco-elastic damped wave model ($\sigma = 1$) with the damping term $-b(t)\Delta u_t$. These two models have been investigated, see e.g. [12], [13], [10], and [11]. The case of structurally damped waves with time-dependent coefficient $b(t) = \mu(1+t)^{\delta}, \delta \in (-1,1)$ was studied in [7] and [9]. The main goal was to discover the parabolic effect for solutions to

$$u_{tt} - \Delta u + \mu(1+t)^{\delta}\big(-\Delta\big)^{\sigma} u_t = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{1.2}$$

Under the parabolic effect we will understand that energies of higher order of the solutions to (1.2) decay faster with increasing order, although the classical energy might have no decay behaviour. It is proved in [7] and [9] that in our language this parabolic effect appears for $\sigma \in (0,1)$. Up to now a *diffusion phenomenon* for (1.1), that is, a connection from the point of view of decay estimates for solutions of parabolic type Cauchy problem is to the authors knowledge still open. Such a diffusion phenomenon is true for $\sigma = 0$. The optimality of these decay estimates remained open. *The first goal of this paper is to prove optimality for a class of scale invariant models. The second goal of this paper is to explain the smoothing effect for solutions to structurally damped waves.* This smoothing effect and a related parabolic profile of solutions was already studied by using a spectral theoretical approach in an abstract setting in [4] and [5]. Applications are given to mixed problems for the case of interior domains.

The main tools of our considerations will be elliptic as well as hyperbolic WKB-analysis to obtain solution representations. For proving the optimality of energy decay estimates we will use ideas from scattering theory and component-wise analysis. In order to comprehend the qualitative behaviour of the solutions to (1.2) for $\sigma \geq 1$ the case $\sigma = 1$ is also described and some new effects will be sketched.

2 Precise estimates for the energies of higher order

We are interested in the structurally damped Cauchy problem (1.2). But now we restrict ourselves to scale invariant models, that is, if $u(1+t,x)$ is a solution of

$$u_{tt} - \Delta u + \mu(1+t)^{\delta}\big(-\Delta\big)^{\sigma} u_t = 0,$$

then $v(t,x) = u\big(\lambda(1+t), \lambda x\big)$ is a solution as well for every $\lambda \neq 0$. This convention leads to the family of scale invariant models

$$u_{tt} - \Delta u + \mu(1+t)^{2\sigma-1}\big(-\Delta\big)^{\sigma} u_t = 0,\ \sigma \in [0,1]\,,\ \mu > 0.$$

Theorem 2.1. *Let us consider the Cauchy problem*

$$\begin{aligned} &u_{tt} - \Delta u + \mu(1+t)^{2\sigma-1}\big(-\Delta\big)^{\sigma} u_t = 0,\ \sigma \in (0,1)\,,\ \mu > 0, \\ &u(0,x) = u_0(x),\ u_t(0,x) = u_1(x). \end{aligned} \tag{2.1}$$

Then the following estimate for the energy

$$E_m(u)(t) = \sum_{|\beta|=m} \big\|\nabla^{\beta} u(t,\cdot)\big\|^2_{L^2(\mathbb{R}^n)} + \sum_{|\tilde{\beta}|=m-1} \Big\|\nabla^{\tilde{\beta}} u_t(t,\cdot)\Big\|^2_{L^2(\mathbb{R}^n)}$$

of order $m \geq 1$ holds and is optimal:

$$E_m(u)(t) \leq C_m\left((1+t)^{-2m}\,\|u_0\|^2_{H^m(\mathbb{R}^n)} + (1+t)^{-2(m-1)}\,\|u_1\|^2_{H^{m-1}(\mathbb{R}^n)}\right).$$

Remark 2.1. *Such estimates are proved in [7] and [9] for the solutions to (1.2). Applying these estimates to the solutions for (2.1), then these estimates coincide with those ones from Theorem 2.1. In this sense the optimality of the results from [7] and [9] is shown.*

2.1 Proof of Theorem 2.1

The proof bases on the use of the scale invariant models (2.1) and the application of different methods from the theory of ordinary differential equations.

First we apply partial Fourier transformation to the equation from (2.1). We obtain

$$\hat{u}_{tt} + |\xi|^2 \hat{u} + \mu(1+t)^{2\sigma-1} |\xi|^{2\sigma} \hat{u}_t = 0, \tag{2.2}$$

where $\hat{u} = \hat{u}(t, \xi)$ denotes the Fourier transform of u.

Then we apply the transformation

$$\tau = (1+t)\,|\xi|\,,\; w(\tau) = \hat{u}(1+t, \xi).$$

The scale invariance of the model implies that $\hat{u}$ is constant for t and ξ satisfying $(1+t)\,|\xi| = \tau = \text{const}$. The function $w = w(\tau)$ satisfies the ordinary differential equation

$$w_{\tau\tau} + \mu\tau^{2\sigma-1} w_\tau + w = 0,\; \tau \in \mathbb{R}^+,\; \mu > 0,\; \sigma \in (0,1). \tag{2.3}$$

The information about the data is transformed to the Cauchy conditions $w(|\xi|) = w_0$ and $w_\tau(|\xi|) = w_1$ depending on the parameter $|\xi|$. For this reason we shall study in the following proposition a parameter-dependent family of Cauchy problems with a parameter in the Cauchy data.

Proposition 2.1. *Consider the parameter-dependent family of Cauchy problems*

$$\begin{aligned} w_{\tau\tau} + \mu\tau^{2\sigma-1} w_\tau + w = 0, \quad \mu > 0,\; \sigma \in (0,1)\,, \\ w(r) = w_0,\; w_\tau(r) = w_1,\; r \geq 0. \end{aligned}$$

The following estimates hold for $0 \leq r \leq \tau$ *and* $\sigma \in \left(0, \frac{1}{2}\right]$:

$$\begin{aligned} |w(\tau)| &\leq C \exp\Big(-\frac{\mu}{4\sigma}(\tau^{2\sigma} - r^{2\sigma})\Big)\big(|w(r)| + |w_\tau(r)|\big), \\ |w_\tau(\tau)| &\leq C \exp\Big(-\frac{\mu}{4\sigma}(\tau^{2\sigma} - r^{2\sigma})\Big)\big(|w(r)| + |w_\tau(r)|\big). \end{aligned}$$

Furthermore, we have for $0 \leq r \leq \tau$ *and* $\sigma \in \left(\frac{1}{2}, 1\right)$:

$$\begin{aligned} |w(\tau)| &\leq C \exp\left(-\frac{C'}{\mu(2-2\sigma)}\big(\tau^{2-2\sigma} - r^{2-2\sigma}\big)\right)\big(|w(r)| + |w_\tau(r)|\big), \\ |w_\tau(\tau)| &\leq C \exp\left(-\frac{C'}{\mu(2-2\sigma)}\big(\tau^{2-2\sigma} - r^{2-2\sigma}\big)\right)\big(|w(r)| + |w_\tau(r)|\big). \end{aligned}$$

Here the constant $C = C(\mu, \sigma)$ *is independent of* r *and the constant* C' *is independent of* r, μ *and* σ.

Proof. We introduce the following two zones:

$$Z_1(N) := \{0 \leq \tau \leq N\}\,, \quad Z_2(N) := \{N \leq \tau < \infty\}\,.$$

Here $N \geq 0$ is a constant that should be chosen later. Due to the integrability of the damping term $\mu\tau^{2\sigma-1}$ in the first zone $Z_1(N)$ we can show for the energy $W = \big(w, w_\tau\big)^T$ the following estimate:

$$\|W(\tau)\| \leq C_N \|W(r)\|$$

for every $0 \leq r \leq \tau \leq N$ and $\sigma \in (0,1)$. To examine the second zone $Z_2(N)$ we distinguish between the cases $r \leq N$ and $r \geq N$. At first we examine the case $r \geq N$. Later we will use the estimate from above for $\|W\|$ in order to obtain results for $r \leq N$. We apply the Liouville transformation

$$v(\tau) := w(\tau) \exp\Big(\frac{\mu}{4\sigma}\tau^{2\sigma}\Big), \quad N \leq r \leq \tau$$

to equation (2.3) and obtain the new Cauchy problem

$$v_{\tau\tau} + \Big(1 - \frac{1}{4}\mu^2\tau^{4\sigma-2}\Big) v - \frac{2\sigma-1}{2}\mu\tau^{2\sigma-2}v = 0, \tag{2.4}$$
$$v_0 = \exp\Big(\frac{\mu}{4\sigma}r^{2\sigma}\Big) w_0 \text{ and } v_1 = \exp\Big(\frac{\mu}{4\sigma}r^{2\sigma}\Big)\Big(\frac{\mu}{2}r^{2\sigma-1}w_0 + w_1\Big).$$

The term $\frac{2\sigma-1}{2}\mu\tau^{2\sigma-2}v$ is considered as a term of lower order from the point of view of WKB analysis. For N sufficiently large the dominating term $1 - \frac{1}{4}\mu^2\tau^{4\sigma-2}$ in (2.4) is positive for $\sigma \in \big(0, \frac{1}{2}\big]$ and negative for $\sigma \in \big(\frac{1}{2}, 1\big)$. We want to distinguish these two cases by emphasizing the hyperbolic and elliptic behaviour of the system for $\sigma \in \big(0, \frac{1}{2}\big]$ and $\sigma \in \big(\frac{1}{2}, 1\big)$, respectively. To do so we rewrite equation (2.4) in the following way:

$$v_{\tau\tau} + a_1^2(\tau)v + b(\tau)v = 0 \quad \text{for} \quad \sigma \in \big(0, \frac{1}{2}\big], \tag{2.5}$$
$$v_{\tau\tau} - a_2^2(\tau)v + b(\tau)v = 0 \quad \text{for} \quad \sigma \in \big(\frac{1}{2}, 1\big), \tag{2.6}$$

where $a_1^2(\tau) = 1 - \frac{1}{4}\mu^2\tau^{4\sigma-2}$, $a_2^2(\tau) = \frac{1}{4}\mu^2\tau^{4\sigma-2} - 1$ and $b(\tau) = \frac{2\sigma-1}{2}\mu\tau^{2\sigma-2}$ with $\mu > 0$. In both equations the term a_1^2 or a_2^2, respectively, denotes the dominant part. The term b is subdominant for $\sigma \in (0,1)$ concerning the asymptotic behaviour of the coefficient in τ.

Beginning with the Cauchy problem for (2.5) we set up the system of equations for the micro-energy $V(\tau) = \big(a_1(\tau)v(\tau), v_\tau(\tau)\big)^T$ and diagonalize it by using the diagonalizer $M = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. We obtain for $V_1(\tau) := M^{-1}V(\tau)$ the system

$$d_\tau V_1 = A_1(\tau)V_1 + B_1(\tau)V_1, \quad V_{1,0} = V_1(r) = M^{-1}V(r) \tag{2.7}$$

with $N \leq r \leq \tau$ and the matrices

$$A_1(\tau) = \begin{pmatrix} -ia_1(\tau) & 0 \\ 0 & ia_1(\tau) \end{pmatrix} \text{ and } B_1(\tau) = \frac{1}{a_1(\tau)}\begin{pmatrix} d_\tau a_1(\tau) & 0 \\ b(\tau) & 0 \end{pmatrix}.$$

Here the matrix A_1 describes the hyperbolicity of our model (2.5) while B_1 is integrable over (N, ∞). Since we want to obtain estimates for $|w|$ we have to estimate $\|V_1(\tau)\|$.

Using the theory of matrizants and the integrability of B_1 on (N,∞) we obtain the estimate $\|V_1(\tau)\| \le C_N \|V_1(r)\|$ and, analogously, $\|V(\tau)\| \le C_N \|V(r)\|$ for $N \le r \le \tau$. Summarizing, we have for $\sigma \in \big(0, \frac{1}{2}\big]$ the estimates

$$\begin{aligned}
|w(\tau)| &\le |v(\tau)| \exp\Big(-\frac{\mu}{4\sigma}\tau^{2\sigma}\Big) \\
&\le C_N a_1^{-1}(\tau) \exp\Big(-\frac{\mu}{4\sigma}\tau^{2\sigma}\Big) |a_1(r)v(r) + v_\tau(r)| \\
&\le C_N a_1^{-1}(\tau) \exp\Big(-\frac{\mu}{4\sigma}\big(\tau^{2\sigma} - r^{2\sigma}\big)\Big) \Big|\Big(a_1(r) + \frac{\mu}{2} r^{2\sigma-1}\Big) w(r) + w_\tau(r)\Big| \\
&\le C_N \exp\Big(-\frac{\mu}{4\sigma}\big(\tau^{2\sigma} - r^{2\sigma}\big)\Big) \big(|w(r)| + |w_\tau(r)|\big), \\
|w_\tau(\tau)| &\le \big(a_1(\tau)\,|v(\tau)| + |v_\tau(\tau)|\big) \exp\Big(-\frac{\mu}{4\sigma}\tau^{2\sigma}\Big) \\
&\le C_N \exp\Big(-\frac{\mu}{4\sigma}\big(\tau^{2\sigma} - r^{2\sigma}\big)\Big) \big(a_1(r)\,|v(r)| + |v_\tau(\tau)|\big) \\
&\le C_N \exp\Big(-\frac{\mu}{4\sigma}\big(\tau^{2\sigma} - r^{2\sigma}\big)\Big) \big(|w(r)| + |w_\tau(r)|\big).
\end{aligned}$$

Now let us devote to the case $\sigma \in \big(\frac{1}{2}, 1\big)$. Based on equation (2.6) we set up the system for the micro-energy $V(\tau) := \big(a_2(\tau)v(\tau), v_\tau(\tau)\big)$ and diagonalize it two times. To do so we use the diagonalizer $M(\tau) = MN_1(\tau)$, where $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is the diagonalizer for the first step. Applying the second step of diagonalization we use the matrix

$$N_1(\tau) = \begin{pmatrix} 1 & \frac{b(\tau) - d_\tau a_2(\tau)}{4a_2^2(\tau) + 2b(\tau)} \\ \frac{b(\tau) + d_\tau a_2(\tau)}{4a_2^2(\tau) + 2b(\tau)} & 1 \end{pmatrix}$$

proposed by Yagdjian (see [14]) which is invertible for $\tau \ge N$ large enough. We obtain the following system for $V_2(\tau) = N_1^{-1}(\tau)M^{-1}V(\tau)$:

$$d_\tau V_2(\tau) = A_2(\tau)V_2(\tau) + B_2(\tau)V_2(\tau), \;\; V_{2,0} = V_2(r) = N_1^{-1}(r)M^{-1}V(r) \tag{2.8}$$

with $N \le r \le \tau$ and

$$A_2(\tau) = \begin{pmatrix} -a_2(\tau) + \frac{d_\tau a_2(\tau) - b(\tau)}{2a_2(\tau)} & 0 \\ 0 & a_2(\tau) + \frac{d_\tau a_2(\tau) + b(\tau)}{2a_2(\tau)} \end{pmatrix}.$$

The term $B_2(\tau)$ is subdominant in (2.8) and its norm is asymptotically equivalent to $\tau^{-2\sigma-1}$, thus, $B_2(\tau)$ is integrable over (N,∞). Using the definition $V_2(\tau) = E_2(\tau,r)V_2(r)$ and (2.8) we can write

$$d_\tau E_2(\tau,r) = \big(A_2(\tau) + B_2(\tau)\big)E_2(\tau,r). \tag{2.9}$$

Here $E_2 = E_2(\tau,r)$ is a propagator to V_2, thus, as a first statement we have the propagator property $E_2(r,r) = I$. Due to the ellipticity of the system for V_2 the propagator consists of increasing and decreasing components. Thus we are interested in sharp estimates for the increasing components in order to get information about the behaviour of $\|V_2\|$. We will obtain these estimates using a weight-function and the Lemma of Gronwall.

We make the following ansatz

$$E_2(\tau,r)=E_{2,d}(\tau,r)Q(\tau,r)$$

with $E_{2,d}(\tau,r)$ is the propagator to $d_\tau - A_2(\tau)$. Using the system for Q we can lead back to a system of integral equations for E_2:

$$E_2(\tau,r)=E_{2,d}(r,\tau)+\int_r^\tau E_{2,d}(\tau,s)B_2(s)E_2(s,r)ds.$$

Due to the form of $E_{2,d}$ we can not apply the Lemma of Gronwall yet. Hence, we modify the system of integral equations introducing a multiplier:

$$E_2^{(1)}(\tau,r):=E_2(\tau,r)\exp\left(-\int_r^\tau \omega(s)ds\right),\quad E_{2,d}^{(1)}(\tau,r):=E_{2,d}(\tau,r)\exp\left(-\int_r^\tau \omega(s)ds\right)$$

with $\omega(s):=a_2(s)+\frac{d_\tau a_2(s)+b(s)}{2a_2(s)}$. With this choice we have

$$E_2^{(1)}(\tau,r)=E_{2,d}^{(1)}(r,\tau)+\int_r^\tau E_{2,d}^{(1)}(s,\tau)B_2(s)E_2^{(1)}(s,r)ds.$$

As we can prove boundedness of $\left\|E_{2,d}^{(1)}\right\|$ we can apply the Lemma of Gronwall. Using the propagator-property of $E_2(\tau,r)$ we obtain

$$\|V(\tau)\|\le C_N\|V_2(\tau)\|\le C_N\exp\left(\int_r^\tau a_2(s)+\frac{d_\tau a_2(s)+b(s)}{2a_2(s)}ds\right)\|V(r)\|. \tag{2.10}$$

Summarizing, we obtain the estimates

$$\begin{aligned}
|w(\tau)| &= \exp\left(-\frac{\mu}{4\sigma}\tau^{2\sigma}\right)|v(\tau)|\\
&\le C_N(a_2(\tau))^{-1}\exp\left(\int_r^\tau \omega(s)ds-\frac{\mu}{4\sigma}\tau^{2\sigma}\right)\big(a_2(r)|v(r)|+|v_\tau(r)|\big)\\
&= C_N\exp\left(\int_r^\tau\left(\omega(s)-\frac{\mu}{2}s^{2\sigma-1}-\frac{d_\tau a_2(s)}{a_2(s)}\right)ds-\frac{\mu}{4\sigma}r^{2\sigma}\right)\\
&\quad\times\big(a_2(r)\big)^{-1}\big(a_2(r)|v(r)|+|v_\tau(r)|\big)\\
&\le C_N\exp\left(\int_r^\tau\left(a_2(s)-\frac{d_\tau a_2(s)-b(s)}{2a_2(s)}-\frac{\mu}{2}s^{2\sigma-1}\right)ds-\frac{\mu}{4\sigma}r^{2\sigma}\right)\\
&\quad\times\big(|v(r)|+(a_2(r))^{-1}|v_\tau(r)|\big).
\end{aligned}$$

At this point we use the asymptotic equivalences

$$a_2(s)\sim\frac{\mu}{2}s^{2\sigma-1}-\frac{1}{\mu}s^{1-2\sigma}\ \text{and}\ \frac{d_\tau a_2(s)-b(s)}{2a_2(s)}\sim(2\sigma-1)s^{-1},$$

and get

$$
\begin{aligned}
|w(\tau)| &\le C_N \left(\frac{r}{\tau}\right)^{2\sigma-1} \exp\left(-C\int_r^\tau \frac{1}{\mu}s^{1-2\sigma}ds - \frac{\mu}{4\sigma}r^{2\sigma}\right)\big(|v(r)| + (a_2(r))^{-1}|v_\tau(r)|\big)\\
&\le C_N \exp\left(-\frac{C}{\mu(2-2\sigma)}\big(\tau^{2-2\sigma}-r^{2-2\sigma}\big) - \frac{\mu}{4\sigma}r^{2\sigma}\right)\\
&\quad\times\big(|v(r)| + (a_2(r))^{-1}|v_\tau(r)|\big)\\
&\le C_N \exp\left(-\frac{C}{\mu(2-2\sigma)}\big(\tau^{2-2\sigma}-r^{2-2\sigma}\big)\right)\\
&\quad\times\Big(|w(r)| + (a_2(r))^{-1}|w_\tau(r)| + (a_2(r))^{-1}\frac{\mu}{2}r^{2\sigma-1}|w(r)|\Big)\\
&\le C_N \exp\left(-\frac{C}{\mu(2-2\sigma)}\big(\tau^{2-2\sigma}-r^{2-2\sigma}\big)\right)\big(|w(r)|+|w_\tau(r)|\big).
\end{aligned}
$$

For $|w_\tau(\tau)|$ we can estimate in the same way

$$
|w_\tau(\tau)| \le C_N \exp\left(-\frac{C}{\mu(2-2\sigma)}\big(\tau^{2-2\sigma}-r^{2-2\sigma}\big)\right)\big(|w(r)|+|w_\tau(r)|\big).
$$

To complete the proof we use for $r \le N \le \tau$ the estimate

$$
\left\|\begin{pmatrix} w(N) \\ w_\tau(N)\end{pmatrix}\right\| \le C_N \left\|\begin{pmatrix} w(r) \\ w_\tau(r)\end{pmatrix}\right\|,
$$

and, finally, we may conclude for $\tau \ge 0$ and $\sigma \in \big(0, \frac{1}{2}\big]$ the estimates

$$
\begin{aligned}
|w(\tau)| &\le C \exp\left(-\frac{\mu}{4\sigma}(\tau^{2\sigma}-r^{2\sigma})\right)\big(|w(r)|+|w_\tau(r)|\big),\\
|w_\tau(\tau)| &\le C \exp\left(-\frac{\mu}{4\sigma}(\tau^{2\sigma}-r^{2\sigma})\right)\big(|w(r)|+|w_\tau(r)|\big).
\end{aligned}
$$

Furthermore, we have for $\tau \ge 0$ and $\sigma \in \big(\frac{1}{2}, 1\big)$ the estimates

$$
\begin{aligned}
|w(\tau)| &\le C \exp\left(-\frac{C'}{\mu(2-2\sigma)}\big(\tau^{2-2\sigma}-r^{2-2\sigma}\big)\right)\big(|w(r)|+|w_\tau(r)|\big),\\
|w_\tau(\tau)| &\le C \exp\left(-\frac{C'}{\mu(2-2\sigma)}\big(\tau^{2-2\sigma}-r^{2-2\sigma}\big)\right)\big(|w(r)|+|w_\tau(r)|\big).
\end{aligned}
$$

This completes the proof. □

The next step is now to prove the optimality of the estimates from Proposition 2.1. With these optimal estimates for w we can deduce to optimal estimates for the energy $E_m(u)(t)$ of order $m \ge 1$. Due to the hyperbolic character of the system for $\sigma \in \big(0, \frac{1}{2}\big]$ we will use *ideas from scattering theory* in order to obtain estimates from above and below for $E_m(u)(t)$. The main idea is to compare the system $d_\tau V_1 = A_1(\tau)V_1 + B_1(\tau)V_1$ with the system $d_\tau V_1^{(1)} = A_1(\tau)V_1^{(1)}$ in order to prove that the asymptotic profile of V_1 and $V_1^{(1)}$ coincide for $\tau \to \infty$.

Proposition 2.2. *Let us consider the following Cauchy problem for equation* (2.5) *for* $\sigma \in \left(0, \frac{1}{2}\right]$:

$$v_{\tau\tau} + a_1^2(\tau)v + b(\tau)v = 0,$$
$$v(r) = v_0, \quad v_\tau(r) = v_1$$

with $N \leq \tau$ *and* $r \leq \tau$. *Defining the micro-energy*

$$V(\tau) := \big(a_1(\tau)v(\tau), v_\tau(\tau)\big)^T$$

we obtain for N *large enough*

$$C_1(r,N)\,\|V(r)\| \leq \|V(\tau)\| \leq C_2(r,N)\,\|V(r)\|.$$

Therefore, the estimate $\|V(\tau)\| \leq C_N\,\|V(r)\|$ *is optimal.*

Proof. We want to use the notations of the proof for Proposition 2.1. We set up the systems $d_\tau V_1 = A_1(\tau)V_1 + B_1(\tau)V_1$ and $d_\tau V_1^{(1)} = A_1(\tau)V_1^{(1)}$. We immediately obtain the propagator $E_{1,d} = E_{1,d}(\tau,r)$ for $V_1^{(1)}$, i.e. $V_1^{(1)}(\tau) = E_{1,d}(\tau,r)V_1^{(1)}(r)$:

$$E_{1,d}(\tau,r) = \begin{pmatrix} \exp\left(-i\int_r^\tau a(s)ds\right) & 0 \\ 0 & \exp\left(i\int_r^\tau a(s)ds\right) \end{pmatrix}, \quad E_{1,d}(r,r) = I$$

for $N \leq \tau$ and $r \leq \tau$. For the propagator $E_1(\tau,r)$ for $V_1(\tau)$ we use the approach $E_1(\tau,r) = E_{1,d}(\tau,r)Q(\tau,r)$. This leads to

$$d_\tau Q(\tau,r) = E_{1,d}(r,\tau)B_1(\tau)E_{1,d}(\tau,r)Q(\tau,r).$$

Let us denote $\mathcal{B}_1(\tau,r) := E_{1,d}(r,\tau)B_1(\tau)E_{1,d}(\tau,r)$. Then we can also obtain for the inverse of Q the system

$$d_\tau Q^{-1}(\tau,r) = -Q^{-1}(\tau,r)\mathcal{B}_1(\tau,r). \tag{2.11}$$

The goal now is to prove the existence of the so-called Møller wave operator $W_+ = W_+(r)$. This operator maps the data of the system for $V_1^{(1)}$ to the data of the system for V_1 in $\tau = r$ so that the asymptotic profile of V_1 and $V_1^{(1)}$ coincide in the following way:

$$\lim_{\tau\to\infty}\left\|V_1^{(1)}(\tau) - V_1(\tau)\right\| = 0. \tag{2.12}$$

Following this construction the Møller wave operator can be expressed as

$$W_+(r) := \lim_{\tau\to\infty} E_1(r,\tau)E_{1,d}(\tau,r) = \lim_{\tau\to\infty} Q^{-1}(\tau,r).$$

To prove the existence of the operator W_+ we use (2.11) with

$$\|\mathcal{B}_1(\tau,r)\| \leq \left|\frac{d_\tau a_1(\tau) + b(\tau)}{a_1(\tau)}\right| \in L^1(N,\infty).$$

We obtain for $\tau, \tau' \geq R$

$$\left\|Q^{-1}(\tau,r) - Q^{-1}(\tau',r)\right\| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_{\tau'}^{\tau} \|\mathcal{B}_1(\tau_1,r)\| \, d\tau_1 \right)^n \leq \varepsilon(R).$$

Thus, the limit $\lim\limits_{\tau\to\infty} Q^{-1}(\tau,r)$ exists. We can prove the invertibility of W_+ analogously. We therefore obtain with (2.12) and energy conservation for $V_1^{(1)}$

$$\begin{aligned}
\big(1-\varepsilon(N)\big) \left\|V_1^{(1)}(\tau)\right\| &\leq \|V_1(\tau)\| \leq \big(1+\varepsilon(N)\big) \left\|V_1^{(1)}(\tau)\right\|, \\
\big(1-\varepsilon(N)\big) \left\|W_+^{-1}(r)V_1(r)\right\| &\leq \|V_1(\tau)\| \leq \big(1+\varepsilon(N)\big) \left\|W_+^{-1}(r)V_1(r)\right\|, \\
C_1(r,N)\,\|V(r)\| &\leq \|V(\tau)\| \leq C_2(r,N)\,\|V(r)\|.
\end{aligned}$$

Therefore, we can not expect any decay or blow up for $V(\tau)$ for $\tau \to \infty$. □

For $\sigma \in \left(\frac{1}{2}, 1\right)$ we can show the following statement:

Proposition 2.3. *Let us consider the following Cauchy problem for* (2.6) *for* $\sigma \in \left(\frac{1}{2}, 1\right)$:

$$\begin{aligned}
&v_{\tau\tau} - a_2^2(\tau)v + b(\tau)v = 0, \\
&v(r) = v_0, \quad v_\tau(r) = v_1
\end{aligned}$$

with $N \leq \tau$ *and* $r \leq \tau$. *Defining the micro-energy*

$$V(\tau) := \big(a_2(\tau)v(\tau), v_\tau(\tau)\big)^T$$

we obtain for N *large enough and* $\omega(\tau) = \frac{d_\tau a_2(\tau) + b(\tau)}{2a_2(\tau)}$ *the estimate*

$$C_1(r,N) \exp\left(\int_r^\tau \omega(s)ds \right) \|V(r)\| \leq \|V(\tau)\| \leq C_2(r,N) \exp\left(\int_r^\tau \omega(s)ds \right) \|V(r)\|.$$

Therefore, the estimate $\|V(\tau)\| \leq C_N \exp\left(\int_r^\tau \omega(s)ds\right) \|V(r)\|$ *is optimal.*

Sketch of the proof: The main idea of the proof of Theorem 2.3 is to analyse every component of the propagator $E_2(\tau,r)$ for the system $d_\tau - A_2(\tau) - B_2(\tau) = 0$ in order to solve the occurring ordinary differential equations. This leads to integral equations for the entries of

$$E_2^{(1)}(\tau,r) = \exp\left(-\int_r^\tau \omega(s)ds \right) E_2(\tau,r)$$

so that we can apply Gronwall's inequality in order to obtain estimates from below and above. These investigations lead to the optimality of the estimates of Proposition 2.1.

As we have sketched on page 89 we want to obtain estimates of the increasing components of E_2. With this information optimal estimates for $\|V_2\|$ can be conceived. Thus the following investigations restrict ourselves to $\big(E_2\big)_{22}$. With (2.8) we define

$$A_2(\tau) =: \begin{pmatrix} \alpha_1(\tau) & 0 \\ 0 & \alpha_2(\tau) \end{pmatrix}, \quad B_2(\tau) = \begin{pmatrix} 0 & \beta_1(\tau) \\ \beta_2(\tau) & 0 \end{pmatrix}$$

and obtain the ordinary differential equations

$$d_\tau \big(E_2\big)_{21}(\tau,r) = \alpha_1(\tau)\big(E_2\big)_{12}(\tau,r) + \beta_1(\tau)\big(E_2\big)_{22}(\tau,r),$$
$$d_\tau \big(E_2\big)_{22}(\tau,r) = \alpha_2(\tau)\big(E_2\big)_{22}(\tau,r) + \beta_2(\tau)\big(E_2\big)_{12}(\tau,r).$$

Integrating these two equations and applying each representation into each other leads to an integral equation for

$$q_{22}(\tau,r) := \exp\left(-\int_r^\tau \alpha_2(s)ds\right)\big(E_2\big)_{22}(\tau,r).$$

Note that $\exp\big(-\int_r^\tau \alpha_2(s)ds\big) \le C_N \exp\big(-\int_r^\tau \omega(s)ds\big)$ for $N \le r \le \tau$. Thus, simultaneously we are investigating $\big(E_2^{(1)}\big)_{22}$. We have

$$q_{22}(\tau,r) = 1 + \int_r^\tau \varphi(\tau,s)q_{22}(s,r)ds$$

with

$$\varphi(\tau,s) := \beta_1(\tau)\int_r^\tau \beta_2(s')\exp\left(\int_{s'}^s \big(\alpha_1(s'')-\alpha_2(s'')\big)ds''\right)ds'.$$

Using the representation

$$q_{22}(\tau,r) = 1 + \sum_{n=1}^\infty \int_r^\tau \varphi(\tau,\tau_1)\int_r^{\tau_1}\varphi(\tau_1,\tau_2)\cdots\int_r^{\tau_{n-1}}\varphi(\tau_{n-1},\tau_n)d\tau_n\cdots d\tau_1$$

we can show $|q_{22}(\tau,r)| \ge \frac{1}{2}$. With mathematical induction it follows

$$\left|\int_r^\tau \varphi(\tau,\tau_1)\int_r^{\tau_1}\varphi(\tau_1,\tau_2)\cdots\int_r^{\tau_{n-1}}\varphi(\tau_{n-1},\tau_n)d\tau_n\cdots d\tau_1\right| \le \frac{1}{3^n}, \quad n \ge 0$$

and thus $|q_{22}(\tau,r)| \ge 1 - \sum\limits_{n=1}^\infty \frac{1}{3^n} = \frac{1}{2}$. This implies the estimates $\left|\big(E_2^{(1)}\big)_{22}\right| \ge C_N$ and $\left|\big(E_2\big)_{22}\right| \ge C_N\exp\big(\int_r^\tau \omega(s)ds\big)$ with $N \le r \le \tau$ and $C_N > 0$. Together with (2.10) the optimality of the estimate in Proposition 2.3 is shown.

Now the last step is to obtain estimates for the energy $E_m(u)(t)$. Let $|\beta| \ge 1$. By Proposition 2.1 we have

$$\begin{aligned}|\xi|^{|\beta|}\,|\hat{u}(t,\xi)| &= |\xi|^{|\beta|}\,\big|w\big((1+t)\,|\xi|\,\big)\big| \\ &\le C\,|\xi|^{|\beta|}\exp\left(-\frac{\mu}{4\sigma}\big((1+t)^{2\sigma}-1\big)\,|\xi|^{2\sigma}\right)\left(\big|w(|\xi|)\big| + \frac{1}{|\xi|}\big|w_\tau(|\xi|)\big|\right)\end{aligned}$$

for $\sigma \in \big(0,\frac{1}{2}\big]$ and

$$\begin{aligned}|\xi|^{|\beta|}\,|\hat{u}(t,\xi)| \le C\,|\xi|^{|\beta|}&\exp\left(-\frac{C'}{\mu(2-2\sigma)}\big((1+t)^{2-2\sigma}-1\big)\,|\xi|^{2-2\sigma}\right) \\ &\times\left(\big|w(|\xi|)\big| + \frac{1}{|\xi|}\big|w_\tau(|\xi|)\big|\right)\end{aligned}$$

for $\sigma \in \big(\frac{1}{2},1\big)$. By Propositions 2.2 and 2.3 we know that these estimates are optimal. At this point we distinguish two cases:

1. For $t \in [0,1]$ we obtain for $\sigma \in (0,1)$
$$|\xi|^{|\beta|}\,|\hat{u}(t,\xi)| \leq C\,|\xi|^{|\beta|}\left(|w(|\xi|)| + \frac{1}{|\xi|}\,|w_\tau(|\xi|)|\right)$$
$$\leq C(\sigma,\mu,|\beta|)\,|\xi|^{\beta}\,|\hat{u}(0,\xi)| + C(\sigma,\mu,|\beta|)\,|\xi|^{|\beta|-1}\,|\hat{u}_t(0,\xi)|\,. \quad (2.13)$$
This result for small t affects the regularity of the data.

2. For $t \in [1,\infty)$ we have for $\sigma \in \left(0,\frac{1}{2}\right]$
$$|\xi|^{|\beta|}\,|\hat{u}(t,\xi)| \leq C\,|\xi|^{|\beta|}\exp\left(-\frac{\mu}{4\sigma}(t\,|\xi|)^{2\sigma}\right)\left(\big|w(|\xi|)\big| + \frac{1}{|\xi|}\big|w_\tau(|\xi|)\big|\right)$$
due to the asymptotic equivalence $(1+t)^{2\sigma}-1 \sim t^{2\sigma}$. For $\sigma \in \left(\frac{1}{2},1\right)$ we can estimate
$$|\xi|^{|\beta|}\,|\hat{u}(t,\xi)| \leq C\,|\xi|^{|\beta|}\exp\left(-\frac{1}{\mu(2-2\sigma)}(t\,|\xi|)^{2-2\sigma}\right)\left(|w(|\xi|)| + \frac{1}{|\xi|}\,|w_\tau(|\xi|)|\right)$$
due to $(1+t)^{2-2\sigma}-1 \sim t^{2-2\sigma}$.
Now the terms
$$(t\,|\xi|)^{|\beta|}\exp\left(-\frac{1}{\mu(2-2\sigma)}(t\,|\xi|)^{2\sigma}\right) \text{ and } (t\,|\xi|)^{|\beta|}\exp\left(-\frac{1}{\mu(2-2\sigma)}(t\,|\xi|)^{2-2\sigma}\right)$$
are bounded so that we have for $\sigma \in (0,1)$
$$\begin{aligned}|\xi|^{|\beta|}\,|\hat{u}(t,\xi)| \;&\leq C(\sigma,\mu,|\beta|)(1+t)^{-|\beta|}\,|\hat{u}(0,\xi)|\\ &\quad + C(\sigma,\mu,|\beta|)(1+t)^{-(|\beta|-1)}\,|\hat{u}_t(0,\xi)|\,.\end{aligned} \quad (2.14)$$
This estimate concerning large times proves the decay of the elastic energy.

Merging the behaviour of $\hat{u}$ in (2.13) and (2.14) we obtain
$$|\xi|^{|\beta|}\,|\hat{u}(t,\xi)| \leq C\big((1+t)^{-|\beta|}\,\langle\xi\rangle^{|\beta|}\,|\hat{u}(0,\xi)| + (1+t)^{-(|\beta|-1)}\,\langle\xi\rangle^{|\beta|-1}\,|\hat{u}_t(0,\xi)|\big).$$
Our first goal is to get conclusions about components of the elastic energy:
$$\left\|\nabla^\beta u(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2|\beta|}\,|\hat{u}(t,\xi)|^2\,d\xi$$
$$\leq C\left((1+t)^{-2|\beta|}\,\|u(0,\cdot)\|^2_{H^{|\beta|}(\mathbb{R}^n)} + (1+t)^{-2(|\beta|-1)}\,\|u_t(0,\cdot)\|_{H^{|\beta|-1}(\mathbb{R}^n)}\right).$$
Analogously, we can study the kinetic energy. Hence with $\left|\tilde{\beta}\right| = |\beta|-1$ we obtain
$$\left\|\nabla^{\tilde{\beta}} u_t(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)}\} \leq C\left((1+t)^{-2|\beta|}\,\|u(0,\cdot)\|^2_{H^{|\beta|}(\mathbb{R}^n)} + (1+t)^{-2(|\beta|-1)}\,\|u_t(0,\cdot)\|_{H^{|\beta|-1}(\mathbb{R}^n)}\right).$$
Summarizing, the energy
$$E_m(u)(t) = \sum_{|\beta|=m}\left\|\nabla^\beta u(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)} + \sum_{|\tilde{\beta}|=m-1}\left\|\nabla^{\tilde{\beta}} u_t(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)}$$
of order $m \geq 1$ satisfies
$$E_m(u)(t) \leq C\left((1+t)^{-2m}\,\|u(0,\cdot)\|^2_{H^m(\mathbb{R}^n)} + (1+t)^{-2(m-1)}\,\|u_t(0,\cdot)\|^2_{H^{m-1}(\mathbb{R}^n)}\right).$$
This completes the proof of Theorem 2.1.

2.2 The case $\sigma = 1$

The case $\sigma = 1$ denotes the visco-elastic case of the structurally damped wave model, the so called visco-elastic damped wave model. Recalling the statement in Theorem 2.1 it is obvious that for $\sigma \in (0,1)$ the rate of decay of the energy increases as the order of the energy increases. This effect is called parabolic effect. We are now interested if this effect is also observable in the scale invariant case with $\sigma = 1$.

Theorem 2.2. *Let us consider the Cauchy problem*

$$
\begin{aligned}
&u_{tt} - \Delta u - \mu(1+t)\Delta u_t = 0, \;\; \mu > 0,\\
&u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x).
\end{aligned}
$$

Then the following estimate for the energy

$$
E_m(u)(t) = \sum_{|\beta|=m} \left\|\nabla^\beta u(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)} + \sum_{|\tilde{\beta}|=m-1} \left\|\nabla^{\tilde{\beta}} u_t(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)}
$$

of order $m \geq 1$ holds and is optimal:

$$
E_m(u)(t) \leq C_m \left((1+t)^{C'} \|u_0\|^2_{H^m(\mathbb{R}^n)} + (1+t)^{C'} \|u_1\|^2_{H^{m-1}(\mathbb{R}^n)}\right).
$$

The constant C' is independent of m.

Proof. The proof can be found in Diplomarbeit Matthes, Section 2.5, [8]. □

Remark 2.2. *The study of the scale invariant models in Theorem 2.1 brings optimal estimates for the energies of higher order for $\sigma \in (0,1)$. Moreover, the parabolic effect can be shown. For $\sigma = 1$ we only know that the scale invariant model does not possess the property of parabolic effect. This observation does not exclude the parabolic effect for other visco-elastic models.*

3 The smoothing effect

In this section we will explain another property of solutions to structurally damped waves, the so-called *smoothing effect*. To explain this effect we introduce the following definition:

Definition 3.1. *A function $u : \mathbb{R}^n \to \mathbb{R}$ belongs to the Gevrey space $\Gamma^s(\mathbb{R}^n)$ if it satisfies*

$$
\hat{u}(\xi) e^{C\langle\xi\rangle^{1/s}} \in L^2(\mathbb{R}^n)
$$

with suitable constants $C > 0$ and $s \geq 1$.
More precisely, a function $u : \mathbb{R}^n \to \mathbb{R}$ belongs to the Gevrey-Sobolev space $\Gamma^{s,\varrho}(\mathbb{R}^n)$ if it satisfies

$$
\hat{u}(\xi)\, \langle\xi\rangle^{\varrho}\, e^{C\langle\xi\rangle^{1/s}} \in L^2(\mathbb{R}^n)
$$

with suitable constants $C > 0$, $s \geq 1$ and $\varrho \in \mathbb{R}$.

Theorem 3.1. *Let us consider the Cauchy problem*

$$\begin{aligned} &u_{tt} - \Delta u + (1+t)^\delta \big(-\Delta\big)^\sigma u_t = 0,\ \sigma \in (0,1),\ |\delta| > 1, \\ &u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \end{aligned} \tag{3.1}$$

with $\big(u_0, u_1\big) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. *Then for* $t > 0$ *the solution* $u = u(t, \cdot)$ *belongs to the Gevrey-Sobolev space. More precisely, we have* $u(t,\cdot) \in \Gamma^{s(\sigma),\varrho(\sigma)}$ *with*

$$s(\sigma) = \begin{cases} \frac{1}{2\sigma}, & \text{for } \sigma \in \big(0, \frac{1}{2}\big] \\ \frac{1}{2-2\sigma}, & \text{for } \sigma \in \big(\frac{1}{2}, 1\big) \end{cases} \quad \text{and} \quad \varrho(\sigma) = \begin{cases} -2\sigma, & \text{for } \sigma \in \big(0, \frac{1}{2}\big] \\ \frac{1}{2} - 3\sigma, & \text{for } \sigma \in \big(\frac{1}{2}, 1\big) \end{cases}$$

for $\delta < -1$ *and* $\varrho(\sigma) = -2\sigma$ *for* $\delta > 1$.

3.1 Outline of the proof

The proof consists of the following three main steps:

1. Determine the characteristic roots of the partial Fourier transformed operator of (3.1),

2. set up the zones in the extended phase space,

3. obtain estimates for $|\hat{u}|$ using elliptic as well as hyperbolic WKB analysis.

The characteristic roots of the partial Fourier transformed operator of (3.1) are

$$\lambda_{1,2}(t,\xi) := -\frac{1}{2} b(t) |\xi|^{2\sigma} \pm \sqrt{\frac{1}{4} b(t)^2 |\xi|^{4\sigma} - |\xi|^2}$$

with $b(t) = (1+t)^\delta$. Thus the principal behavior of the system is ruled by the behavior of the discriminant $\frac{1}{4} b(t)^2 |\xi|^{4\sigma} - |\xi|^2$. We therefore set up zones in the extended phase space in which we can observe elliptic and hyperbolic behaviour as well. We define with $\varepsilon > 0$

$$\begin{aligned} \text{the elliptic zone} \quad & Z_{\text{ell}}(\varepsilon) := \left\{ (t,\xi) : |\xi|^2 \le \varepsilon \frac{1}{4} b(t)^2 |\xi|^{4\sigma} \right\}, \\ \text{the hyperbolic zone} \quad & Z_{\text{hyp}}(\varepsilon) := \left\{ (t,\xi) : \frac{1}{4} b(t)^2 |\xi|^{4\sigma} \le \varepsilon |\xi|^2 \right\}, \\ \text{the reduced zone} \quad & Z_{\text{red}}(\varepsilon) := \left\{ (t,\xi) : \varepsilon |\xi|^2 \le \frac{1}{4} b(t)^2 |\xi|^{4\sigma} \le \frac{1}{\varepsilon} |\xi|^2 \right\}. \end{aligned}$$

Due to our investigations on regularity we have to devote to high frequencies, i.e. $|\xi| \ge N$ with N large enough. Figures 1 and 2 give a geometrical idea of the zones. Choosing a fixed $t > 0$ we can choose N large enough such that we

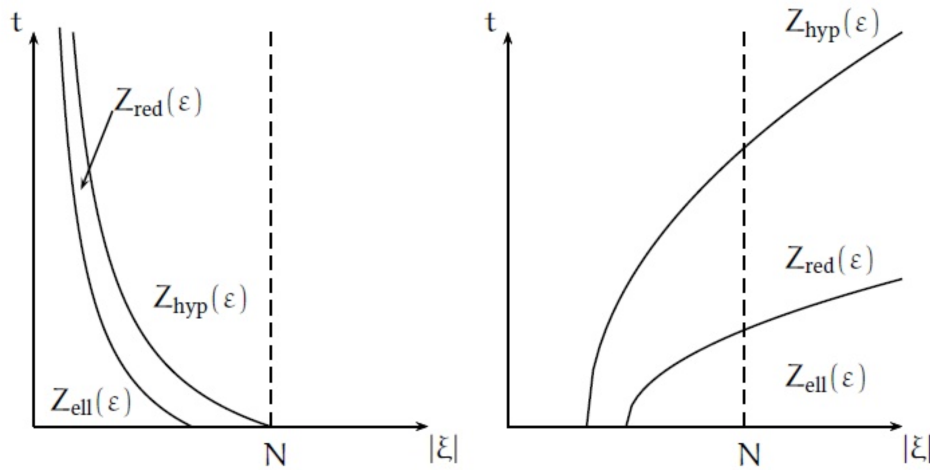

Figure 1: Principal shape of the zones for decreasing and decaying dissipation for $\sigma \in \left(0, \frac{1}{2}\right)$ (left) and $\sigma \in \left(\frac{1}{2}, 1\right)$ (right).

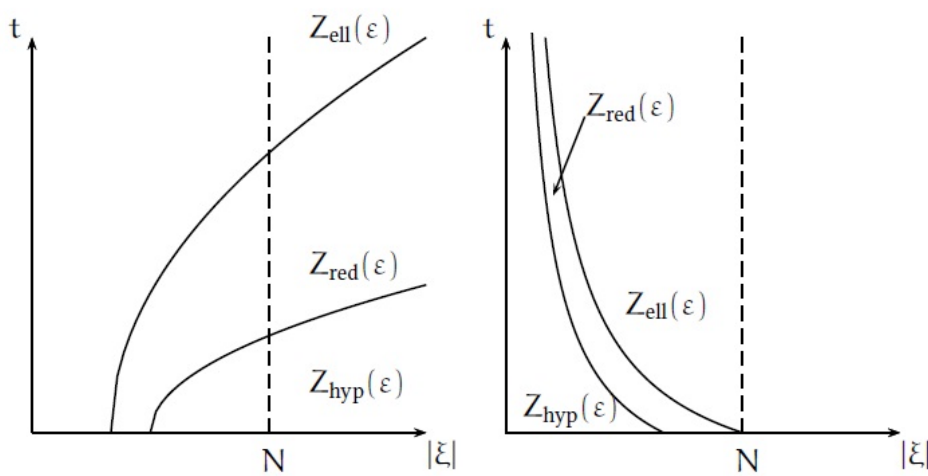

Figure 2: Principal shape of the zones for increasing and unbounded dissipation for $\sigma \in \left(0, \frac{1}{2}\right)$ (left) and $\sigma \in \left(\frac{1}{2}, 1\right)$ (right).

can completely work in one zone. Depending on the zone we have to apply the corresponding WKB analysis. The hyperbolic WBK analysis we are applying for $\sigma \in \left(0, \frac{1}{2}\right)$ involves one step of diagonalization while the elliptic WKB analysis involves two steps.

3.2 The proof for $\sigma \in \left(0, \frac{1}{2}\right]$

As we have shown in the last section we can restrict ourselves to the case when $|\xi|^2$ dominates such that we will work completely in the zone $Z_{\text{hyp}}(\varepsilon)$. We will write the characteristic roots in the following way to emphasize their behaviour in this situation:

$$\lambda_{1,2}(t,\xi) = \underbrace{-\frac{1}{2}(1+t)^{-a}\,|\xi|^{2\sigma}}_{=:m(t,\xi)} \pm i \underbrace{\sqrt{|\xi|^2 - \frac{1}{4}\,|\xi|^{4\sigma}\,(1+t)^{-2a}}}_{=:d_1(t,\xi)}.$$

Here and in the following we restrict ourselves to $a > 1$. Here we can see that the term $m(t,\xi)$ may determine the regularity in some sense. Thus, we apply the Liouville transformation

$$\hat{u}(t,\xi) = \exp\left(\frac{1}{2}\int_0^t m(t',\xi)dt'\right) v(t,\xi) = \exp\left(-\frac{1}{2}|\xi|^{2\sigma}\int_0^t (1+t')^{-a}dt'\right) v(t,\xi)$$

to obtain the Cauchy problem

$$v_{tt} + \underbrace{\left(|\xi|^2 - \frac{1}{4}|\xi|^{4\sigma}(1+t)^{-2a}\right)}_{=d_1^2(t,\xi)} v + \underbrace{\frac{a}{2}|\xi|^{2\sigma}(1+t)^{-a-1}}_{=:\beta(t,\xi)} v = 0$$

with the data $v_0(\xi) := v(0,\xi) = \hat{u}_0(\xi)$ and $v_1(\xi) := v_t(0,\xi) = -\frac{1}{2}|\xi|^{2\sigma}\hat{u}_0(\xi) + \hat{u}_1(\xi)$. We define the micro-energy $V = V(t,\xi) := \big(d_1(t,\xi)v(t,\xi), v_t(t,\xi)\big)^T$ and set up its corresponding Cauchy problem

$$\begin{aligned} \partial_t V &= \begin{pmatrix} 0 & d_1 \\ -d_1 & 0 \end{pmatrix} V + \tfrac{1}{d_1}\begin{pmatrix} \partial_t d_1 & 0 \\ \beta & 0 \end{pmatrix} V, \\ V_0(\xi) &:= V(0,\xi) = \big(d_1(0,\xi)v_0(\xi), v_1(\xi)\big)^T. \end{aligned} \tag{3.2}$$

Now the goal is to investigate V for $t \to \infty$. To understand the terms in (3.2) we consider

$$\begin{aligned} \frac{\partial_t d_1(t,\xi)}{2d_1(t,\xi)} &= \frac{1}{2}a|\xi|^{4\sigma-2}(1+t)^{-2a-1}\frac{1}{1-\underbrace{\frac{1}{4}|\xi|^{4\sigma-2}(1+t)^{-2a}}_{\leq\varepsilon \text{ in } Z_1(\varepsilon)}} \\ &\leq \frac{1}{2}a|\xi|^{4\sigma-2}(1+t)^{-2a-1}\big(1+\varepsilon+\varepsilon^2+\dots\big), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{\beta(t,\xi)}{2d_1(t,\xi)} &= \frac{1}{2}a|\xi|^{2\sigma-1}(1+t)^{-a-1}\frac{1}{\sqrt{1-\frac{1}{4}|\xi|^{4\sigma-2}(1+t)^{-2a}}} \\ &\leq \frac{1}{2}a|\xi|^{2\sigma-1}(1+t)^{-a-1}\left(1+\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^2+\dots\right). \end{aligned} \tag{3.4}$$

Taking into consideration this information and the definition of the hyperbolic zone we can interpret the first matrix in (3.2) as “principal part”. Moreover, we see that the two terms from above are integrable with respect to t. This will be very helpful. Thus, we choose the diagonalizer $M = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ in the following diagonalization procedure. We define $V(t,\xi) = MV^{(1)}(t,\xi)$ and get

$$\begin{aligned} \partial_t V^{(1)} &= \underbrace{\begin{pmatrix} -id_1 & 0 \\ 0 & id_1 \end{pmatrix}}_{=:D_1(t,\xi)} V^{(1)} + \underbrace{\frac{\partial_t d_1}{2d_1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{=:R_{1,r}(t,\xi)} V^{(1)} + \underbrace{\frac{i\beta}{2d_1}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}}_{=:R_{1,i}(t,\xi)} V^{(1)}, \\ V_0^{(1)}(\xi) &:= V^{(1)}(0,\xi) = M^{-1}V_0(\xi). \end{aligned} \tag{3.5}$$

To find a representation of the solution to (3.5) we begin with the fundamental solution of the diagonal part $\partial_t - D_1(t,\xi)$. This is the linear propagator $E_{d,1} = E_{d,1}(t,s,\xi)$ satisfying $\partial_t E_{d,1}(t,s,\xi) - D_1(t,\xi)E_{d,1}(t,s,\xi) = 0$ and $E_{d,1}(s,s,\xi) = I$. Due to the structure of $D_1(t,s,\xi)$ we get

$$E_{d,1}(t,s,\xi) := \begin{pmatrix} \exp\left(-i\int_s^t d_1(t',\xi)dt'\right) & 0 \\ 0 & \exp\left(i\int_s^t d_1(t',\xi)dt'\right) \end{pmatrix}.$$

For the fundamental solution $E_1 = E_1(t,s,\xi)$ of the full system we make the ansatz

$$E_1(t,s,\xi) = E_{d,1}(t,s,\xi)Q(t,s,\xi).$$

The matrix Q satisfies $Q(s,s,\xi) = I$ and

$$\partial_t Q(t,s,\xi) = \underbrace{E_{d,1}(s,t,\xi)\big(R_{1,r}(t,\xi) + R_{1,i}(t,\xi)\big)E_{d,1}(t,s,\xi)}_{=:\mathcal{R}_1(t,s,\xi)}Q(t,s,\xi).$$

Using the matrizant representation for Q and the special structure of $E_{d,1}$ we arrive immediately at

$$\|Q(t,s,\xi)\| = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_s^t \|\mathcal{R}_1(t',s,\xi)\|\,dt'\right)^n,$$

$$\|Q(t,s,\xi)\| \le \exp\left(\int_s^t \|\mathcal{R}_1(t',s,\xi)\|\,dt'\right), \tag{3.6}$$

respectively. At this point we can use the integrability of $\mathcal{R}_1$ with respect to t, i.e. $\mathcal{R}_1(\cdot,s,\xi) \in L^1(0,\infty)$. This follows from the integrability of $R_{1,r}$ and $R_{1,i}$ with respect to t. Having the estimates (3.3) and (3.4) in mind a direct consequence is

$$\|Q(t,s,\xi)\| \le \exp\left(C|\xi|^{4\sigma-2}\right) \le C$$

for $\sigma \in \left(0,\frac{1}{2}\right]$ and large $|\xi|$. An interpretation of this inequality is that v does not contribute to any smoothing effect for u. We will show this in detail in the following lines. We obtain with $\|E_{d,1}(t,s,\xi)\| = 1$ the estimate

$$\|V(t,\xi)\| = \left\|ME_1(t,0,\xi)M^{-1}V_0(\xi)\right\| \le C\|V_0(\xi)\|.$$

Consequently, we have

$$\begin{aligned}
|\hat{u}(t,\xi)| &= \exp\left(-\frac{1}{2}|\xi|^{2\sigma}\int_0^t(1+t')^{-a}dt'\right)|v(t,\xi)| \\
&\le C\frac{d_1(0,\xi)}{d_1(t,\xi)}\exp\left(-\frac{1}{2}|\xi|^{2\sigma}\int_0^t(1+t')^{-a}dt'\right)\big(|v_0(\xi)| + |v_1(\xi)|\big) \\
&\le C\exp\left(-\frac{1}{2}|\xi|^{2\sigma}\int_0^t(1+t')^{-a}dt'\right)\big(|v_0(\xi)| + |v_1(\xi)|\big) \\
&\le C|\xi|^{2\sigma}\exp\left(-\frac{1}{2}|\xi|^{2\sigma}\int_0^t(1+t')^{-a}dt'\right)\big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\big)
\end{aligned}$$

for $|\xi| \ge N$. Finally, we obtain for those ξ the desired estimate

$$|\hat{u}(t,\xi)| \le C\langle\xi\rangle^{2\sigma}\exp\left(-\frac{1}{2}\langle\xi\rangle^{2\sigma}\int_0^t(1+t')^{-a}dt'\right)\big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\big).$$

3.3 The proof for $\sigma \in \left[\frac{1}{2}, 1\right)$

In the case $\sigma \in \left(\frac{1}{2}, 1\right)$ we have to recall two facts concerning our investigations:

1. We are interested in large frequencies, therefore we can choose a constant N large enough and $|\xi| \geq N$.

2. We make statements about the smoothing effect for every fixed time t. Therefore, some crucial constants may also depend on t.

These two facts and the geometrical representation of the zones for $\sigma \in \left(\frac{1}{2}, 1\right)$ depicted in Figures 1 and 2 lead us to the conclusion that we can restrict ourselves to the zone $Z_{\text{ell}}(\varepsilon)$. For every t we can choose $N = N(t)$ such that for $|\xi| \geq N$ we work in $Z_{\text{ell}}(\varepsilon)$. In this zone we have $\frac{1}{4}|\xi|^{4\sigma}(1+t)^{-2a}$ as the dominating term in the square root of the characteristic roots. The definition of this zone is

$$\frac{1}{\varepsilon}|\xi|^2 \leq \frac{1}{4}|\xi|^{4\sigma}(1+t)^{-2a}. \tag{3.7}$$

The upper bound for t satisfies

$$\frac{1}{\varepsilon}|\xi|^2 = \frac{1}{4}|\xi|^{4\sigma}(1+t_{\xi,1})^{-2a}.$$

Following the main principles of construction from the last section we first apply the dissipative transformation

$$\hat{u}(t,\xi) = \exp\left(-\frac{1}{2}|\xi|^{2\sigma}\int_0^t (1+t')^{-a}dt'\right)v(t,\xi) \text{ for } t \in (0, t_{\xi,1}]$$

in order to obtain the differential equation for $v(t,\xi)$:

$$v_{tt} - \underbrace{\left(\frac{1}{4}|\xi|^{4\sigma}(1+t)^{-2a} - |\xi|^2\right)}_{=:d_2^2(t,\xi)} v + \underbrace{\frac{a}{2}|\xi|^{2\sigma}(1+t)^{-a-1}}_{=:\beta(t,\xi)} v = 0$$

with the Cauchy data $v_0(\xi) := v(0,\xi) = \hat{u}_0(\xi), v_1(\xi) := v_t(0,\xi) = -\frac{1}{2}|\xi|^{2\sigma}\hat{u}_0(\xi) + \hat{u}_1(\xi)$. We define the energy $V(t,\xi) := \big(d_2(t,\xi)v(t,\xi), v_t(t,\xi)\big)^T$ and obtain the system of differential equations

$$\partial_t V = \begin{pmatrix} 0 & d_2 \\ d_2 & 0 \end{pmatrix} V + \frac{1}{d_2}\begin{pmatrix} \partial_t d_2 & 0 \\ -\beta & 0 \end{pmatrix} V, \quad V_0(\xi) := V(0,\xi) = \big(d_2(0,\xi)v_0(\xi), v_1(\xi)\big).$$

We want to diagonalize the first part of this system. Hence, we are using the diagonalizer $M = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $V(t,\xi) = MV^{(1)}(t,\xi)$ to get

$$\partial_t V^{(1)} = \underbrace{\begin{pmatrix} -d_2 & 0 \\ 0 & d_2 \end{pmatrix}}_{=:D_2(t,\xi)} V^{(1)} + \underbrace{\frac{\partial_t d_2}{2d_2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} V^{(1)} + \frac{\beta}{2d_2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}}_{=:R_2(t,\xi)} V^{(1)}$$

with the Cauchy data $V_0^{(1)}(\xi) := V^{(1)}(0,\xi) = M^{-1}V_0(\xi)$. In order to make statements about solutions of this system of differential equations we will need the estimates

$$\begin{aligned}\left|\frac{\partial_t d_2(t,\xi)}{d_2(t,\xi)}\right| &= a(1+t)^{-1}\frac{1}{1-\underbrace{4|\xi|^{2-4\sigma}(1+t)^{2a}}_{\leq\varepsilon}}\\ &\leq a(1+t)^{-1}\big(1+\varepsilon+\varepsilon^2+\dots\big) \quad \text{and}\\ \left|\frac{\beta(t,\xi)}{d_2(t,\xi)}\right| &= a(1+t)^{-1}\frac{1}{\sqrt{1-4|\xi|^{2-4\sigma}(1+t)^{2a}}}\\ &\leq a(1+t)^{-1}\left(1+\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^2+\dots\right).\end{aligned}$$

Here we used (3.7) and the definition of the zone $Z_{\text{ell}}(\varepsilon)$. To describe the solution $V^{(1)}(t,\xi)$ for $t\in(0,t_{\xi,1}]$ we introduce the propagator $E_2 = E_2(t,s,\xi)$ and get $V^{(1)}(t,\xi) = E_2(t,s,\xi)V^{(1)}(s,\xi)$, $E_2(s,s,\xi) = I$ for any $t,s\in(0,t_{\xi,1}]$. After setting

$$\varphi_1(t,\xi) := -d_2(t,\xi)+\frac{\partial_t d_2(t,\xi)+\beta(t,\xi)}{2d_2(t,\xi)}, \varphi_2(t,\xi) := d_2(t,\xi)+\frac{\partial_t d_2(t,\xi)-\beta(t,\xi)}{2d_2(t,\xi)},$$
$$r_{12}(t,\xi) := \frac{-\partial_t d_2(t,\xi)-\beta(t,\xi)}{2d_2(t,\xi)}, r_{21}(t,\xi) := \frac{-\partial_t d_2(t,\xi)+\beta(t,\xi)}{2d_2(t,\xi)}$$

we obtain

$$\partial_t E_2(t,s,\xi) = \begin{pmatrix}\varphi_1(t,\xi) & 0\\ 0 & \varphi_2(t,\xi)\end{pmatrix}E_2(t,s,\xi)+\begin{pmatrix}0 & r_{12}(t,\xi)\\ r_{21}(t,\xi) & 0\end{pmatrix}E_2(t,s,\xi). \tag{3.8}$$

In the following we want to show that

$$\left\|V^{(1)}(t,\xi)\right\| \leq \exp\left(\int_0^t \varphi_2(t_1,\xi)dt_1\right)\left\|V_0^{(1)}(\xi)\right\|$$

holds and is optimal. Now we diagonalize (3.8) a second time in order to obtain integrability of the remainder terms. For that we use the transformation

$$E_2^{(1)}(t,\xi) = N_1(t,\xi)E_2(t,\xi)$$

with the diagonalizer

$$N_1(t,\xi) := \begin{pmatrix}1 & -\frac{\partial_t d_2(t,\xi)+\beta(t,\xi)}{4d_2^2(t,\xi)-2\beta(t,\xi)}\\ \frac{\partial_t d_2(t,\xi)-\beta(t,\xi)}{4d_2^2(t,\xi)-2\beta(t,\xi)} & 1\end{pmatrix}$$

proposed by Yagdjian (see [14], Proposition 2.2.8). We obtain the following new system of equations:

$$E_2^{(1)}(t,\xi) = \begin{pmatrix}\varphi_1(t,\xi)+r_{11}^{(1)}(t,\xi) & 0\\ 0 & \varphi_2(t,\xi)+r_{22}^{(1)}(t,\xi)\end{pmatrix}E_2^{(1)}(t,\xi)$$

$$+\begin{pmatrix} 0 & r_{12}^{(1)}(t,\xi) \\ r_{21}^{(1)}(t,\xi) & 0 \end{pmatrix} E_2^{(1)}(t,\xi) \tag{3.9}$$

$$=: \begin{pmatrix} \varphi_1^{(1)}(t,\xi) & 0 \\ 0 & \varphi_2^{(1)}(t,\xi) \end{pmatrix} E_2^{(1)}(t,\xi) + \begin{pmatrix} 0 & r_{12}^{(1)}(t,\xi) \\ r_{21}^{(1)}(t,\xi) & 0 \end{pmatrix} E_2^{(1)}(t,\xi) \tag{3.10}$$

with

$$R_2^{(1)}(t,\xi) = \begin{pmatrix} r_{11}^{(1)}(t,\xi) & r_{21}^{(1)}(t,\xi) \\ r_{21}^{(1)}(t,\xi) & r_{22}^{(1)}(t,\xi) \end{pmatrix} =: N_1^{-1}(t,\xi)\big(R_2(t,\xi)(N_1(t,\xi)-I) - \partial_t N_1(t,\xi)\big).$$

Using this representation and the definition of the zone $Z_{\mathrm{ell}}(\varepsilon)$ we obtain the estimate

$$\left\| R_2^{(1)}(t,\xi) \right\| \le C(1+t)^{-2} |\xi|^{-1}.$$

Having this in mind we can follow the construction of component-wise analysis as we did in Section 2.1 and get for

$$q_{11}^{(1)}(t,\xi) := E_{11}^{(1)}(t,\xi) \exp\left(-\int_0^t \varphi_2^{(1)}(t_1,\xi)dt_1 \right)$$

the integral inequality

$$\begin{aligned} \left| q_{11}^{(1)}(t,\xi) \right| &\le \left| \alpha_1^{(1)}(t,\xi) \right| + \int_0^t \left| \alpha_2^{(1)}(t,s,\xi) \right| |q_{11}(s,\xi)|\, ds, \qquad \text{where} \\ \alpha_1^{(1)}(t,\xi) &:= \exp\left(\int_0^t \big(\varphi_1^{(1)}(t_1,\xi) - \varphi_2^{(1)}(t_1,\xi)\big) dt_1 \right) \qquad \text{and} \\ \alpha_2^{(1)}(t,s,\xi) &:= r_{21}^{(1)}(s,\xi) \int_s^t r_{12}^{(1)}(t_1,\xi) \exp\left(\int_{t_1}^t \big(\varphi_1^{(1)}(t_2,\xi) - \varphi_2^{(1)}(t_2,\xi)\big) dt_2 \right) dt_1. \end{aligned}$$

Analysing the terms $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ we see

$$\left| \alpha_1^{(1)}(t,\xi) \right| \le 1 \text{ and } \left| \alpha_2^{(1)}(t,s,\xi) \right| \le |\xi|^{-1} (1+s)^{-2}$$

which leads to

$$\left| q_{11}^{(1)}(t,\xi) \right| \le C$$

by using Gronwall's inequality. Going through these steps for every entry of $Q^{(1)}(t,\xi) := \begin{pmatrix} q_{11}^{(1)}(t,\xi) & q_{12}^{(1)}(t,\xi) \\ q_{21}^{(1)}(t,\xi) & q_{22}^{(1)}(t,\xi) \end{pmatrix}$ we obtain

$$Q^{(1)}(t,\xi) = A_1^{(1)}(t,\xi) + \int_0^t A_2^{(1)}(s,\xi) \circ Q^{(1)}(t,s,\xi) ds$$

with the Hadamard product $\circ$ and the matrices $A_1^{(1)}(t,\xi)$ and $A_2^{(1)}(t,\xi)$. The modulus of every entry of these two matrices can be estimated by C and $C|\xi|^{-1}(1+s)^{-2}$ respectively such that we can apply Gronwall's inequality to every component of $Q^{(1)}$ in order to obtain $|q_{ij}(t,\xi)| \leq C$ for $i,j = 1,2$. Now we can show $\left|q_{22}^{(1)}(t,\xi)\right| \geq \frac{1}{2}$ such that we have the equivalence of $\|E_2\|$ to the weight-function $\exp\left(\int_0^t \varphi_2(t_1,\xi) + r_{22}^{(1)}(t_1,\xi)dt_1\right)$. We know that

$$q_{22}^{(1)}(t,\xi) = 1 + \int_0^t \alpha_2^{(1)}(t,s,\xi) q_{22}^{(1)}(s,\xi)ds,$$

where $\left|\alpha_2^{(1)}(t,\xi)\right| \leq C|\xi|^{-1}(1+t)^{-2}$. Using the matrizant representation we obtain the following representation for $q_{22}^{(1)}$:

$$q_{22}^{(1)}(t,\xi) = 1 + \sum_{n=0}^{\infty} \int_0^t \alpha_2^{(1)}(t,t_1,\xi) \times \int_0^{t_1} \alpha_2^{(1)}(t_1,t_2,\xi) \cdots \int_0^{t_{n-1}} \alpha_2^{(1)}(t_{n-1},t_n,\xi)dt_n dt_{n-1}\cdots dt_1.$$

Our goal is to show $\left|q_{22}^{(1)}(t,\xi)\right| \geq \frac{1}{2}$ for a constant N large enough. Here N is the constant we have used to constrain $|\xi|$. We can do this by verifying the inequality

$$\left|\int_0^t \alpha_2^{(1)}(t,t_1,\xi) \int_0^{t_1} \alpha_2^{(1)}(t_1,t_2,\xi) \cdots \int_0^{t_{n-1}} \alpha_2^{(1)}(t_{n-1},t_n,\xi)dt_n dt_{n-1}\cdots dt_1\right| \leq \frac{1}{3^n}$$

by using induction principle. In detail $\left\|V^{(1)}(t,\xi)\right\|$ behaves like

$$\begin{aligned}\left\|V^{(1)}(t,\xi)\right\| &\leq \exp\left(\int_0^t \left(d_2(t_1,\xi) + \frac{\partial_t d_2(t_1,\xi) - \beta(t_1,\xi)}{2d_2(t_1,\xi)} + r_{22}^{(1)}(t_1,\xi)\right)dt_1\right)\left\|V_0^{(1)}(\xi)\right\| \\ &\leq C_N\sqrt{\frac{d_2(t,\xi)}{d_2(0,\xi)}}\exp\left(\int_0^t d_2(t_1,\xi)dt_1\right)\left\|V_0^{(1)}(\xi)\right\|.\end{aligned}$$

With this we have

$$\begin{aligned}|\hat{u}(t,\xi)| &\leq \exp\left(-\int_0^t \frac{1}{2}|\xi|^{2\sigma}(1+t_1)^{-a}dt_1\right)|v(t,\xi)| \\ &\leq C\sqrt{\frac{d_2(0,\xi)}{d_2(t,\xi)}}\exp\left(\int_0^t \left(d_2(t_1,\xi) - \frac{1}{2}|\xi|^{2\sigma}(1+t_1)^{-a}\right)dt_1\right) \\ &\quad \times \left(|v_0(\xi)| + |v_1(\xi)|\right) \\ &\leq C(1+t)^{\frac{a}{2}}\exp\left(\int_0^t \left(d_2(t_1,\xi) - \frac{1}{2}|\xi|^{2\sigma}(1+t_1)^{-a}\right)dt_1\right) \\ &\quad \times \left(|v_0(\xi)| + |v_1(\xi)|\right) \\ &\leq C|\xi|^{3\sigma-\frac{1}{2}}\exp\left(-\int_0^t \frac{1}{2}|\xi|^{2\sigma}(1+t_1)^{-a}\left(1-\sqrt{1-4|\xi|^{2-4\sigma}(1+t_1)^{2a}}\right)dt_1\right) \\ &\quad \times \left(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\right).\end{aligned}$$

Furthermore, we see $1-\sqrt{1-4|\xi|^{2-4\sigma}(1+t_1)^{2a}} \geq 2|\xi|^{2-4\sigma}(1+t_1)^{2a}$ using the Taylor series for the square root. It follows

$$|\hat{u}(t,\xi)| \leq C|\xi|^{3\sigma-\frac{1}{2}}\exp\left(-\frac{1}{2}|\xi|^{2-2\sigma}\int_0^t (1+t_1)^a dt_1\right)\left(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\right).$$

Looking for the regularity of u at $t = t_0$ we now can choose $N = N(t_0)$ large enough to work completely in $Z_{\mathrm{ell}}(\varepsilon)$ (see Figures 1 and 2). Due to the structure of $Z_{\mathrm{ell}}(\varepsilon)$ this can be done for every $t_0 > 0$. Consequently,

$$\begin{aligned} |\hat{u}(t_0,\xi)| &\le C\,|\xi|^{3\sigma-\frac{1}{2}} \exp\left(-\frac{1}{2}\,|\xi|^{2-2\sigma}\int_0^{t_0}(1+t_1)^a dt_1\right)\big(\,|\hat{u}_0(\xi)|+|\hat{u}_1(\xi)|\,\big) \\ &\le C\,|\xi|^{3\sigma-\frac{1}{2}} \exp\big(-C(t_0)\,|\xi|^{2-2\sigma}\big)\big(\,|\hat{u}_0(\xi)|+|\hat{u}_1(\xi)|\,\big) \end{aligned}$$

holds and it is optimal for $|\xi| \ge N$. Summarizing, for all $\xi \in \mathbb{R}^n$ we obtain

$$|\hat{u}(t_0,\xi)| \le C\,\langle\xi\rangle^{3\sigma-\frac{1}{2}} \exp\big(-C(t_0)\,\langle\xi\rangle^{2-2\sigma}\big)\big(\,|\hat{u}_0(\xi)|+|\hat{u}_1(\xi)|\,\big).$$

Remark 3.1. *Following the approach from this section we are even able to prove the following result:*
Let us consider

$$u_{tt} - \Delta u + (1+t)^{-a}\big(-\Delta\big)^{\sigma} u_t = 0,\ \ \sigma\in(0,1),\ \ u(0,x)=u_0(x),\ \ u_t(0,x)=u_1(x)$$

with the Cauchy data from $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Then we have for every $t>0$ the optimal smoothing of solution

$$u(t,\cdot)\in\Gamma^{s(\sigma),\varrho(\sigma)}(\mathbb{R}^n),$$

where

$$s(\sigma)=\begin{cases} \frac{1}{2\sigma} & \text{for } \sigma\in\big(0,\frac{1}{2}\big]\,, \\ \frac{1}{2-2\sigma} & \text{for } \sigma\in\big(\frac{1}{2},1\big)\,, \end{cases}$$

and

$$\varrho(\sigma)=\begin{cases} -2\sigma & \text{for } \sigma\in\big(0,\frac{1}{2}\big]\,, \\ \frac{1}{2}-3\sigma & \text{for } \sigma\in\big[\frac{1}{2},1\big)\,. \end{cases}$$

As we see from the approach the constant $C(t_0)$ in the exponential term is increasing like $(1+t_0)^{a+1}$ for $a>-1$ such that the regularity remains Gevrey for any $t_0>0$ but improves concerning the exponential term.

3.4 The case $\sigma = 1$

Let us consider the Cauchy problem for the visco-elastic damped wave equation

$$\begin{aligned} &u_{tt}-\Delta u-(1+t)^{\delta}\Delta u_t=0,\ \ |\delta|>1, \\ &u(0,x)=u_0(x),\ \ u_t(0,x)=u_1(x). \end{aligned} \tag{3.11}$$

Can we observe a smoothing effect for this model, too?

Theorem 3.2. *Let us consider the Cauchy problem* (3.11) *with* $\big(u_0(x),u_1(x)\big)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. *Then we can not observe any smoothing effect for the solution $u(t,\cdot)$ for $t>0$.*

In the proof of this statement we are going through the three steps as we have done in the proof to Theorem 3.1. We obtain for the Fourier transform $\hat{u}(t_0,\cdot)$ of the solution $u=u(t_0,\cdot)$ only the estimate

$$|\hat{u}(t_0,\xi)|\le C\,\langle\xi\rangle^2\big(\,|\hat{u}_0(\xi)|+|\hat{u}_1(\xi)|\,\big),\ \ |\xi|\ge N$$

with a fixed $t_0>0$ and N large enough. This leads to the desired statement. More explanations can be find in Diplomarbeit Matthes, Section 3.4, [8].

4 Some concluding remarks

We have shown optimal higher order energy estimates for the solutions of the structurally damped wave equation with the special dissipation term $b(t) = \mu(1+t)^{2\sigma-1}$ with $\mu > 0$ and $\sigma \in (0,1)$. We used statements from the scattering theory and the component wise analysis in order to show the optimality. Furthermore, we observed a smoothing effect to the solutions of the structurally damped wave equation with the dissipation term $b(t) = (1+t)^{\delta}$ with $|\delta| > 1$. Analysing the characteristic roots of the structurally damped wave equation we conceived zones where we can observe elliptic and hyperbolic behaviour of the solutions. These zones crucially depend on the behaviour of $b(t)$. Thus, it is interesting and congruous to extend these results for general dissipation terms. We expect the following statement:
Let us consider the Cauchy problem

$$\begin{aligned} &u_{tt} - \Delta u + b(t)\big(-\Delta\big)^{\sigma} u_t = 0, \ \sigma \in (0,1), \\ &u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) \end{aligned} \tag{4.1}$$

with $\big(u_0, u_1\big) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *under suitable assumptions for the coefficient* $b(t)$*. Then for* $t > 0$ *the solution* $u = u(t,\cdot)$ *belongs to a Gevrey-Sobolev space. More precisely, we have* $u(t,\cdot) \in \Gamma^{s(\sigma),\varrho(\sigma)}$ *with*

$$s(\sigma) = \begin{cases} \frac{1}{2\sigma}, & \textit{for } \sigma \in \big(0, \frac{1}{2}\big], \\ \frac{1}{2-2\sigma}, & \textit{for } \sigma \in \big(\frac{1}{2}, 1\big), \end{cases} \quad \textit{and} \quad \varrho(\sigma) = \begin{cases} -2\sigma, & \textit{for } \sigma \in \big(0, \frac{1}{2}\big], \\ \frac{1}{2} - 3\sigma, & \textit{for } \sigma \in \big(\frac{1}{2}, 1\big). \end{cases}$$

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Received: 30.07.2012