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ON SPECTRAL PROPERTIES OF A PERIODIC PROBLEM WITH AN INTEGRAL PERTURBATION OF THE BOUNDARY CONDITION

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Abstract. In this paper we consider the spectral problem for the Schrödinger equation with an integral perturbation in the periodic boundary conditions. The unperturbed problem is assumed to have the system of eigenfunctions and associated functions forming a Riesz basis in $L_2(0,1)$. We construct the characteristic determinant of the spectral problem. We show that the basis property of the system of root functions of the problem may fail to be satisfied under an arbitrarily small change in the kernel of the integral perturbation.

1 Introduction

It is well known that the system of root functions of an ordinary differential operator with arbitrary strongly regular boundary conditions is a Riesz basis in $L_2(0,1)$. In the case in which the boundary conditions are regular but not strongly regular, the basis property of the system of root function, unlike the completeness property, is not even determined by the boundary conditions. V.A. Il'in [2] was the first to note this effect [2] and he constructed a related example for a second order differential operator of general form. As shown in [2], in this case, in addition to the boundary conditions, the coefficients of the differential operator also affect the basis property. Moreover, this property can change under an arbitrarily small change in coefficients in the metric of those spaces in which they are defined. In the present paper we consider the problem of instability of the basis property under integral perturbation of the boundary conditions for the periodic boundary value problem.

2 Statement of the problem

In the space $L_2(0,1)$ we consider the operator L_0 generated by the ordinary differential expression

$$l(u) = -u''(x) + q(x)u(x),\ 0 < x < 1 \tag{2.1}$$

and the periodic boundary conditions

$$U_1(u) \equiv u'(0) - u'(1) = 0,\ U_2(u) \equiv u(0) - u(1) = 0. \tag{2.2}$$

Let L_1 be the operator in $L_2(0,1)$, given by expression (1) and the "perturbed" boundary conditions:

$$U_1(u) = \int_0^1 \overline{p(x)}u(x)dx,\ U_2(u) = 0,\ p(x) \in L_2(0,1). \tag{2.3}$$

In the present paper the unperturbed operator L_0 is assumed to have the system of eigen- and associated functions (EAF) forming a Riesz basis in $L_2(0,1)$ and we construct the characteristic determinant of the spectral problem for the operator L_1. On the basis of this formula we make conclusions about instability of the Riesz basis property of the EAF of the problem with the integral perturbation of the boundary condition. The case $q(x) \equiv 0$ was studied in [4].

The problem of the basis of eigen- and associated functions of the operator L_1 with more general integral boundary conditions was positively solved in [10] where the Riesz basis with parentheses property under the conditions of Birkhoff regularity [8, pp. 66–67] of the boundary conditions of the unperturbed problem was proved. Moreover, under the additional assumption of strong regularity the Riesz basis property of EAF was proved.

For the unperturbed operator L_0 the Riesz basis with parentheses property of EAF in case of the regular boundary conditions was established in [11]. If the boundary conditions are strongly regular then EAF form a Riesz basis [7, 3]. For the second order equation the Riesz basis property under regular but not strongly regular boundary conditions was considered in [5]. In our case boundary conditions (2.2) are regular but not strongly regular boundary conditions. So we cannot apply the results of [10] for it and an additional investigation is required.

Problem (2.1), (2.2) is a periodic boundary problem, moreover, it is also self-adjoint for a real-valued coefficient $q(x)$. The problem of the Riesz basis property of the system of root functions of the periodic problem with a complex-valued coefficient $q(x)$ was investigated in [6, 13]. In particular, it was shown that the set of potentials $q(x)$ such that the system of root functions of problem (2.1), (2.2) is a Riesz basis in $L_2(0,1)$, is dense in $L_2(0,1)$. As follows from the results in [11], the set of potentials $q(x)$ for which problem (2.1), (2.2) has an asymptotically multiple spectrum, is dense in $L_2(0,1)$. A particular case of the potential $q(x)$, when all the eigenvalues of (2.1), (2.2) are multiple, investigated in our study [9].

3 Characteristic determinant of a spectral problem

We additionally assume that the potential $q(x)$ is chosen in such a way that the unperturbed periodic problem (2.1), (2.2) has the system of EAF forming a Riesz basis in $L_2(0,1)$. Let λ_k^0 be eigenvalues (numbered in decreasing order of their modules) of the operator L_0 of multiplicity m_k+1 to which correspond the eigenfunctions $u_{k0}^0(x)$ and chains of the adjoint functions $u_{kj}^0(x),\ j=\overline{1,m_k}$. Then the biorthogonal system consists of the eigenfunctions $v_{km_k}^0(x)$ and the associated functions $v_{kj}^0(x), j=\overline{0,m_k-1}$ of the operator L_0^* corresponding to the eigenvalues $\overline{\lambda_k^0}$. Obviously the system of EAF $\left\{v_{kj}^0(x),\ j=\overline{0,m_k},\ k=\overline{1,\infty}\right\}$ of the operator L_0^* also forms a Riesz basis in $L_2(0,1)$.

Applying integration by parts, we obtain the Lagrange formula:

$$\int_0^1 l(u)\overline{v(x)}dx-\int_0^1 u(x)\overline{l^*(v)}dx=\left[u'(0)-u'(1)\right]\overline{v(0)}+$$
$$+u'(1)\left[\overline{v(0)}-\overline{v(1)}\right]-\left[u(0)-u(1)\right]\overline{v'(0)}-u(1)\left[\overline{v'(0)}-\overline{v'(1)}\right]. \tag{3.1}$$

Here $l^*(v)$ is the adjoint differential expression:

$$l^*(v)=-v''(x)+\overline{q(x)}v(x),\ 0<x<1. \tag{3.2}$$

Consequently the operator L_0^* corresponding to the operator L_0 is given by differential expression (3.2) and the boundary conditions

$$U_1(v)\equiv v'(0)-v'(1)=0,\ U_2(v)\equiv v(0)-v(1)=0. \tag{3.3}$$

Also the operator L_1^* corresponding to the operator L_1 is given by the loaded differential expression

$$l_1^*(v)=-v''(x)+\overline{q(x)}v(x)+p(x)v(0),\ 0<x<1. \tag{3.4}$$

and periodic boundary conditions (3.3).

Now we construct the characteristic determinant of the spectral problem. Let $u_1(x,\lambda),u_2(x,\lambda)$ be a system of fundamental solution of the equation $l(u)=\lambda u$ satisfying the conditions $u_j^{(k-1)}(0,\lambda)=\delta_{jk},\ j,k=1,2$. Here δ_{jk} is the Kronecker symbol. Assuming that the general solution $u(x,\lambda)=C_1u_1(x,\lambda)+C_2u_2(x,\lambda)$, satisfies boundary conditions (2.3), we obtain the following linear system with respect to the coefficients C_k:

$$C_1\left[-u_1'(1,\lambda)-\int_0^1\overline{p(x)}u_1(x,\lambda)dx\right]+C_2\left[1-u_2'(1,\lambda)-\int_0^1\overline{p(x)}u_2(x,\lambda)dx\right]=0,$$
$$C_1[1-u_1(1,\lambda)]+C_2[1-u_2(1,\lambda)]=0.$$

The determinant of this system is the characteristic determinant of problem (2.1), (2.3):

$$\Delta_1(\lambda)=\begin{vmatrix} -u_1'(1,\lambda)-\int_0^1\overline{p(x)}u_1(x,\lambda)dx & 1-u_1(1,\lambda) \\ 1-u_2'(1,\lambda)-\int_0^1\overline{p(x)}u_2(x,\lambda)dx & -u_2(1,\lambda)\end{vmatrix} \tag{3.5}$$

It is easy to see that the characteristic determinant of unperturbed problem (2.1), (2.2) is obtained by (3.5) with $p(x)=0$. We denote it by $\Delta_0(\lambda)$. We expand the function $p(x)$ along the basis $\left\{v_{kj}^0(x),\ j=\overline{0,m_k},\ k=\overline{1,\infty}\right\}$:

$$p(x)=\sum_{k=1}^{\infty}\left[\sum_{j=0}^{m_k}a_{kj}v_{kj}^0(x)\right]. \tag{3.6}$$

Using (3.6), we shall obtain a more convenient representation of the determinant $\Delta_1(\lambda)$. To do all this we first calculate

$$\int_0^1 \overline{p(x)} u_s(x,\lambda)dx = \sum_{k=1}^{\infty}\left[\sum_{j=0}^{m_k} \overline{a_{kj}}\left(u_s(x,\lambda), v_{kj}^0(x)\right)\right], \ s=1,2. \tag{3.7}$$

In order to avoid the problem of choice of the associated functions we assume that EAF of the adjoint problem are constructed by the following formulas:

$$L_0^* v_{km_k^0}^0 = \overline{\lambda_k^0} v_{km_k^0}^0, \ L_0^* v_{kj}^0 = \overline{\lambda_k^0} v_{kj}^0 + \sqrt{\overline{\lambda_k^0} v_{kj+1}^0}, \ j=\overline{0, m_k-1}.$$

It is easy to check the following chain of equalities

$$(\lambda-\lambda_k^0)(u_s(x,\lambda), v_{kj}^0(x)) = \left(\lambda u_s(x,\lambda), v_{kj}^0(x)\right) - \left(u_s(x,\lambda), \overline{\lambda_k^0} v_{kj}^0(x)\right)$$

$$= (l(u_s), v_{kj}^0) - \left(u_s, L_0^* v_{kj}^0 - \left(\overline{\lambda_k^0}\right)^{\frac{1}{2}} v_{kj+1}^0\right) = (l(u_s), v_{kj}^0) - \left(u_s, L_0^* v_{kj}^0\right)$$

$$+ \left(\lambda_k^0\right)^{\frac{1}{2}} (u_s, v_{kj+1}^0).$$

Here we use Lagrange formula (3.1) and boundary conditions (3.3). Then for all $j=\overline{0, m_k-1}$ we get

$$(\lambda-\lambda_k^0)(u_s(x,\lambda), v_{kj}^0(x)) = B_{ks}(j) + \left(\lambda_k^0\right)^{\frac{1}{2}} (u_s, v_{kj+1}^0),$$

where we denote

$$B_{ks}(j) = [u_s'(0) - u_s'(1)]\, \overline{v_{kj}^0(0)} - [u_s(0) - u_s(1)]\, \overline{v_{kj}^{0\,\prime}(0)}. \tag{3.8}$$

Repeating such calculations $(m_k - 1 - j)$ times, we obtain

$$(u_s(x,\lambda), v_{kj}^0(x)) = \sum_{r=0}^{m_k-1-j} B_{ks}(j+r) \frac{(\lambda_k^0)^{\frac{r}{2}}}{[\lambda-\lambda_k^0]^{r+1}} + \left(\frac{(\lambda_k^0)^{\frac{1}{2}}}{\lambda-\lambda_k^0}\right)^{m_k-j} (u_s, v_{k\ m_k}^0)$$

Similarly for the eigenfunction $v_{k\ m_k}^0$ we get

$$(\lambda-\lambda_k^0)(u_s(x,\lambda), v_{k\ m_k}^0(x)) = B_{ks}(m_k)$$

Combining two last equations, we get

$$(u_s(x,\lambda), v_{kj}^0(x)) = \sum_{r=0}^{m_k-j} B_{ks}(j+r) \frac{(\lambda_k^0)^{\frac{r}{2}}}{[\lambda-\lambda_k^0]^{r+1}}.$$

Substituting here the explicit form of $B_{ks}(j+r)$ given by (3.8), we find

$$\begin{aligned}(u_s(x,\lambda), v_{kj}^0(x)) &= [u_s'(0) - u_s'(1)] \left[\sum_{r=0}^{m_k-j} \frac{(\lambda_k^0)^{\frac{r}{2}}}{[\lambda-\lambda_k^0]^{r+1}} \overline{v_{k\ j+r}^0(0)}\right] \\ &-[u_s(0) - u_s(1)] \left[\sum_{r=0}^{m_k-j} \frac{(\lambda_k^0)^{\frac{r}{2}}}{[\lambda-\lambda_k^0]^{r+1}} \overline{v_{k\ j+r}^{\prime\,0}(0)}\right]\end{aligned} \tag{3.9}$$

Now we can substitute (3.9) in formula (3.7). Then

$$\int_0^1 \overline{p_m(x)} u_s(x,\lambda)dx = [u_s'(0) - u_s'(1)]\, A_1(\lambda) - [u_s(0) - u_s(1)]A_2(\lambda),$$

where we denote

$$A_i(\lambda) = \sum_{k=1}^{\infty} \left[\sum_{j=0}^{m_k} \overline{a_{k\ j}} \left(\sum_{r=0}^{m_k-j} \frac{(\lambda_k^0)^{\frac{r}{2}}}{[\lambda - \lambda_k^0]^{r+1}} \overline{v_{k\ j+r}^{0\ (i-1)}(0)} \right) \right]. \tag{3.10}$$

Using (3.5), after elementary transformations, we get:

$$\Delta_1(\lambda) = \Delta_0(\lambda) - \Delta_0(\lambda)A_1(\lambda) = \Delta_0(\lambda)\Big(1 - A_1(\lambda)\Big). \tag{3.11}$$

Substituting here the value of $A_1(\lambda)$ given by (3.10), we find the following representation of the characteristic determinant of the operator L_1:

$$\Delta_1(\lambda) = \Delta_0(\lambda) \left(1 - \sum_{k=1}^{\infty} \left[\sum_{j=0}^{m_k} \overline{a_{k\ j}} \left(\sum_{r=0}^{m_k-j} \frac{(\lambda_k^0)^{\frac{r}{2}}}{[\lambda - \lambda_k^0]^{r+1}} \overline{v_{k\ j+r}^{0}(0)} \right) \right] \right). \tag{3.12}$$

Let us formulate the obtained result in the form of a theorem.

Theorem 3.1. *Let problem* (2.1), (2.2) *have the eigenvalues* λ_k^0 *and a system of EAF forming a Riesz basis. Then the characteristic determinant of problem* (2.1), (2.3) *with the perturbed boundary conditions can be represented in form* (3.12)*, where* $\Delta_0(\lambda)$ *is the characteristic determinant of problem* (2.1), (2.2)*;* $\{v_{kj}^0\}$ *are EAF of the adjoint unperturbed problem;* a_{kj} *are the Fourier coefficients of the biorthogonal expansion* (3.6) *of function* $p(x)$ *along this system.*

In representation (3.11) the function $A_1(\lambda)$ has poles at $\lambda = \lambda_k^0$ of maximal order $m_k + 1$. But at the same points the function $\Delta_0(\lambda)$ has zeros of order $m_k + 1$. So the function $\Delta_1(\lambda)$, given by formula (3.12), is an entire analytic function of the variable λ.

4 Particular cases of the characteristic determinant

Consider the cases in which formula (3.12) is simpler. First suppose that the potential $q(x)$ is chosen in such a way that the eigenvalues λ_k^0 of the unperturbed periodic boundary problem (2.1), (2.2) are all double except for the simple root λ_0^0; the corresponding root subspace consists of two eigenfunctions $c_k(x), s_k(x)$; and the system $\{c_0(x),\ c_k(x), s_k(x), k = \overline{1,\infty}\}$ is a Riesz basis in $L_2(0,1)$. Then the biorthogonal adjoint system $\{c_0^*(x),\ c_k^*(x), s_k^*(x)\}$ consists of the eigenfunctions of the adjoint boundary problem (3.2), (3.3) and is a Riesz basis in $L_2(0,1)$ as well. Without loss of generality, we choose the biorthogonal systems in such a way that $s_k^*(0) = 0$. Then in this particular case formula (3.12) of the representation of the characteristic determinant of the perturbed problem has the form

$$\Delta_1(\lambda) = \Delta_0(\lambda) \left(1 - \sum_{k=0}^{\infty} \overline{a_k} \frac{\overline{c_k^*(0)}}{\lambda - \lambda_k^0} \right),$$

where a_k are the Fourier coefficients of biorthogonal expansion (3.6) which in our particular case has the form

$$p(x) = \sum_{k=0}^{\infty} a_k c_k^*(x) + \sum_{k=1}^{\infty} b_k s_k^*(x).$$

In particular, if $q(x) \equiv 0$, then chosing the basis of the unperturbed problem to be $c_0^*(x) = 1,\ c_k^*(x) = \cos 2k\pi x,\ s_k^*(x) = \sin 2k\pi x$, the characteristic determinant will be written as

$$\Delta_1(\lambda) = \sin^2 \frac{\sqrt{\lambda}}{2} \left(1 - \sum_{k=0}^{\infty} \frac{\overline{a_k}}{\lambda - (2k\pi)^2}\right).$$

From the analysis of the last formula it follows that $\lambda_k^1 = \lambda_k^0 = (2k\pi)^2$ is always an eigenvalue of problem (2.1), (2.3) with $q(x) \equiv 0$. This fact clarifies the results of paper [4].

Another case of simpler form of characteristic determinant (3.12) is the case in which $p(x)$ is represented as a finite sum in (3.6), that is when there exists such number N that $a_{kj} = 0$ for all $k > N$. In this case the formula (3.12) has the form

$$\Delta_1(\lambda) = \Delta_0(\lambda) \left(1 + \sum_{k=1}^{N} \left[\sum_{j=0}^{m_k} \overline{a_{k\ j}} \left(\sum_{r=0}^{m_k - j} \frac{(\lambda_k^0)^{\frac{r}{2}}}{[\lambda - \lambda_k^0]^{r+1}} \overline{v_{k\ j+r}^0(0)}\right)\right]\right). \qquad (4.1)$$

From this special case of formula (3.12) it is easy to prove the following

Corollary 4.1. *Let problem* (2.1), (2.2) *have the eigenvalues* λ_k^0 *and a system of EAF forming a Riesz basis. Then for any preassigned numbers* $\hat{\lambda}$ *(complex) and* $\hat{m}$ *(natural) there exists a function* p *such that* $\hat{\lambda}$ *is an eigenvalue of problem* (2.1), (2.3) *of multiplicity* $\hat{m}$.

5 Perturbations preserving the basis property

From the analysis of formula (4.1) it is also easy to see that $\Delta_1(\lambda_k^0) = 0$ for all $k > N$. Hence all the eigenvalues $\lambda_k^0,\ k > N$ of the unperturbed problem (2.1), (2.2) are eigenvalues of perturbed problem (2.1), (2.3). Also it is not hard to see that the multiplicity $(m_k + 1)$ of the eigenvalues $\lambda_k^0,\ k > N$ is also preserved.

Moreover from the biorthogonality condition of the system of EAF of the adjoint problems it follows that in this case

$$\int_0^1 \overline{p(x)} u_{kj}^0(x) dx = 0,\ j = \overline{0, m_k}, k > N.$$

So EAF $u_{kj}^0(x)$ of problem (2.1), (2.2) at $k > N$ satisfy the boundary conditions (2.3) and hence, are EAF of problem (2.1), (2.3). Thus in this case the system of EAF of problem (2.1), (2.3) and the system of EAF of problem (2.1), (2.2) (forming a Riesz basis) coincide except for a finite number of the first terms. Consequently, the system of EAF of problem (2.1), (2.3) also is a Riesz basis in $L_2(0, 1)$.

By the Riesz basis property in $L_2(0, 1)$ of the system of EAF $\{v_{kj}^0(x)\}$ of the adjoint unperturbed problem, the set of functions $p(x)$, represented by finite sums of (3.6) is dense in $L_2(0, 1)$. Hence the following statement is proved.

Theorem 5.1. *Let problem* (2.1), (2.2) *have a system of EAF forming a Riesz basis. Then the set of all functions* $p \in L_2(0,1)$, *for which the system of EAF of problem* (2.1), (2.3) *is a Riesz basis in* $L_2(0,1)$, *is dense in* $L_2(0,1)$.

Note that in [4] an analogue of Theorem 5.1 was proved for a particular case of the integral perturbation of the periodic boundary conditions for the operator of double differentiation (that is for the case $q(x) \equiv 0$).

6 Perturbations that do not preserve the basis property

Moreover, in [4] it was proved that the set of all functions $p \in L_2(0,1)$ for which the system of EAF of the problem (with perturbed periodic boundary conditions) does not form even an ordinary basis in $L_2(0,1)$ is also dense in $L_2(0,1)$. We prove a similar result for more general case $q(x) \not\equiv 0$.

Theorem 6.1. *Let periodic problem* (2.1), (2.2) *have a system of EAF forming a Riesz basis in* $L_2(0,1)$, *let the eigenvalues of the problem be asymptotically double, and let root subspaces corresponding to double eigenvalues consist of two eigenfunctions. Then the set of all functions* $p \in L_2(0,1)$, *for which the system of EAF of problem* (2.1), (2.3) *does not form even an ordinary basis in* $L_2(0,1)$, *is dense in* $L_2(0,1)$.

Proof. Asymptotic doubleness of the eigenvalues means the existence of such a number N_0 that all eigenvalues λ_k^0 for $k > N_0$ are double. As the root subspaces for $k > N_0$ consist of two eigenfunctions $c_k(x),\ s_k(x)$ for the operator L_0 and two eigenfunctions $c_k^*(x),\ s_k^*(x)$ for the operator L_0^*, we may choose the eigenfunctions inside the root subspace in a way convenient for us. Without loss of generality, we assume that the functions $c_k(x),\ s_k(x)$ are normalized, mutually orthogonal and $s_k^*(0)$ for all $k > N_0$.

So the system of EAF of problem (3.2), (3.3) has the form:

$$\{v_{kj}^0(x),\ j = \overline{0, m_k},\ k = \overline{1, N_0};\ c_k^*(x),\ s_k^*(x),\ k > N_0\},$$

and expansion (3.6) may be represented in the form:

$$p(x) = \sum_{k=1}^{N_0} \left[\sum_{j=0}^{m_k} a_{kj} v_{kj}^0(x) \right] + \sum_{k=N_0+1}^{\infty} \left[a_k c_k^*(x) + b_k s_k^*(x) \right]. \tag{6.1}$$

In this case (taking into account that $s_k^*(0) = 0$) the characteristic determinant (3.12) has the form:

$$\begin{aligned}\Delta_1(\lambda) = \Bigg(1 - \textstyle\sum_{k=1}^{N_0} \left[\sum_{j=0}^{m_k} \overline{a_{kj}} \left(\sum_{r=0}^{m_k - j} \frac{(\lambda_k^0)^{\frac{r}{2}}}{[\lambda - \lambda_k^0]^{r+1}} \overline{v_{kj+r}^0(0)} \right) \right] + \\ + \textstyle\sum_{k=N_0+1}^{\infty} \overline{a_k} \frac{\overline{c_k^*(0)}}{\lambda - \lambda_k^0} \Bigg) \Delta_0(\lambda).\end{aligned} \tag{6.2}$$

From the analysis of formula (6.2) it is easy to see $\Delta_1(\lambda_k^0) = 0$ for all $k > N_0$; i.e., all eigenvalues $\lambda_k^0,\ k > N_0$ of unperturbed problem (2.1), (2.2) are also eigenvalues of perturbed problem (2.1), (2.3). So $\overline{\lambda_k^0}$, $k > N_0$ are eigenvalues of problem (3.2), (3.3). As $s_k^*(0) = 0$, the functions $v_k^0(x) = \beta_k s_k^*(x)$ are eigenfunctions of problem (3.2), (3.3), corresponding to the eigenvalues $\overline{\lambda_k^0}$, $k > N_0$.

If for any $n > N_0$ simultaneously $a_n \neq 0$ and $b_n \neq 0$, then by direct calculation it is easy to verify that the function $u_n^0(x) = \overline{b_n} c_n(x) - \overline{a_n} s_n(x)$ is an eigenfunction of problem (2.1), (2.2), corresponding to the eigenvalue λ_n^0.

From the condition of biorthogonality of the system of EAF corresponding to the adjoint problems it follows that

$$1 = (v_n^0, u_n^0) = (\beta_n s_n^*(x),\ \overline{b_n} c_n(x) - \overline{a_n} s_n(x)) = -\beta_n a_n.$$

Hence we find the coefficient $\beta_n = -1/a_n$. Therefore,

$$u_n^0(x) = \overline{b_n} c_n(x) - \overline{a_n} s_n(x) \text{ and } v_k^0(x) = -(1/a_n) s_n^*(x) \tag{6.3}$$

are pairs in biorthogonal systems. Since the eigenfunctions $c_n(x), s_n(x)$ are chosen to be mutually orthogonal and normalized, we have

$$\|u_n^0(x)\|^2 = |b_n|^2 \|c_n(x)\|^2 + |a_n|^2 \|s_n(x)\|^2 = |b_n|^2 + |a_n|^2. \tag{6.4}$$

From the biorthogonality condition we obtain

$$1 = (s_n(x), s_n^*(x)) \leq \frac{1}{2}\|s_n\|^2 + \frac{1}{2}\|s_n^*\|^2 = \frac{1}{2} + \frac{1}{2}\|s_n^*\|^2.$$

Therefore $\|s_n^*\| \geq 1$. This, together with (6.4), implies the inequality

$$\|u_n^0(x)\| \cdot \|v_n^0(x)\| \geq \sqrt{1 + |b_n/a_n|^2}$$

for biorthogonal pair (6.3).

Consequently, for all functions p whose coefficients of the expansions of the functions (6.1) contain a subsequence $a_n \neq 0,\ b_n \neq 0$ such that $\lim_{n\to\infty} |b_n/a_n| = \infty$, we get

$$\lim_{n\to\infty} \|u_n^0(x)\| \cdot \|v_n^0(x)\| = \infty;$$

i.e., the uniform minimality condition [1, p. 66] fails to be satisfied for the system of root functions of the operator L_1 and, therefore, it is not even an ordinary basis in $L_2(0,1)$.

Denote by $\sigma_N(x)$ the partial sum of series (6.1). Suppose $N > N_0$. Due to the Riesz basis property of the system of EAF system of problem (3.2), (3.3), the linear manifold of all functions p, which can be presented as a series

$$p(x) = \sigma_N(x) + \sum_{k=N+1}^{\infty} [a_k c_k^*(x) + b_k s_k^*(x)],$$

where $a_k = 1/(k2^k)$, $b_k = 1/(2^k)$, $k > N$ is dense in $L_2(0,1)$. Since $b_k/a_k = k \to \infty$, for such functions p the system of EAF of problem (2.1), (2.3) it is not even an ordinary basis in $L_2(0,1)$. □

Note that, in the proof of the theorem, we have justified that problem (2.1), (2.3) and problem (3.3) for loaded equation (3.4) are adjoint. Since adjoint operators simultaneously have the Riesz basis property of root functions, we have the following assertion.

Corollary 6.1. *The set P of all functions $p \in L_2(0,1)$, for which the system of root functions of problem* (3.3) *for loaded equation* (3.4) *is a Riesz basis in $L_2(0,1)$, is dense in $L_2(0,1)$. The set $L_2(0,1)\diagdown P$ is dense in $L_2(0,1)$ as well.*

The results of the present work, in contrast to [10], demonstrate the possibility of instability of the basis properties of system of EAF of problems with integral perturbation of the boundary conditions, which are regular but not strongly regular.

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