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NEW EXAMPLES OF POMPEIU FUNCTIONS

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Abstract. For given sequence of real numbers $\{x_k\}_1^\infty \subset I := [0,1]$ the explicitly defined function $\varphi : I \to I$ is constructed such that $\varphi(x_k) = 0,\ k \in \mathbb{N},\ \ \varphi(x) > 0$ a.e. and all $x \in I$ are Lebesgue points of $\varphi(\cdot)$. So its primitive $f(\cdot)$ is an everywhere differentiable strictly increasing function with $f'(x_k) = 0,\ k \in \mathbb{N}$.

1 Introduction

It was A. Köpcke who first constructed in 1887-1889 a strictly increasing function $f(x)$, differentiable at every point of the segment $I := [0,1]$ and such that the zero set $X_0(f')$ of its derivative f' is a dense subset of this segment.

Later ([1], 1906) Romanian mathematician Dimitrie Pompéiu (a student of Henri Poincaré) has given much a simpler example of such function with **bounded** derivative.

Further in papers by A. Denjoy [2], A. Bruckner [3] and others this class of functions was named Pompéiu functions. In Russian language the Pompéiu example seemed to be exposed firstly in the last edition of the known problem book by B.M. Makarov et al [4] (2004, Problem 3.11).

The author got acquainted with this book for the first time at the end of March 2012. Mathematicians of the Eurasian National University (Astana, Republic of Kazakhstan) Mukhtarbay Otelbaev and Erlan Nursultanov, who posed in August 2011 the problem of existence of such functions as an unsolved one, were also unaware of that book.

Our aim is to modify one of Denjoy examples in a way that allows obtaining *new explicit* examples of Pompeiu functions f with $X_0(f')$ containing arbitrarily given countable dense set.

2 Formulation of the main result

Let us remind the definition of some classical notions.

Definition 2.1. *([5], Chapter IX, § 4) A number $x \in \mathbb{R}$ is called a Lebesgue point of a measurable function $\varphi(\cdot)$, summable in some neighborhood of this point if the following* *limit relationship*

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |\varphi(t) - \varphi(x)|\, dt = 0 \tag{2.1}$$

holds.

H. Lebesgue has proved that for any function $\varphi(\cdot)$ summable on $\mathbb{R}$ *almost all* points satisfy (2.1), and this condition implies, that at the point x the function $\varphi(x)$ coincides with the derivative of its indefinite Lebesgue integral:

$$\varphi(x) = f'(x); \quad f(x) := \int_0^x \varphi(t)\ dt. \tag{2.2}$$

In turn, the continuity of $\varphi(\cdot)$ at the point x is *sufficient* for x to be the Lebesgue point.

Definition 2.2. *([5], Chapter XV, § 4) The function $\varphi(\cdot)$ is called to be upper semi-continuous at the point $x \in \mathbb{R}$, if the upper limit of $\varphi(t)$ as $t \to x$ equals $\varphi(x)$, i.e. for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\varphi(t) < \varphi(x) + \varepsilon$ for all $t \in U_\delta(x) := (x - \delta, x + \delta)$.*

Theorem *Let $\{\gamma_k\}_{k=1}^\infty$ be a sequence of positive numbers such that $\gamma_{k+1} > \gamma_k + \varepsilon_0$ for some $\varepsilon_0 > 0$ and for all $k \in \mathbb{N}$. Let $X = \{x_k\}_{k=1}^\infty$ be a countable set dense on I. Then the function $\varphi : I \to I$ defined by the formula*

$$\varphi(x) = \inf_{k\in\mathbb{N}} |x - x_k|^{1/\gamma_k}, \tag{2.3}$$

possesses the following properties:
(i) *$\varphi(x_k) = 0$ for all $k \in \mathbb{N}$,*
(ii) *$\varphi(x) > 0$ a. e. on I,*
(iii) *$\varphi(\cdot)$ is upper semi-continuous at every $x \in I$,*
(iv) *the set of continuity points of $\varphi(\cdot)$ coincides with the set*

$$X_0(\varphi) := \{x \in I : \varphi(x) = 0\},$$

(v) *every $x \in I$ is a Lebesgue point for $\varphi(\cdot)$,*
(vi) *thus for $\varphi(\cdot)$ and all $x \in I$ relationship (2.2) holds with $f(\cdot)$ being strictly increasing, differentiable on I,*

$$0 \le f'(x) \le 1 \quad \text{and} \quad X \subset X_0(f').$$

Remark 1. *One might take $\gamma_k := k$ in the Theorem, but more general choice of $\{\gamma_k\}_{k=1}^\infty$ will be useful to derive some corollaries (see Section 4).*

Remark 2. *In his "first example" A. Denjoy ([3], Sections 24 - 25) defined the function by the formula (a little bit resembling (2.3))*

$$g(x) := \sum_{k=1}^\infty \frac{u_k}{|x - x_k|^{1/k}}; \quad u_k > 0, \quad \sum_{k=1}^\infty u_k < +\infty \tag{2.3*}$$

and proved that this series converges to the finite sum a.e. on I, *the function* $\psi(x) := 1/g(x)$ *being approximatively continuous and upper semi-continuous at each* $x \in I$, *whereas the set of its ordinary continuity points coincides with*

$$\{x : \psi(x) = 0\} = \{x : g(x) = +\infty\},$$

whose cardinality is continuum.

3 Proof of the Theorem

Steps (i)-(ii). Directly by (2.3) it follows that $\varphi(x_k) = 0$ for all $k \in \mathbb{N}$, and if $a \in (0,1)$ then

$$\text{meas } X_0(\varphi) \le \sum_k \text{meas}\{x : |x - x_k|^{1/\gamma_k} < a\}$$

$$= \sum_{k=1}^{\infty} a^{\gamma_k} \le \frac{a^{\gamma_1}}{1 - a^{\varepsilon_0}} \to 0, \quad a \to 0. \tag{3.1}$$

Hence $X \subset X_0(\varphi)$ and meas $X_0(\varphi) = 0$.

Step (iii). According to Definition 2.2 the operation of taking infimum of arbitrary family of functions uniformly bounded from below in some neighbourhood $U_\delta(x)$ and upper semi-continuous at x, always yields a function upper semi-continuous at x.

Step (iv). For every $x \in X_0(\varphi)$ the upper semi-continuity implies continuity, because $\varphi(x) \ge 0, \ x \in I$.

Conversely, if $\varphi(x) = a > 0$, then by the construction (see (2.3)) the function $\varphi(\cdot)$ has at this point the discontinuity of the second kind, and because the sequence $\{x_k\}$ is dense on I, the set of partial limits $\varphi(t), \ t \to x$, coincides with the whole segment $[0, a]$.

Step (v). To prove that every point x of *discontinuity* of $\varphi(\cdot)$ is also its Lebesgue point, one has to check (see Definition 2.1) that for the integral $(x \notin X_0(\varphi),$ i.e. $\varphi(x) = a > 0, \ h > 0)$ the relationship

$$I(h, x; f) := \int_{U_h(x)} |\varphi(t) - \varphi(x)| \, dt = o(h), \quad h \to +0, \tag{3.2}$$

holds. This is be the most difficult part of the proof of the Theorem.

First, we rewrite the integral $I(h, x; f)$ as follows:

$$I(h, x; f) = \int_{U_h(x)} (\varphi(t) - \varphi(x))_+ \, dt + \int_{U_h(x)} (\varphi(x) - \varphi(t))_+ \, dt$$

$$=: I_1(h, x; \varphi) + I_2(h, x; \varphi); \quad (y)_+ := \max(y, 0), \ y \in \mathbb{R}. \tag{3.3}$$

According to Definition 2.2, the upper semi-continuity of $\varphi(\cdot)$ implies $I_1(h, x; \varphi) = o(h), \ h \to +0$.

To estimate $I_2(h, x; \varphi)$ we need one technical auxillary assertion.

Lemma. *For* $0 < z < 0,5,\ 0 < \tau < 1$, *the inequality*

$$w(z,\tau) := 1 - (1-z)^{\tau} < 2z(1-0.5^{\tau}) < (\ln 4)\ \tau z \tag{3.4}$$

holds.

Proof of Lemma. Indeed, for τ being fixed, the second derivative

$$w''_{zz}(z,\tau) = \tau(1-\tau)(1-z)^{\tau-2} > 0,$$

i.e. $w(\cdot,\tau)$ is *strictly convex* on $(0,1)$. Therefore for $0 < z < 0.5$ the graph of $w(\cdot,\tau)$ is situated below the segment of the line joining the points $(0,0)$ and $(0.5; 1-0.5^{\tau})$, and this is exactly the first inequality in (3.4). In turn, the function $v(\tau) := 1 - 0.5^{\tau}$ is *concave* on $(0,+\infty)$, and thus $v(\tau) < v'(0)\tau = (\ln 2)\ \tau$

Continuing Step (v) of the proof of the Theorem let us choose such $n := n(a) \in \mathbb{N}$, that for $m \geq n$ the minimum among the numbers $\gamma_k a^{\gamma_k},\ k \in \{1,2,\ldots,m\}$ equals the last of them.

Let now suppose that $h \in (0; 0.5a^{\gamma_n})$ and introduce the *uniquely defined* integer $m = m(h) \geq n$ such that $a^{\gamma_{m+1}} \leq 2h < a^{\gamma_m}$.

Further, for any $k \in \mathbb{N}$ there holds

$$|x_k - x| \geq a^{\gamma_k},$$

because otherwise in accordance with (2.3) one would have $\varphi(x) \leq |x_k - x|^{1/\gamma_k} < a$.

From these observations it follows that for all $t \in U_h(x),\ \ k \leq m$ the relationships

$$|x_k - t| \geq |x_k - x| - |x - t| > a^{\gamma_k} - h$$

hold. Applying (3.4) with $z := a^{-\gamma_k}h \in (0; 0.5)$,
$\tau := 1/\gamma_k$ one comes to

$$(a - |x_k - t|^{1/\gamma_k})_+ \leq a - (a^{\gamma_k} - h)^{1/\gamma_k} = a(1 - (1 - a^{-\gamma_k}h)^{1/\gamma_k})$$

$$< \frac{2ah}{a^{\gamma_k}\gamma_k} \leq \frac{2ah}{a^{\gamma_m}\gamma_m} \leq \frac{a}{\gamma_m}; \quad k \leq m,\ a^{\gamma_{m+1}} \leq 2h < a^{\gamma_m},\ m \geq n(a). \tag{3.5}$$

Now let us consider integers $k > m,\ h$ being the same. For this case the following chain of relationships:

$$\int_{U_h(x)} \sup_{k>m} \left(a - |x_k - t|^{1/\gamma_k}\right)_+ dt \leq \sum_{k>m} \int_{U_h(x)} \left(a - |t - x_k|^{1/\gamma_k}\right)_+ dt$$

$$= \sum_{k>m} \int_{|t|<a^{\gamma_k}} \left(a - |t|^{1/\gamma_k}\right) dt = 2\sum_{k>m} \frac{a^{\gamma_k+1}}{\gamma_k + 1}$$

$$< \left(\frac{2a}{1 - a^{\varepsilon_0}}\right) \frac{a^{\gamma_{m+1}}}{\gamma_{m+1}} < C(a,\varepsilon_0)\frac{h}{\gamma_m}. \tag{3.6}$$

holds. Here we have taken into consideration that $a^{\gamma_{m+1}} \leq 2h$ by virtue of the choice of the number m.

Recall that according to formula (2.3) and the accepted notation $\varphi(x) =: a$ one has:

$$(\varphi(x) - \varphi(t))_+ = \sup_{k\in\mathbb{N}} \left(a - |x_k - t|^{1/\gamma_k}\right)_+ \tag{3.7}$$

Combining estimates (3.5) and (3.6) for the cases $k \le m$ and $k > m$, we conclude that

$$I_2(h, x; \varphi) < \frac{2a}{\gamma_m} h + \int_{U_h} \sup_{k>m} \left(a - |x_k - t|^{1/\gamma_k}\right)_+ dt$$

$$< (2a + C(a, \varepsilon_0)) \frac{h}{\gamma_m} = o(h), \tag{3.8}$$

since $\gamma_m = \gamma_{m(h)} \to +\infty,\ h \to +0,$ and this completes the proof of the Theorem. □

4 Corollaries

As was proved by D. Pompeiu himself if the zero set $X_0(f')$ is dense on I then its cardinality is necessarily continuum. Therefore following refinements of the Theorem may seem interesting.

Corollary 4.1. *If another countable set $Y = \{y_m\}_{m=1}^{\infty}$, disjoint with X, is given, then by properly choosing the sequence $\{\gamma_k\}$ one may guarantee that function (2.3) will in addition satisfy: $\varphi(y_m) > 0,\ m \in \mathbb{N}$.*

Proof. Let us put in (2.3) $\gamma_1 := 1$, and choose such $\gamma_2 \ge 1 + \gamma_1$ that

$$|x_2 - y_1|^{1/\gamma_2} \ge |x_1 - y_1|^{1/\gamma_1}.$$

Further, if $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k$ are chosen, we find such $\gamma_{k+1} \ge \gamma_k + 1$, that for all $m \in \{1, 2, \ldots, k\}$ the inequalities

$$|x_{k+1} - y_m|^{1/\gamma_{k+1}} \ge \min_{1\le s\le k} |x_s - y_m|^{1/\gamma_s} \tag{4.1}$$

hold. This is possible because for $\alpha \in (0, 1)$ $\alpha^{1/\gamma} \to 1, \gamma \to +\infty$. Now by the construction one has the relationship

$$\varphi(y_m) := \inf_{k\in\mathbb{N}} |x_k - y_m|^{1/\gamma_k} = \min_{1\le s\le m} |x_s - y_m|^{1/\gamma_s} > 0 \tag{4.2}$$

for all $m \in \mathbb{N}$ (see (2.3)). □

Corollary 4.2. *Let there be given three pair-wise disjoint sets X, Y^+, Y^-, whose cardinality $\le \aleph_0$, X being dense in I; then one may construct a function $f(\cdot)$, differentible at every point of I, such that $|f'(x)| \le 1$ for all $x \in I$ and*

$$f'(x) = 0,\ x \in X; \quad f'(x) > 0,\ x \in Y^+; \quad f'(x) < 0,\ x \in Y^-. \tag{4.3}$$

Proof. By virtue of Corollary 4.1 there exist two funtions $\varphi_1(\cdot), \varphi_2(\cdot)$ such that $0 \le \varphi_1(x), \varphi_2(x) \le 1 \ \forall x \in I,$ for which every point of I is a Lebesgue point and

$$\varphi_1(x) = 0, \ x \in X \cup Y^-, \quad \varphi_1(x) > 0, \ x \in Y^+;$$
$$\varphi_2(x) = 0, \ x \in X \cup Y^+, \quad \varphi_2(x) > 0, \ x \in Y^-. \tag{4.4}$$

Now the function

$$f(x) := \int_0^x (\varphi_1(t) - \varphi_2(t))\, dt$$

possesses all the properties stated in the Corollary 4.2. □

Remark 3. *Certainly, if the sets Y^+, Y^- are also dense in I, then the function of Corollary 4.2 is not monotonic on any interval $(a, b) \subset I$.*

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