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ADDITION OF LOWER ORDER TERMS PRESERVING ALMOST HYPOELLIPTICITY OF POLYNOMIALS

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Abstract. A linear differential operator $P(D)$ with constant coefficients is called *almost hypoelliptic* if all derivatives $P^{(\nu)}(\xi)$ of the characteristic polynomial $P(\xi)$ can be estimated above via $P(\xi)$. In this paper we describe the collection of lower order terms addition of which to an almost hypoelliptic operator $P(D)$ (polynomial $P(\xi)$) preserves its almost hypoellipticity and its strength.

1 Introduction

We shall use the following standard notation: N- the set of all natural numbers, $N_0 = N \cup \{0\}$, $N_0^n = N_0 \times ... \times N_0$- the set of all n-dimensional multi-indices, R^n - the n-dimensional Euclidean space. $R_0^n = \{\xi \in R^n; \xi_1 \cdots \xi_n \neq 0\}$. For $\xi = (\xi_1, \cdots, \xi_n) \in R^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in N_0^n$ we put $|\xi| = \sqrt{\xi_1^2 + ... + \xi_n^2}$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$, where $D_j = \frac{\partial}{\partial \xi_j}$ $(j = 1, \cdots, n)$.

For a linear differential operator with constant coefficients $P(D) = \sum\limits_\alpha \gamma_\alpha D^\alpha$, let $P(\xi) = \sum\limits_\alpha \gamma_\alpha \xi^\alpha$ be its characteristic polynomial (complete symbol), where the sum extends over a finite collection of multi-indices $(P) = \{\alpha \in N_0^n, \gamma_\alpha \neq 0\}$.

Definition 1.1. The least convex polyhedron $\Re = \Re(P)$ containing all points $\alpha \in (P)$ is called **the Newton** or **the characteristic polyhedron** of an operator $P(D)$ (a polynomial $P(\xi)$).

A polyhedron $\Re$ with vertices from N_0^n is called **complete** (see [19]), if $\Re$ has a vertex at the origin and also it has vertices on each coordinate axis.

Let $\Re$ be a complete polyhedron. A set $\Gamma \subset \Re$ is called a face of $\Re$, if there exist a (unit) vector $\lambda = (\lambda_1, \cdots, \lambda_n)$ and a number $d = d(\lambda, \Gamma) \geq 0$ such that $(\lambda, \alpha) \equiv (\lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n) = d$ for all points $\alpha \in \Gamma$, wile $(\lambda, \beta) < d$ for $\beta \in \Re \setminus \Gamma$. The unit vector λ is called an outward normal ($\Re-$normal) of the face Γ. The set of all $\Re-$normals of Γ we denote by $\Lambda(\Gamma)\cdot$

It is clear, that for a $k-$dimensional face $(0 \leq k \leq n-1)$ of Γ of a complete polyhedron $\Re$ the set $\Lambda(\Gamma)$ is an open $(n-k)-$dimensional cone. Observe, that the $\Re-$normal of $(n-1)-$dimensional face is determined uniquely.

Definition 1.2. A face Γ of a complete polyhedron $\Re$ is called **principal**, if there exists $\lambda \in \Lambda(\Gamma)$ with at least one positive coordinate. If in $\Lambda(\Gamma)$ there exists a λ with nonnegative (positive) coordinates, then the face Γ we call **proper (completely proper)** (see [21], [14] or [17].) A complete polyhedron $\Re$ is called **proper (completely proper)** if all its $(n-1)-$dimensional non-coordinate faces are proper (completely proper). A point $\alpha \in \Re$ is called principal (proper, completely proper) if α belongs to a principal (proper, completely proper) face of $\Re$.

A monomial ξ^α is called a lower order monomial with respect to the polynomial P, if 1) $\alpha \in \Re(P)$, 2) α is non-principal point of $\Re(P)$. A polynomial Q is called a lower order term with respect to the polynomial P, if every monomial ξ^α with $\alpha \in (Q)$ is a lower order monomial with respect to P.

Definition 1.3. We say that an operator $P(D)$ is more powerful than an operator $Q(D)$ (a polynomial $P(\xi)$ is more powerful than a polynomial $Q(\xi)$) and write $Q < P$, if for some constant $C > 0$

$$|\,Q(\xi)| \leq C[1 + |P(\xi)|\,] \quad \forall \xi \in R^n.$$

Definition 1.4. A polynomial $P(\xi)$ is called **almost hypoelliptic** if $D^\alpha P < P$ for all $\alpha \in N_0^n$.

It is well known that one can add any lower order term to an elliptic or semielliptic polynomial without violating its ellipticity or semiellipticity. However adding some lower order term to hypoelliptic (by L. Hörmander [16], by L. Gårding - B. Malgrange [8], by V.I. Burenkov [2] - [5], by V.V. Grushin [15] and others), to hyperbolic (by I.G. Petrovski [23], or by L. Gårding [7], see also [14]) or to almost hypoelliptic (see [19] or [12]) polynomials can violate their (almost) hypoellipticity or hyperbolicity.

The aim of the paper is finding algebraic conditions on a lower order term Q with respect to an almost hypoelliptic polynomial P under which the polynomial $P(\xi) + Q(\xi)$ is almost hypoelliptic.

Let $\Re = \Re(P)$ be the Newton polyhedron of a polynomial $P(\xi)$, and let $\Re_i^j$ $(i = 1, \cdots, M_j; j = 0, 1, \cdots, n-1)$ be principal faces of $\Re\cdot$ With each $\Re_i^j$ we associate a subpolynomial

$$P^{i,j}(\xi) = \sum_{\alpha \in \Re_i^j} \gamma_\alpha \xi^\alpha.$$

It is easy to check (see, for instence, [21]) that the polynomial $P^{i,j}$ is $\lambda-$ homogeneous (generalized homogeneous) for every vector $\lambda \in \Lambda(\Re_i^j)$, that is, there exists a number $d_i^j(\lambda)$, satisfying the conditions

$$P^{i,j}(\xi) = \sum_{(\lambda,\alpha) = d_i^j(\lambda)} \gamma_\alpha \xi^\alpha; \quad P^{i,j}(t^\lambda \xi) = t^{d_i^j(\lambda)} P^{i,j}(\xi), \tag{1.1}$$

for every $t > 0$ and for all $\xi \in R^n$, where $t^\lambda \xi = (t^{\lambda_1}\xi_1, \cdots, t^{\lambda_n}\xi_n)\cdot$

Definition 1.5. A face $\Re_i^j$ $(1 \leq i \leq M_j; 0 \leq j \leq n-1)$ of the Newton polyhedron $\Re = \Re(P)$ is called **regular or non-degenerate** (see [21]) if $P^{i,j}(\xi) \neq 0$ for $\xi \in R_0^n\cdot$ If $P^{i,j}(\eta^0) = 0$ for some point $\eta^0 \in R_0^n$, then face $\Re_i^j$ is called **irregular or degenerate**. A polynomial P is called **regular** if all its principal faces are regular.

In [22] S.M. Nikolskii showed that if the Newton polyhedron $\Re = \Re(P)$ is completely proper and if for some constant $C > 0$

$$1 + |P(\xi)| \geq C \sum_{\nu \in \Re} |\xi^\nu| \quad \forall \xi \in \Re,$$

then P is hypoelliptic. In [21] V.P. Mikhailov proved that every regular polynomial with complete Newton polyhedron $\Re$ satisfies this inequality.

Applying this fact and Lemma 2 in [21] one can easely prove

Lemma 1.1. *If P is a regular polynomial with a complete Newton polyhedron $\Re$ and Q is any lower term with respect to P then*

a) $P < P + Q < P$,

b) if the polyhedron $\Re$ is proper and P is almost hypoelliptic then $P + Q$ is almost hypoelliptic.

Therefore it suffices to consider only the case in which the polynomial P is irregular.

2 Almost hypoellipticity of polynomials in terms of powers of polynomials

Let P be an almost hypoelliptic polynomial with constant (generally complex) coefficients. In this section we obtain some conditions under which the relation $Q < P$ ensures almost hypoellipticity of the polynomial $P + Q\cdot$

Lemma 2.1. *Let R_1 and R_2 be $\lambda-$homogeneous polynomials ($\lambda \in R^n$) with $\lambda-$orders $d_1 > d_2$ such that $R_2 < R_1\cdot$ Then*

$$|R_2(\xi)|/|R_1(\xi)| \to 0 \text{ as } |R_1(\xi)| \to \infty, \tag{2.1}$$

$$|R_2(\xi)| \leq C(1 + |R_1(\xi)|^{\frac{d_2}{d_1}}) \quad \forall \xi \in R^n\cdot \tag{2.2}$$

Proof. It is obvious that (2.2) implies (2.1). To prove (2.2) note that for $\xi \in R^n$: $R_1(\xi) = 0$, inequality (2.2) immediately follows by the homogeneity of polynomials R_1 and R_2 and the condition $R_2 < R_1\cdot$ Let $R_1(\xi) \neq 0$ and $t = |R_1(\xi)|^{-\frac{1}{d_1}}$ then by the $\lambda-$homogeneity of polynomials R_1 and R_2 and by the conditon $R_2 < R_1$ we obtain

$$t^{d_2}|R_2(\xi)| = |R_2(t^\lambda \xi)| \leq C[1 + |R_1(t^\lambda \xi)|] = C[1 + t^{d_1}|R_1(\xi)|] = 2C,$$

i.e. $|R_1(\xi)|^{-\frac{d_2}{d_1}} |R_2(\xi)| \leq 2C$, which proves (2.2). □

Corollary 2.1. *Let $R(\xi)$ be a $\lambda-$homogeneous polynomial with $\lambda-$order $d = d_R$, $Q_j(\xi)$ be $\lambda-$homogeneous polynomials with $\lambda-$orders δ_j $(j = 0, 1, \cdots, M)$, $d > \delta_0 > \delta_1 > \cdots > \delta_M$, $Q(\xi) = Q_0(\xi) + \cdots + Q_M(\xi)$ and $Q < R$. Then*

$$|Q(\xi)|/|R(\xi)| \to 0 \quad \text{as} \quad |R(\xi)| \to \infty\cdot \tag{2.3}$$

Proof. From the following system of linear algebraic equations with respect to the $\{Q_j(\xi)\}$

$$\sum_{i=0}^{M} t^{\delta_i} Q_i(\xi) = Q(t^\lambda \xi) \quad t = 1, \cdots, M+1$$

(with a nonzero determinant) we obtain that for some positive constants C_1, C_2, C_3

$$|Q_i(\xi)| \le C_1 \sum_{t=1}^{M} |Q(t^\lambda \xi)| \le C_2 \sum_{t=1}^{M} [1 + |R(t^\lambda \xi)|\,] = C_2(1 + [\sum_{t=1}^{M} t^d]|R(\xi)|)$$

$$\le C_3[1 + |R(\xi)|\,] \quad \forall \xi \in R^n, \quad i = 0, 1, \cdots, M.$$

This means that $Q_i < R$ $(i = 0, 1, \cdots, M)$ and (2.3) follows by Lemma 2.1. □

We denote by I_n the set of all polynomials $P(\xi) = P(\xi_1, \cdots, \xi_n)$ such that

$$|P(\xi)| \to \infty \;\text{ as }\; |\,\xi| \to \infty.$$

In [9] and [10] in the case $n = 2$ there were obtained necessary and sufficient conditions ensuring that $P \in I_2$. In [12] it is proved that all the solutions of the equation $P(D)u = f$, where f and all its derivatives are square integrable with a certain exponential weight, which are square integrable with the same weight, are also such that all their derivatives are square integrable with this weight, if and only if the operator $P(D)$ is almost hypoelliptic. In [13] the existence of a constant $\kappa_0 > 0$ is proved such that all solutions of a class of a partially hypoelliptic (with respect to the hyperplane $x'' = (x_2, ..., x_n) = 0$ of the space E^n) (see [16], Definition 11.2.4 and Theorem 11.1.1) equation $P(D)u = 0$ in the strip $\Omega_\kappa = \{(x_1, x'') = (x_1, x_2, ..., x_n) \in E^n;\ |x_1| < \kappa\}$ are infinitely differentable when $\kappa \ge \kappa_0$ and $D^\alpha u \in L_2(\Omega_\kappa)$ for all multi-indices $\alpha = (0, \alpha'') = (0, \alpha_2, ..., \alpha_n)$ in the Newton polyhedron of te operator $P(D)$.

Theorem 2.1. *Let a polynomial $P \in I_n$ be almost hypoelliptic and $Q < P\cdot$ Denote by C_0 the smallest positive constant for which*

$$|\,Q(\xi)| \le C_0[1 + |\,P(\xi)|\,] \;\; \forall \xi \in R^n$$

and put $\Delta = 1/C_0\cdot$ Then, for any complex number a such that $|\,a| < \Delta$, the polynomial $P + a\,Q$ is almost hypoelliptic.

Proof. Since $P \in I_n$, $Q < P$ and $|a| < \Delta$ there exist some positive numbers M and ε such that

$$1 - |a| \frac{|Q(\xi)|}{|P(\xi)|+1} \, [1 + \frac{1}{|P(\xi)|}] \geq \varepsilon \quad \forall \xi \in R^n : |\xi| \geq M \cdot \tag{2.4}$$

On the other hand since P is almost hypoelliptic and $Q < P$, by Theorem 10.4.3 in [14] we get that for a certain constant $C_1 > 0$

$$|\frac{D^\nu P(\xi)}{P(\xi)}| + |a| \, |\frac{D^\nu Q(\xi)}{P(\xi)}| \leq C_1 \quad \forall \xi \in R^n \cdot \tag{2.5}$$

Let now $\nu \in N_0^n$ and $|\xi| \geq M \cdot$ Then inequalities (2.4) and (2.5) imply that

$$\frac{|D^\nu [P(\xi) + a\, Q(\xi)]\,|}{|P(\xi) + a\, Q(\xi)|} \leq \frac{|\,|D^\nu P(\xi)| + |a|\,|D^\nu Q(\xi)|\,|}{|\,|P(\xi)| - |a|\,|Q(\xi)|\,|} =$$

$$= \frac{|\frac{D^\nu P(\xi)}{P(\xi)}| + |a| \, |\frac{D^\nu Q(\xi)}{P(\xi)}|}{1 - |a| \frac{|Q(\xi)|}{|P(\xi)|+1} \, [1 + \frac{1}{|P(\xi)|}]} \leq C_1/\varepsilon,$$

which completes the proof. □

Example 1. The following example shows that in general in Theorem 2.1 the number Δ cannot be replaced by a larger one. Let $n = 2$, $P(\xi) = (\xi_1^2 - \xi_2^2)^2 + \xi_1^2 + \xi_2^2 + 1$, and $Q(\xi) = \xi_1^2 + \xi_2^2 \cdot$ It is obvious that $Q < P$, where $\Delta = C_0 = 1$. By Theorem 2.1 in [17] P is almost hypoelliptic. By the same theorem $P + a\,Q$ is almost hypoelliptic for any $a : -1 < a < 1$. On the other hand for $a = -1$, $P(\xi) + a\,Q(\xi) = (\xi_1^2 - \xi_2^2)^2 + 1$, is not almost hypoelliptic since $D^{(2,\,0)}(P + aQ)(t, t) = 8t^2 \to \infty$ as $t \to \infty$, whereas $(P + aQ)(t, t) = 1$ for any $t \in R^1 \cdot$

Lemma 2.2. *Let P be an almost hypoelliptic polynomial and $P < Q < P \cdot$ Then Q is almost hypoelliptic.*

Proof. By Lemma 10.4.2 in [16] for any polynomial $S(\xi) = S(\xi_1, \cdots, \xi_n)$ there exists a constant $c = c(S) > 0$ such that

$$c^{-1} \, \tilde{S}(\xi) \leq \sup_{|\eta| \leq 1} |\, S(\xi + \eta)| \leq c \, \tilde{S}(\xi) \quad \forall \xi \in R^n,$$

where $\tilde{S}$ is L. Hörmander's function, defined by formula

$$\tilde{S}(\xi) = \sqrt{\sum_{|\nu| \geq 0} |D^\nu S(\xi)\,|^2} \cdot$$

This, together with the assumptions of the lemma, imply that for some positive constants C_j $(j = 1, \cdots, 5)$

$$\tilde{Q}(\xi) \leq C_1 \sup_{|\eta| \leq 1} |\, Q(\xi + \eta)| \leq C_2 \sup_{|\eta| \leq 1} [\,1 + |\, P(\xi + \eta)|\,]$$

$$\leq C_3 \, \tilde{P}(\xi) \leq C_4 [\,1 + |\, P(\xi)|\,] \leq C_5 [\,1 + |\, Q(\xi)|\,] \quad \forall \xi \in R^n \cdot$$

□

Below the notation $Q << P$ means that

$$|\,Q(\xi)/P(\xi)| \to 0 \quad \text{as} \quad |\,\xi| \to \infty\cdot$$

As a complement to Theorem 2.1 we prove the following statement.

Theorem 2.2. *Let a polynomial $P \in I_n$ be almost hypoelliptic and $Q << P\cdot$ Then for any complex number a the polynomial $P + aQ$ is almost hypoelliptic.*

Proof. First note that the condition $Q << P$ implies that $P + aQ < P$ for any a. On the other hand, since $P \in I_n$ and $Q << P$, for any $a \neq 0$ there exist a number $M = M(a) > 0$ such that

$$|\,Q(\xi)| \leq \frac{1}{2\,|\,a|}\,|\,P(\xi)| \quad \forall \xi \in R^n : |\,\xi| \geq M.$$

Then for all $\xi \in R^n : \ |\xi| \geq M$

$$|P(\xi)| \leq |P(\xi) + aQ(\xi)| + |\,a|\,|Q(\xi)| \leq |P(\xi) + aQ(\xi)| + \frac{1}{2}|\,P(\xi)|,$$

which means that $P < P + aQ$. Thus $P < P + aQ < P$, hence (see Lemma 2.2) $P + aQ$ is almost hypoelliptic. □

Theorem 2.3. *Let $P_0,\ P_1$ be $\lambda-$homogeneous polynomials with real coefficients and with $\lambda-$orders $d_0 > d_1$, $\{Q_j\}$ be $\lambda-$homogeneous polynomials with real coefficients and with $\lambda-$orders $\{\delta_j\}$ $(j = 0, 1, \cdots, M)$ and $Q(\xi) = Q_0(\xi) + \cdots + Q_M(\xi)$.*

If the polynomial $P = P_0 + P_1 \in I_n$ is almost hypoelliptic, $d_0 > \delta_0 > \delta_1 > \cdots > \delta_M > d_1$ and $Q < P_0$, then

1) $P_0 < P$, $P_1 < P$;

2) $P < P + Q < P$;

3) the polynomial $P + Q$ is almost hypoelliptic.

Proof. 1. Since $|P_1| < |P_0| + |P|$, to prove statement 1) it suffices to show that $P_0 < P$.

It is proved in [17] that for the polynomial $P = P_0 + P_1 \in I_n$ with real coefficients $P_0(\xi) \geq 0\,(\leq 0)$ for all $\xi \in R^n$ and $P_1(\eta) > 0\,(< 0)$ for all $\eta \in \Sigma(P_0) \equiv \{\xi : \xi \in R^n_0,\ P_0(\xi) = 0\}$. We can assume that $P_0(\xi) \geq 0$ for all $\xi \in R^n$ and $P_1(\eta) > 0$ for all $\eta \in \Sigma(P_0)$.

Let $0 \neq \xi \in R^n$, we put

$|\,\xi, \lambda| = [\sum_{j=1}^{n} |\,\xi_j|^{2/\lambda_j}]^{1/2}$, $\eta_j = \xi_j/|\,\xi, \lambda|^{\lambda_j}$ $(j = 1, ..., n)$.

Then $|\,\eta, \lambda| = 1$ and for $t = |\,\xi, \lambda|$ we have $\xi = t^\lambda \eta = (t^{\lambda_1}\eta_1, ..., t^{\lambda_n}\eta_n)$, where $|\,\xi, \lambda| = 0$ if and only if $|\,\xi| = 0$.

We divide the unit sphere Σ in R^n into two classes: $\Sigma = \Sigma' \cup \Sigma''$, where $\Sigma' = \{\xi : P_1(\xi) > 0\}$, $\Sigma'' = \{\xi : P_1(\xi) \leq 0\}$ and put $p_0 = \min\{P_0(\xi); \xi \in \Sigma''\}$, $p_1 = \max\{|P_1(\xi)|; \xi \in \Sigma''\}$, $p_2 = \max\{|P_0(\xi)|; \xi \in \Sigma''\}\cdot$ The set Σ'' is closed, $\Sigma(P_0) \subset \Sigma'$ and $P_0(\xi) > 0$ for $\xi \in \Sigma''$, hence $p_0 > 0\cdot$

If $\eta \in \Sigma'$ then by the definition of Σ' we have

$$P(\xi) = P_0(\xi) + t^{d_1}\,P_1(\eta) \geq P_0(\eta)\cdot \tag{2.6}$$

Let $\eta \in \Sigma''$. Since $d_0 > d_1$ there exists a number $t_0 > 0$ such that $p_1 t^{d_1} \leq \frac{p_0}{2} t^{d_0}$ for all $t \geq t_0$. Then for $t \geq t_0$ we have

$$P(\xi) = t^{d_0} P_0(\eta) + t^{d_1} P_1(\eta) \geq p_0 t^{d_0} - p_1 t^{d_1}$$

$$\geq \frac{p_0}{2} t^{d_0} = \frac{1}{2} \frac{p_0}{p_2} p_2 t^{d_0} \geq \frac{1}{2} \frac{p_0}{p_2} P_0(\eta) t^{d_0} = \frac{1}{2} \frac{p_0}{p_2} P_0(\xi)\cdot \tag{2.7}$$

For the points $\xi \in R^n$ for which $t \leq t_0$ we have

$$P_0(\xi) \leq p_3 \equiv \max\{P_0(\xi); |\lambda, \xi| \leq t_0\}\cdot \tag{2.8}$$

By (2.6) - (2.8) it follows that $P_0 < P$ and statement 1) is proved.

2. The right-hand side of statement 2) follows immediately by statement 1). To prove the left-hand side of statement 2) note that the inequality $|P(\xi)| \leq C[1+|P(\xi)+Q(\xi)|\,]$, with a positive constant C, is obvious when $|P(\xi)|$ is bounded. Therefore it is required to consider only the case in which $|P(\xi)|$ is unbounded. In this case by Corollary 2.1 there exists a number $\varepsilon_1 \in (0.1)$ such that $|Q(\xi)| \leq \varepsilon_1 |P_0(\xi)|$ for sufficiently large $|P(\xi)|$. Since by proved statement 1) $P_0 < P$, this means that $|Q(\xi)| \leq \varepsilon |P(\xi)|$ for a number $\varepsilon \in (0.1)$ and for sufficiently large $|P(\xi)|$. Then

$$|P(\xi)| \leq |P(\xi) + Q(\xi)| + |Q(\xi)| \leq |P(\xi) + Q(\xi)| + \varepsilon |P(\xi)|$$

and statement 2) follows.

3. To prove statement 3) we need to show that $D^\nu(P+Q) < P+Q$ for any $\nu \in N_0^n$. By statement 2) $P < P+Q$, hence it suffices to show that $D^\nu(P+Q) = D^\nu P + D^\nu Q < P\cdot$ The relation $D^\nu P < P$ follows from almost hypoellipticity of $P\cdot$ To prove the relations $D^\nu Q < P$ for all $\nu \in N_0^n$ we show that $\tilde{Q} < P$, where $\tilde{Q}$ is L. Hörmander's function of the polynomial Q. By Theorem 10.4.3 in [16] for this purpose it is suffices to show that for a constant $C_1 > 0$ $|Q(\xi)| \leq C_1 \tilde{P}(\xi)$ $\forall \xi \in R^n\cdot$ Since the polynomial P is almost hypoelliptic, this inequality is equivalent to the relation $Q < P$. By the condition $Q < P_0$ of the theorem and by alredy proved statement 1) $P_0 < P$, which completes the proof. □

Remark 1. *By Theorem 10.4.3 in [16] one can replace the condition* $Q < P_0$ *of Theorem 2.3 by the weaker one:* $Q < P$.

3 Comparison of powers of polynomials

Our purpose in this section is finding conditions under which the polynomial $P \in I_n$ with real coefficients is more powerfull than a polynomial Q, i.e. for which $Q < P$. First note that any polynomial $P \in I_n$ with real coefficients preserves sign for sufficiently large $|\xi|\cdot$ Therefore, without loss of generality, in the sequel we assume that $P(\xi) \geq 0$ for all $\xi \in R^n\cdot$

For $\lambda \in R^n$ and a $\lambda-$homogeneous polynomial $R(\xi)$ we set $\Sigma(R) = \{\eta \in R^{n,0}, |\eta| = 1, R(\eta) = 0\}$ and for a point $\eta \in \Sigma(R)$ denote $\aleph(\eta, R) = \{\nu \in N_0^n, D^\nu R(\eta) \neq 0\}$,

$$\Delta(\eta, R) \equiv \Delta(\eta, R, \lambda) = \min_{\nu\in \aleph(\eta,R)} (\lambda, \nu)\cdot \tag{3.1}$$

Let $\Re_i^j$ $(1 \le i \le M_j; 0 \le j \le n-1)$ be a principal face of the complete Newton polyhedron $\Re = \Re(P)$ of a polynomial P. It is clear that for every vector $\lambda \in \Lambda(\Re_i^j)$ there exists a natural number $M = M(\lambda, \Re_i^j)$ and non-negative numbers $d_j = d_j(\lambda, \Re_i^j)$ $(j = 0, 1, \cdots, M)$, such that P can be represented as a sum of non-zero $\lambda-$homogeneous polynomials P_j with $\lambda-$degree d_j:

$$P(\xi) = \sum_{j=0}^{M} P_j(\xi) \equiv \sum_{j=0}^{M} P_{d_j}(\xi) = \sum_{j=0}^{M} \sum_{(\lambda,\alpha)=d_j} \gamma_\alpha \xi^\alpha, \tag{3.2}$$

where $d_0 > d_1 > \cdots > d_M \ge 0$. It is also clear that $P_{d_0}(\xi) \equiv P^{i,j}(\xi)$ for every $\lambda \in \Lambda(\Re_i^j)\cdot$

Let the principal face $\Gamma \equiv \Re_l^k$ $(1 \le l \le M_k; 0 < k \le n-1)$ be irregular and $\lambda \in \Lambda(\Gamma) \equiv \Lambda(\Re_l^k)$. With the $\lambda-$homogeneous polynomials P_{d_j} $j = 0, 1, \cdots, M = M(\Lambda, \Gamma)$ (with the face $\Gamma \equiv \Re_l^k$) we associate the sets $\Sigma(\lambda, P_j)$, $\aleph(\eta, P_j)$ and the numbers $\Delta(\eta, P_j, \lambda)$, defined by (3.1).

Remark 2. *Let an irregular principal face* $\Gamma \equiv \Re_l^k$ $(1 \le l \le M_k; 0 < k \le n-1)$ *of a polynomial* $P \in I_n$, *a vector* $\lambda \in \Lambda(\Gamma)$ *and a point* $\eta \in \Sigma(P^{l,k})$ *be fixed and* P *be represented in form (3.2). Since* $P \in I_n$, *there exists a number* $j_0 = j_0(\Gamma, \lambda, \eta) : 0 < j_0 \le M$ *such that a)* $P_{j_0}(\xi) \ne const$, *b)* $P_j(\eta) = 0$ $(j = 0, 1, \cdots, j_0-1)$, $P_{j_0}(\eta) \ne 0\cdot$ *Thus with any triplet* (Γ, λ, η) *we associate a unique number* $j_0 = j_0(\Gamma, \lambda, \eta)$, *which we use in this section.*

Similarly to (3.2) for every $\lambda \in \Lambda(\Gamma)$ *one can represent a polynomial* Q *as the sum of non-zero* $\lambda-$*homogeneous polynomials:*

$$Q(\xi) = \sum_{j=0}^{M(\lambda,Q)} Q_j(\xi)\cdot$$

If $Q_j < P$ *for all* $j = 0, 1, \cdots, M(\lambda, Q)$, *then it is clear that* $Q < P$.

Therefore in order to simplify the formulations of results it is convenient to agree that

a) Q *is a* $\lambda-$*homogeneous polynomial,*

b) an irregular principal face Γ *is* $(n-1)-$*dimensional (in the two-dimensional case only this is possible) with the* $\Re-$*normal* λ, *which is defined uniquely,*

c) $j_0(\Gamma, \lambda, \eta) = 1$ *for all* $\eta \in \Sigma(\Gamma)$.

A generalized homogeneous polynomial R is called a polynomial with characteristics of constant multiplicity (see [6] or [24]) if for each $\eta \in \Sigma(R)$ there exists a neighborhood $U(\eta)$, sufficiently smooth generalized homogeneous functions $q(\xi) = q(\xi, \eta)$, $r(\xi) = r(\xi, \eta)$ and a natural number $m = m(\eta)$, which does not depend on $\xi \in U(\eta)$, such that $q(\eta) = 0$, $r(\eta) \ne 0$, $grad\, q(\eta) \ne 0$ and

$$R(\xi) = [q(\xi)]^m\, r(\xi) \quad \forall \xi \in U(\eta)\cdot \tag{3.3}$$

Remark 3. *In the case $n = 2$, every generalized homogeneous polynomial R is a polynomial with characteristics of constant multiplicity, i.e. it has a representation of form (3.3) in some neighborhood of each $\eta \in \Sigma(R)$ (see [17], Lemma 2.1). On the other hand, for $n \geq 3$ not every generalized homogeneous polynomial can be represented in form (3.3). Let $n = 3$, and let $R(\xi) = (\xi_1 - \xi_2)(\xi_2 - \xi_3)$. It is easy to see that this homogeneous polynomial is not representable in form (3.3) in any neighborhood of the point $\eta = (1, 1, 1) \in \Sigma(R)$.*

Theorem 3.1. *Let $\Re = \Re(P)$ be the complete Newton polyhedron of a polynomial $P(\xi) = P(\xi_1, \cdots, \xi_n) \in I_n$ all principal faces of which are regular except for one principal $(n-1)-$dimensional irregular face $\Gamma \equiv \Re_l^{n-1}\cdot$ Let λ be the $\Re-$normal of this face, and let $(\lambda, \alpha) = d_0$ be the equation of the $(n-1)-$dimensional supporting hyperplane going through this face. Let $Q(\xi)$ be a $\lambda-$homogeneous polynomial of $\lambda-$degree d_Q : $d_1 < d_Q \leq d_0$, $\Re(Q) \subset \Re(P)$. Let us represent the polynomial P in form (3.2) with the above vector λ, where $P_1(\eta) \neq 0$ for all $\eta \in \Sigma(\Gamma) \equiv \Sigma(P^{l, n-1})$.*

Let P_0 and Q be polynomials with characteristics of constant multiplicity, i.e. for each $\eta \in \Sigma(\Gamma)$ the polynomials P_0 and Q can be represented in the form (see (3.3))

$$P_0(\xi) = [q(\xi)]^m \, r(\xi) \quad \forall \xi \in U(\eta), \tag{3.4}$$

$$Q(\xi) = [q(\xi)]^{m_1} \, r_1(\xi) \quad \forall \xi \in U_1(\eta), \tag{3.5}$$

where $q(\xi) = q(\xi, \eta)$, $r(\xi) = r(\xi, \eta)$, $r_1(\xi) = r_1(\xi, \eta)$, $q(\eta) = 0$, $r(\eta) \neq 0, r_1(\eta) \neq 0$, and if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then $D_n P_0(\eta) \neq 0$,

Then $Q < P$ if and only if

$$1) \quad \Sigma(Q) \subseteq \Sigma(P_0),$$

$$2) \quad \frac{d_0 - d_1}{d_Q - d_1} \geq \frac{\Delta(\eta, P_0)}{\Delta(\eta, Q)} \quad \forall \eta \in \Sigma(P_0)\cdot \tag{3.6}$$

Remark 4. *1) It is obvious that one can assume that $U_1(\eta) = U(\eta)$ for all $\eta \in \Delta(\Gamma)$, 2) the fact that in the right-hand side of the representations (3.4), (3.5) $q_1(\xi) \equiv q(\xi)$ is motivated by condition 1) of this theorem and mentioned above Lemma 2.1 in [17].*

Remark 5. *Since a $\lambda-$ homogeneous polynomial is $\sigma\lambda-$ homogeneous for any $\sigma > 0$ and relation (3.6) holds after replacing λ by $\sigma \lambda$, we can assume that the numbers $\Delta(\eta, P_0)$ and $\Delta(\eta, Q)$ are natural and $m(\eta) = \Delta(\eta, P_0)$, $m_1(\eta) = \Delta(\eta, Q)$ for all $\eta \in \Sigma(P_0)\cdot$*

Proof of Theorem 3.1. Necessity of 1) is obvious. Indeed, if $Q(\eta) \neq 0$ for some point $\eta \in \Sigma(P_0)$ then $|P(t^\lambda \eta)| = t^{d_1} |P_1(\eta)| [1 + o(1)]$ and $|Q(t^\lambda \eta)| = t^{d_Q} |Q(\eta)|$ as $t \to \infty\cdot$ Since $d_Q > d_1$ this means that $|Q(t^\lambda \eta)| / |P(t^\lambda \eta)| \to \infty\cdot$

Necessity of 2). For some point $\eta \in \Sigma(P_0)$ let

$$\frac{d_0 - d_1}{d_Q - d_1} < \frac{\Delta(\eta, P_0)}{\Delta(\eta, Q)}\cdot \tag{3.7}$$

For $\vartheta = (\vartheta_1, \cdots, \vartheta_n) \in R^n$, $t > 0$, and $\kappa > 0$ set

$$\xi_i = t^{\lambda_i}\,(\eta_i + \vartheta_i t^{-\kappa\lambda_i}) \quad i = 1, \cdots, n\cdot$$

Then by Taylor's formula we have as $t \to \infty$

$$Q(\xi) = t^{d_Q}\, Q(\eta + \theta t^{-\kappa\lambda}) = t^{d_Q} \sum_{\alpha} t^{-\kappa(\lambda,\, \alpha)}\, \frac{\theta^\alpha}{\alpha!}\, D^\alpha Q(\eta)$$

$$= t^{d_Q - \kappa\Delta(\eta, Q)} \sum_{(\lambda,\alpha)=\Delta(\eta,Q)} \frac{\theta^\alpha}{\alpha!}\, D^\alpha Q(\eta) + o(t^{d_Q - \kappa\Delta(\eta,Q)})\cdot$$

Choose $\theta \in R^n$ in such a way that

$$\sum_{(\lambda,\, \alpha)=\Delta(\eta,\, Q)} \frac{\theta^\alpha}{\alpha!}\, D^\alpha Q(\eta) \neq 0\cdot$$

The existence of such a vector θ obviously follows the definition of $\Delta(\eta, Q)\cdot$ Then for all $t > 0$ with a constant $C_1 > 0$

$$|\, Q(\xi)| \geq C_1\, t^{\,d\,_Q - k\Delta(\eta,\, Q)}. \tag{3.8}$$

For the polynomials P_0 and P_1 we obviously have for a constant $C_2 > 0$

$$|\, P_0(\xi)\,| \leq C_2\, t^{\,d\,_Q - \kappa\Delta(\eta,\, P_0)}, \quad |P_1(\xi)| \leq C_2\, t^{\,d_1}\cdot \tag{3.9}$$

Obvious geometric arguments show that as $t \to \infty$

$$|\, P(\xi) - [P_0(\xi) + P_1(\xi)\,]\,| = o(t^{d_1})\cdot \tag{3.10}$$

Choose a number κ so that $d_0 - \kappa\,\Delta(\eta,\, P_0) = d_1 < d_Q - \kappa\Delta(\eta,\, Q),$ which is possible by (3.7). Then by (3.9), (3.10) and (3.8) we have that for some constant $C_3 > 0$

$$|\, P(\xi)\,| \leq C_3\, t^{d_1}(1 + o(1)) = o(|\, Q(\xi)\,|),$$

as $t \to \infty$, from which it follows that $|\, Q(\xi)\,|/|\, P(\xi)\,| \to \infty$ as $t \to \infty\cdot$

This proves the necessity of condition 2) for $Q < P\cdot$

To prove the sufficiency we will use the method of V.P. Mikhailov, applied by him in [21] for study regular polynomials, modified by us and adapted for non-regular polynomials (see, for instance, [18]).

Suppose, to the contrary, that under the fulfillment of conditions 1) and 2) there exists a sequence $\{\xi^s\}$, such that $|\xi^s| \to \infty$ as $s \to \infty$ and

$$|\, Q(\xi)\,|/[|\, P(\xi)\,| + 1\,] \to \infty\cdot \tag{3.11}$$

It can be assumed without loss of generality that all of the coordinates of the vectors $\{\xi^s\}$ are positive. Let for $s \in N$

$$\rho_s = \exp\sqrt{\sum_{j=1}^{n} (\ln \xi_j^s)^2}; \quad \lambda_i^s = \frac{\ln \xi_i^s}{\ln \rho_s} \quad (i = 1, \cdots, n). \tag{3.12}$$

Then $\xi^s = \rho_s^{\lambda^s}$ $(\xi_i^s = \rho_s^{\lambda_i^s};\ i = 1, \cdots, n)$ and $|\,\lambda^s\,| = 1$ $(s = 1, 2, \cdots)\cdot$

Since the vectors λ^s are in the unit sphere, the sequence $\{\lambda^s\}$ has a limit point λ^∞, and by passing to a subsequence we may assume that $\lambda^s \to \lambda^\infty$ as $s \to \infty$. By the convexity of the polyhedron $\Re$ it follows that λ^∞ is an outward normal to one and only one face of $\Re\cdot$

In R^n consider an orthogonal basis $e^{1,1}, e^{1,2}, \cdots, e^{1,n}$ with $e^{1,1} = \lambda^\infty\cdot$ Then $\lambda^s = \Sigma_{i=1}^n \lambda_{1,\,i}^s e^{1,\,i}\cdot$ Morever, since $\lambda^s \to \lambda^\infty = e^{1,1}$, we have $\lambda_{1,\,1}^s \to 1$ and $\lambda_{1,\,i}^s = o(\lambda_{1,\,1}^s)$ for $i = 2, 3, \cdots, n$ as $s \to \infty\cdot$

If at the expense of an admissible choice of a subsequence we have $\Sigma_{j=2}^n \lambda_{1,\,j}^s\, e^{1,\,j} = 0$, for all sufficiently large s, we denote the basis $e^{1,\,1}, e^{1,\,2}, \cdots, e^{1,\,n}$ by $e^1, \cdots, e^n$. Otherwise, by an appropriate choice of a subsequence we have that $\Sigma_{j=2}^n \lambda_{1,\,j}^s\, e^{1,\,j} \neq 0$ for all $s \in N$ and that as $s \to \infty$

$$\sum_{i=2}^n \lambda_{1,\,i}^s\, e^{1,\,i} / |\sum_{i=2}^n \lambda_{1,\,i}^s\, e^{1,\,i}| \to e^{2,2};\ \ |e^{2,2}| = 1\cdot$$

In the subspace spanned by $e^{1,\,2}, \cdots, e^{1,\,n}$ we pass to a new orthogonal basis $e^{2,\,2}, \cdots, e^{2,\,n}$ with the vector $e^{2,2}$ defined above. Then (for $n \geq 3$)

$$\lambda^s = \lambda_{1,\,1}^s\, e^{1,\,1} + \sum_{i=2}^n \lambda_{2,\,i}^s\, e^{2,\,i},$$

where clearly

$$\lambda_{2,\,2}^s = o(\lambda_{1,\,1}^s),\ \ \lambda_{2,\,i}^s = o(\lambda_{2,\,2}^s),\ \ i = 3, \cdots, n,\ \ s \to \infty\cdot$$

Proceeding analogously in the subspace with basis $e^{2,\,3}, \cdots, e^{2,\,n}$ and so on, we finally obtain (after modifying the notation) to an orthogonal basis $e^1, e^2, \cdots, e^n$ such that

$$\lambda^s = \sum_{i=1}^n \lambda_i^s\, e^i,\ \ \lambda_1^s \to 1,\ \ \lambda_{i+1}^s = o(\lambda_i^s)\ (i = 1, \cdots, n-1),\ \ s \to \infty\cdot$$

Moreover, there exist natural numbers s_0 and m; $(m \leq n)$ such that for all $s \geq s_0$ we have $\lambda_i^s \neq 0$ for $i = 1, \cdots, m$ and $\lambda_i^s = 0$ for $i = m+1, \cdots, n\cdot$ Without loss of generality we may assume that $s_0 = 1$ and that $\lambda_i^s > 0$ $(i = 1, \cdots, m)$ for all $s \in N\cdot$

To relate the constructed basis with the Newton polyhedron $\Re = \Re(P)$ of the polynomial P, we consider the faces $\Re_{i_1}^{k_1}, \Re_{i_2}^{k_2}, \cdots, \Re_{i_m}^{k_m}$ of $\Re$, where $\Re_{i_1}^{k_1}$ lies in the supporting hyperplane of $\Re$ with outward normal e^1 while each face $\Re_{i_j}^{k_j}$ $(2 \leq j \leq m)$ either coincides with the face $\Re_{i_{j-1}}^{k_{j-1}}$ or is a subfece thereof, and in either case $\Re_{i_j}^{k_j}$ lies in that predecessor, orthogonal to the supporting hyperplane for whose points α the quantity (e^j, α) is largest possible.

It is obvious by the construction of the faces $\Re_{i_1}^{k_1}, \Re_{i_2}^{k_2}, \cdots, \Re_{i_m}^{k_m}$ that their dimensions are subject to the relation $k_1 \geq k_2 \geq \cdots, \geq k_m\cdot$

As above, let P^{i_j, k_j} be the subpolynomials of P corresponding to the faces $\Re_{i_j}^{k_j}$ $(j = 1, \cdots, m)$, and $\{Q^{i_j, k_j}\}$ be subpolynomials defined by the polynomial Q and the vectors $\{e^j\}$ analogously to the polynomials $\{P^{i_j, k_j}\}\cdot$

Compare the behaviour of the $P(\xi^s)$ and $Q(\xi^s)$ as $s \to \infty$, i.e. as $\rho_s \to \infty$ and $\xi^s = \rho_s^{\lambda^s} = \rho_s^{\Sigma_{j=1}^n \lambda_j^s e^j} \quad (s = 1, 2, \cdots)\cdot$

Since we may select a subsequence, we can assume that for some r $(1 \leq r \leq m)$

$$\rho_s^{\lambda_r^s} \to \infty, \quad \rho_s^{\lambda_{r+1}^s} \to b \geq 1 \quad \text{as} \quad s \to \infty\cdot \tag{3.13}$$

(For $r = m = n$ set $\lambda_{n+1}^s = 0$ for all $s \in N$ and let e^{n+1} be an arbitrary unit vector).

From the convexity of $\Re$ and their faces and from e^j-homogeneity of polynomials $P^{i\,j,k\,j}$ and $Q^{i\,j,k\,j}$ we obtain that for an arbitrary multi-index α belonging to all $\Re^{i\,j,k\,j}$ (i.e. $\alpha \in \Re_{i_m}^{k_m}$) and for some positive $\varepsilon_1, \cdots, \varepsilon_r$

$$P(\xi^s) = \rho_s^{(\alpha,\, \lambda_1^s e^1)}[\, P^{i\,1,\,k\,1}(\rho_s^{\Sigma_{j=2}^n \lambda_j^s e^j}) + o(\rho_s^{-\varepsilon_1 \lambda_1^s})\,]$$

$$= \rho_s^{(\alpha,\, \lambda_1^s e^1 + \lambda_2^s e^2)}[\, P^{i\,2,\,k\,2}(\rho_s^{\Sigma_{j=3}^n \lambda_j^s e^j}) + o(\rho_s^{-\varepsilon_2 \lambda_2^s})\,] = \ldots$$

$$= \rho_s^{(\alpha,\, \Sigma_{j=1}^r \lambda_j^s e^j)}[\, P^{i\,r,\,k\,r}(\rho_s^{\Sigma_{j=r+1}^{n+1} \lambda_j^s e^j}) + o(\rho_s^{-\varepsilon_r \lambda_r^s})\,]\cdot \tag{3.14}$$

For the polynomial Q we obtain similarly that, for a multiindex $\beta \in \Re(Q)$ and for some positive $\varepsilon_1^{'}, \cdots, \varepsilon_r^{'}$

$$Q(\xi^s) = \rho_s^{(\beta,\, \Sigma_{j=1}^r \lambda_j^s e^j)}[\, Q^{i\,r,\,k\,r}(\rho_s^{\Sigma_{j=r+1}^{n+1} \lambda_j^s e^j}) + o(\rho_s^{-\varepsilon_r^{'} \lambda_r^s})\,]\cdot \tag{3.15}$$

By our assumptions

$$\eta^s \equiv \rho_s^{\Sigma_{j=r+1}^{n+1} \lambda_j^s e^j} \to b^{\,e^{r+1}} \equiv \eta, \quad (0 < \eta_i < \infty; \quad i = 1, \cdots, n)\cdot$$

We consider two cases: $(\alpha, e^1) > 0$ and $(\alpha, e^1) = 0\cdot$ The case $(\alpha, e^1) < 0$ is impossible, as can be seen from the fact that the equation for the hyperplane of support with outword unit normal λ of a proper polyhedron $\Re$ can be written in the form $(\lambda, \alpha) = d$, where $d \geq 0$ is the distance from the origin to the given hyperplane and α is a roving point of the hyperplane (see, for example, [1]). In the case $(\alpha, e^1) > 0$ the face $\Re_{i_r}^{k_r}$ of $\Re(P)$ is principal and in the case $(\alpha, e^1) = 0$, the face $\Re_{i_r}^{k_r}$ is non-principal.

First let $(\alpha, e^1) > 0$ and $P^{i\,r,\,k\,r}(\eta) \neq 0$. Then for the polynomials P and Q we obtain by (3.14) and (3.15) that

$$P(\xi^s) = \rho_s^{(\alpha,\, \Sigma_{j=1}^r \lambda_j^s e^j)}\; P^{i\,r,\,k\,r}(\eta)\,(1 + o(1)),$$

$$Q(\xi^s) = \rho_s^{(\beta,\, \Sigma_{j=1}^r \lambda_j^s e^j)}\; Q^{i\,r,\,k\,r}(\eta)\,(1 + o(1))$$

as $s \to \infty$.

By the condition $\Re(Q) \subset \Re(P)$ in our theorem, by the positivity of λ_j^s $(j = 1, \cdots, r;\ s \in N)$ and by the definition of vectors $e^1, \cdots, e^r$ we easily obtain that $(\beta,\, \Sigma_{j=1}^r \lambda_j^s e^j) \leq (\alpha,\, \Sigma_{j=1}^r \lambda_j^s e^j)$ for all $s \in N$, hence the last two relations together contradict (3.11).

Let now $(\alpha, e^1) > 0$ and $P^{i_r, k_r}(\eta) = 0$. Since the face $\Re_{i_m}^{k_m}$ is principal and we assumed all principal faces of $\Re$ to be regular except of the principal $(n-1)-$dimensional face $\Gamma \equiv \Re_l^{n-1}$, we obtain that the face $\Re_{i_r}^{k_r}$ coincides with the irregular face Γ, i.e.

$$k_r = n-1,\ \Re_{i_r}^{k_r} = \Gamma,\ e^1 = \lambda,\ ,Q^{i_r,k_r}(\xi) \equiv Q(\xi)\cdot$$

We represent P as the sum of $\lambda-$homogeneous polynomials by (see formula (3.2)). Note that $P_0(\eta) = P^{i_r,k_r}(\eta) = 0$, $P_1(\eta) \neq 0$ and by condition 1) $Q(\eta) = 0$.

Representations (3.14) and (3.15) for a $\gamma \in N_0^n$ such that $(\lambda, \gamma) = d_1(\lambda)$ take the form

$$P(\xi^s) = \rho_s^{\lambda_1^s(\lambda,\alpha)} P_0(\eta^s) + \rho_s^{\lambda_1^s(\lambda,\gamma)} P_1(\eta^s) + o(\rho_s^{\lambda_1^s(\lambda,\gamma)}), \tag{3.16}$$

$$Q(\xi^s) = \rho_s^{\lambda_1^s(\lambda,\beta)} Q(\eta^s). \tag{3.17}$$

We will prove that there exists a constant $C > 0$ such that for all $s \in N$ for which $\rho_s \in U(\eta)$ (see representations (3.4) and (3.5), where $U_1(\eta) \equiv U(\eta)$)

$$|\,Q(\eta^s)\,| \leq C\,|\,P_0(\eta^s)\,|^{(d_Q-d_1)/(d_0-d_1)} \quad \forall \rho_s \in U(\eta)\cdot \tag{3.18}$$

By representations (3.4) and (3.5), where $m(\eta) = \Delta(\eta, P_0)$, $m_1(\eta) = \Delta(\eta, Q)$ for all $\eta \in \Sigma(P_0)$ (see Remark 3.4), we obtain

$$\frac{[Q(\eta^s)]^{d_0-d_1}}{[P_0(\eta^s)]^{d_Q-d_1}} = \frac{[q(\eta^s)]^{m_1(d_0-d_1)}\,[r_1(\eta^s)]^{d_0-d_1}}{[q(\eta^s)]^{m(d_Q-d_1)}\,[r(\eta^s)]^{d_Q-d_1}}$$

$$= [q(\eta^s)]^{m_1(d_0-d_1)-m(d_Q-d_1)}\ \frac{[r_1(\eta^s)]^{d_0-d_1}}{[r(\eta^s)]^{d_Q-d_1}}\cdot \tag{3.19}$$

By our assumption $r(\eta) \neq 0$, i.e. $r(\eta^s) \neq 0$ for all $\rho_s \in U(\eta)$, hence there exists a constant $C_1 > 0$ such that

$$|r_1(\eta^s)|^{\,d_0-d_1}/|r(\eta^s)|^{\,d_Q-d_1} \leq C_1 \quad \forall \rho_s \in U(\eta)\cdot \tag{3.20}$$

By condition 2) of our theorem $m_1(d_0-d_1) - m(d_Q-d_1) = \Delta(\eta, Q)(d_0-d_1) - \Delta(\eta, P_0)(d_Q-d_1) \geq 0$. This, together with (3.19) and (3.20), proves (3.18).

By assumption $P \in I_n$ $P_0(\eta^s) \geq 0$ and $P_1(\eta^s) > 0$ for all $\rho_s \in U(\eta)$. Consequently from (3.16) we obtain that for $S(\xi) = P(\xi) - [P_0(\xi) + P_1(\xi)]$ on our sequence $\{\xi^s\}$ we have $|\,S(\xi^s)\,|/|\,P(\xi^s)\,| \to 0$ as $s \to \infty$, which in turn shows that for sufficiently large s

$$|\,P(\xi^s)\,| \geq C_2|\rho_s^{d_0}\,P_0(\eta^s) + \rho_s^{d_1}\,P_1(\eta^s)\,| \tag{3.21}$$

for a constant $C_2 > 0$.

Now for $x_s = \rho_s \geq 1$, $y_s = |P_0(\eta^s)| \in [0,1]$ and $a_1 > 0, a_2 > 0$ apply the inequality

$$x_s^{d_Q} y_s^{(d_Q-d_1)/(d_0-d_1)} \leq C\,(1 + a_1 x_s^{d_0} y_s + a_2 x_s^{d_1}) \quad s = 1, 2, \cdots,$$

(which can be proved simply if we choose $y_s = z_s^{d_0-d_1}$). We obtain for all $s \in N$

$$\rho_s^{d_Q} \mid P_0(\eta^s) \mid^{(d_Q-d_1)/(d_0-d_1)} \leq C_3 \left[|\rho_s^{d_Q} P_0(\eta^s) + \rho_s^{d_1} P_1(\eta^s) \mid + 1 \right]$$

with a constant $C_3 > 0$.

This, together with (3.18), (3.21), contradicts (3.11) and completes the consideration in the case $(e^1, \alpha) > 0$, i.e. when the face $\Re_{i_m}^{k_m}$ is principal.The case $(e^1, \alpha) = 0$, when the face $\Re_{i_r}^{k_r}$ is non-principal can be treated analogously to the corresponding case of Theorem 2 in [17]. □

4 Description of lower order terms preserving almost hypoellipticity

For an arbitrary multi-index $\nu \in N_0^n$ the sets $\Sigma(\lambda, D^\nu P_j)$, $\aleph(\eta, D^\nu P_j)$ and the numbers $\Delta(\eta, D^\nu P_j, \lambda)$ for the $\lambda-$homogeneous polynomials $\{P_j\}$ can be defined similary using representation $D^\nu P$ in the form

$$D^\nu P(\xi) = \sum_\alpha \gamma_{\alpha,\,\nu} \xi^\alpha = \sum_{j=0}^{M} D^\nu P_j(\xi) \equiv \sum_{j=0}^{M} D^\nu P_{d_j}(\xi) \tag{4.1}$$

(see (3.1), (3.2))

In [20] it is proved that if a polynomial $P(\xi) = P(\xi_1, \xi_2)$ is almost hypoelliptic then each principal irregular face of the proper Newton polygon $\Re(P)$ is completely proper. Thus the restriction to completely proper irregular faces below (see Lemma 4.1 and Theorem 4.1) for almost hypoelliptic polynomials is motivated by this circumstance.

Lemma 4.1. *Let $\Gamma \equiv \Re_l^k$ $(1 \leq l \leq M_k; 0 < k \leq n-1)$ be an irregular completely proper face of the proper Newton polyhedron $\Re = \Re(P)$ of an almost hypoelliptic polynomial P. Then for any $\lambda \in \Lambda(\Gamma)$ and $\eta \in \Sigma(P^{l,\,k})$*

$$d_j(\lambda) - \Delta(\eta, \lambda, P_j) \leq d_{j_0} \quad (j = 0, 1, \cdots, j_0 - 1), \tag{4.2}$$

where the number $j_0 = j_0(\Gamma, \lambda, \eta)$ is defined as in Remark 3.1.

Proof. Suppose, to the contrary, that for some $0 \leq j \leq j_0 - 1$ the inequality (4.2) is violated. We denote by j_1 the least of such $\{j\}$. Thus, let

$$d_j(\lambda) - \Delta(\eta, \lambda, P_j) \leq d_{j_0}\, (0 \leq j \leq j_1 - 1); d_{j_1}(\lambda) - \Delta(\eta, \lambda, P_{j_1}) > d_{j_0}. \tag{4.3}$$

Let $\beta \in N_0^n$ is chosen in such way that $D^\beta P_{j_1}(\eta) \neq 0$ and $(\lambda, \beta) = \Delta(\eta, \lambda, P_{j_1})$. We consider the behaviour of the polynomials P and $D^\beta P$ on the sequence $\xi^s = s^\lambda \eta$ $(s = 1, 2, \cdots)\cdot$

Since $d_j(\lambda) > d_{j_1}(\lambda)$ $j = 0, 1, \cdots, j_1 - 1$, it follows by (4.3) that $\Delta(\eta, \lambda, P_j) > \Delta(\eta, \lambda, P_{j_1})$ $j = 0, 1, \cdots, j_1 - 1$. Then $P_j(\eta) = D^\beta P_j(\eta) = 0$ $(j = 0, 1, \cdots, j_0 - 1)$, $D^\beta P_{j_1}(\eta) \neq 0$. Hence by representation (4.1) and inequality (4.3) we obtain $P(\xi^s) = P_{j_0}(\eta) s^{d_{j_0}} + o(s^{d_{j_0}})$ as $s \to \infty$. For the polynomial $D^\beta P$, respectively, for all $s = 1, 2, \cdots$

$$D^\beta P(\xi^s) = s^{d_{j_1}(\lambda)-\Delta(\eta,\,\lambda,\,P_{j_1})}\, D^\beta P_{j_1}(\eta) + \sum_{j=j_1+1}^{M} s^{d_j(\lambda)-\Delta(\eta,\,\lambda,\,P_{j_1})}\, D^\beta P_j(\eta)\cdot$$

Since $d_j(\lambda) < d_{j_1}(\lambda)$ $(j = j_1+1,\cdots,M)$ we have

$$|D^\beta P(\xi^s)| = |D^\beta P_{j_1}(\eta)| s^{d_{j_1}(\lambda)-\Delta(\eta,\,\lambda,\,P_{j_1})} (1+o(1))$$

as $s \to \infty$.

These relations together with (4.3) show that

$$|D^\beta P(\xi^s)|/[\,1+|P(\xi^s)|\,] \to \infty$$

as $s \to \infty$, which contradicts the almost hypoellipticity of P. □

Theorem 4.1. *Let $P(\xi) = P(\xi_1,\cdots,\xi_n) \in I_n$ be an almost hypoelliptic polynomial with the proper Newton polyhedron $\Re$, all the principal faces of which are regular except of one $(n-1)-$dimensional irregular completely proper face $\Gamma = \Re^{n-1}_{i_0}$. Let λ be the $\Re-$normal of this face, and let $(\lambda,\alpha) = d_0$ be the equation of the $(n-1)-$dimensional supporting hyperplane going through this face.*

Let $Q(\xi)$ be a lower order term with respect to P. Using the vector λ represent the polynomials P and Q in form (see (3.2)):

$$P(\xi) = \sum_{j=0}^{M(P)} P_j(\xi) \equiv \sum_{j=0}^{M(P)} P_{d_j}(\xi) = \sum_{j=0}^{M(P)} \sum_{(\lambda,\,\alpha)=d_j} \gamma_\alpha^P \xi^\alpha, \tag{4.4}$$

$$Q(\xi) = \sum_{j=1}^{M(Q)} Q_j(\xi) \equiv \sum_{j=0}^{M(Q)} Q_{\delta_j}(\xi) = \sum_{j=0}^{M(Q)} \sum_{(\lambda,\,\alpha)=\delta_j} \gamma_\alpha^Q \xi^\alpha, \tag{4.5}$$

where

1) $j_0(\eta) = 1$, i.e $P_1(\eta) \neq 0$ for any $\eta \in \Sigma(\Gamma)$,

2) let $d_0 > \delta_1 > \delta_2 \cdots > \delta_l > d_1$ and $\delta_j < d_1$ $j = l+1,\cdots,M(Q)$ for any $\eta \in \Sigma(\Gamma)$, hence P_0 and Q_j $j = 1,\cdots,l$ are polynomials with characteristics of constant multiplicity, i.e. for each $\eta \in \Sigma(\Gamma)$ the polynomials P_0 and Q_j $1 \le j \le l$ can be represented in form (see (3.8)):

$$P_0(\xi) = [q(\xi)]^m\, r(\xi) \;\; \xi \in U(\eta), \tag{4.6}$$

$$Q_j(\xi) = [q(\xi)]^{m_j}\, r_j(\xi) \;\; \xi \in U_j(\eta) \quad j = 1,\cdots,l, \tag{4.7}$$

where $q(\eta) = 0$, $r(\eta) \neq 0, r_j(\eta) \neq 0$, and if $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, then $D_n P_0(\eta) \neq 0$,

3) $Q_j(\xi) \ge 0$ $\forall \xi \in U_j(\eta)$ $j = 0,1,\cdots,l$.

Then the polynomial $T = P+Q$ is almost hypoelliptic if and only if for all $\eta \in \Sigma(\Gamma)$

$$\delta_j - \Delta(\eta, Q_j) \le d_1 \quad (j = 1,\cdots,l)\cdot \tag{4.8}$$

Remark 6. *1) It is obvious that one can assume $U_j(\eta) = U(\eta)$ for all $j = 1, \cdots, l$ and $\eta \in \Delta(\Gamma)$, 2) the consideration of the case $j_0(\eta) = 1$ for any $\eta \in \Delta(\Gamma)$ does not restrict generality, since in the case of more than one P_j $(j > 0)$ such that $P_j(\eta) = 0$ for a point $\eta \in \Delta(\Gamma)$, one can include such P_j in the set of lower order terms, 3) the case of several non-regular faces of polyhedron $\Re(P)$ may be considered analogously, 4) the restriction to $(n-1)-$ dimensionality of a irregular face is motivated first by simplicity and, secondly, by the fact that in the two-dimensional case this is only possibility.*

Proof of Theorem 4.1. The necessity of conditions (4.8) is contained in Lemma 4.1.

To prove the sufficency suppose, to the contrary, that there exists a multi-index $\nu \in N_0^n$ and a sequence $\{\xi^s\}$, such that $|\xi^s| \to \infty$ as $s \to \infty$ and $|D^\nu T(\xi^s)\,|/|T(\xi^s)\,| \to \infty$, where one can assume (see [15],Theorem 1.1) that $|\nu\,| = 1$. To be definite, let $\nu = (1, 0, \cdots, 0)$ and as $s \to \infty$

$$|D_1 T(\xi^s)\,|/|T(\xi^s)\,| \to \infty \quad \text{as} \quad s \to \infty. \tag{4.9}$$

Reasoning as in the proof of Theorem 3.1, we obtain a representation for polynomial T similar to representation (3.14) for the polynomial P. We consider only case $(\alpha, e^1) > 0\cdot$ Since the polyhedron $\Re(Q)$ consists only of non-principal points of $\Re(T) = \Re(P)$, in this case $\Re_{i_r}^{k_r}(T) = \Re_{i_r}^{k_r}(P)$ is a principal face of $\Re(P)\cdot$ First let $T^{i\,r,\,k\,r}(\eta) = P^{i\,r,\,k\,r}(\eta) \neq 0\cdot$ Then we obtain

$$|T(\xi^s)\,| = \rho_s^{\lambda_1^s\,(\alpha,\,e^1)}\,|P^{i_r,\,k_r}(\eta)\,|\,(1 + o(1)) \tag{4.10}$$

as $s \to \infty$. Similary, for the polynomial $D_1 T$, we have that for some constant $C_1 > 0$

$$|D_1 T(\xi^s)\,| \le C_1\,\rho_s^{\lambda_1^s\,[(\alpha,\,e^1) -\,e_1^1]}\,|D_1 P^{i_r,\,k_r}(\eta)\,| + o(\rho_s^{\lambda_1^s\,(\alpha,\,e^1)})\cdot \tag{4.11}$$

If $e_1^1 < 0$, for simple geometric reasons it is clear that the face $\Re_{i_1}^{k_1}$ (and consequently all its subfaces) of the proper polyhedron $\Re(T) = \Re(P)$ lies on hyperplane $\alpha_1 = 0$, i.e. the polynomial $P^{i_r,\,k_r}(\xi)$ is not depend on ξ_1. Then $D_1 P^{i\,r,\,k\,r}(\xi) = 0$ for all $\xi \in R^n$ and, in particular, $D_1 P^{i\,r,\,k\,r}(\eta) = 0$. Reasoning as above (see representations (3.14), (4.4) and (4.5)) we obtain for a certain number $\varepsilon > 0$

$$|D_1 T(\xi^s)\,| = |D_1 Q_1(\eta)\,|\,\rho_s^{\lambda_1^s\,(\delta_1 - \varepsilon)}(1 + o(1))$$

as $s \to \infty$, where $\delta_1 < (\alpha, e^1)$. This, together with (4.10), contradicts (4.9).

Let now $P^{i\,r,\,k\,r}(\eta) = 0$. Then $\Re_{i_r}^{k_r} = \Gamma$, i.e. $k_r = n-1$, $e^1 = \lambda$ is the $\Re-$ normal of Γ, $\eta^s \equiv \rho_s^{\Sigma_{j=2}^n \lambda_j^s e^j} \to b^{\,e^{r+1}} \equiv \eta$ $(0 < \eta_i < \infty;\ \ i = 1, \cdots, n)$ as $s \to \infty$, where $\eta \in \Sigma(P_0)$ (see representation (4.4)).

The polynomials P_j, $D_1 P_j$, $(j = 0, 1, \cdots, M(P))$; Q_j, $D_1 Q_j$ $(j = 1, \cdots, M(Q))$ are $\lambda-$homogeneous and P_0, Q_j $(j = 1, \cdots, l)$ are polynomials with isolated characteristics, i.e. represented in form (4.6) - (4.7). Then

$$D_1 P_0(\xi) = m[q(\xi)]^{m-1}\,D_1 q(\xi)\,r(\xi) + [q(\xi)]^m\,D_1 r(\xi) \tag{4.12}$$

and for any $j = 1, \cdots, l$

$$D_1 Q_j(\xi) = m_j[q(\xi)]^{m_j - 1}\,D_1\,q(\xi)\,r_j(\xi) + [q(\xi)]^{m_j}\,D_1\,r_j(\xi)\cdot \tag{4.13}$$

For convenience we denote $Q_0 = P_0,\ m_0 = m,\ r_0 = r,\ \delta_0 = d_0$ and prove that for each $j = 0, 1, \cdots, l$ there exists a constant $\omega_j > 0$ such that

$$| D_1 Q_j(\xi^s) | \leq \omega_j [|Q_j(\xi^s)| + | P_1(\xi^s) |] \quad \forall s \in N \cdot \tag{4.14}$$

Let the number $s_0 \in N$ be chosen in such a way that $\eta^s \in U(\eta)$ for $s \geq s_0$. Then for $s \geq s_0$ we get

$$Q_j(\xi^s) = \rho_s^{\lambda_1^s \delta_j} Q_j(\eta^s) = \rho_s^{\lambda_1^s \delta_j} [q(\eta^s)]^{m_j} r_j(\eta^s) \quad (j = 0, 1, \cdots, l),$$

$$P_1(\xi^s) = \rho_s^{\lambda_1^s d_1} P_1(\eta^s).$$

For the polynomials $\{D_1 Q_j\},\ D_1 P_1$ respectively, for $s \geq s_0$

$$D_1 Q_j(\xi^s) = \rho_s^{\lambda_1^s (\delta_j - \lambda_1)} \{ m_j [q(\eta^s)]^{m_j - 1} D_1 q(\eta^s) + [q(\eta^s)]^{m_j} D_1 r_j(\eta^s) \},$$

$$D_1 P_1(\xi^s) = \rho_s^{\lambda_1^s (d_1 - \lambda_1)} D_1 P_1(\eta^s),$$

where $\lambda_1^s \to 1$ as $s \to \infty$.

Since $\eta^s \to \eta,\ q(\eta^s) \to q(\eta) = 0$ as $s \to \infty$ and $r_j(\eta) \neq 0,\ P_1(\eta) \neq 0,$ we obtain from the last four representations with some positive constants C_1, C_2 and the same s

$$| Q_j(\xi^s) | \geq C_1 \rho_s^{\lambda_1^s \delta_j} |q(\eta^s)|^{m_j}, \quad (j = 0, 1, \cdots, l) \tag{4.15}$$

$$| P_1(\xi^s) | \geq C_1 \rho_s^{\lambda_1^s d_1}, \tag{4.16}$$

$$| D_1 Q_j(\xi^s) | \leq C_2 \rho_s^{\lambda_1^s (\delta_j - \lambda_1)} |q(\eta^s)|^{m_j - 1}, \quad (j = 0, 1, \cdots, l), \tag{4.17}$$

$$| D_1 P_1(\xi^s) | \leq C_2 \rho_s^{\lambda_1^s (d_1 - \lambda_1)}. \tag{4.18}$$

If $m_{j_0} = 1$ for a j_0, satisfying $0 \leq j_0 \leq l$, then by the conditions $D_n q(\eta) \neq 0$ and $\lambda_n \leq \lambda_1$ we get $\Delta(\eta, P_{j_0}) = \lambda_n \leq \lambda_1$. Therefore $\delta_{j_0} - \lambda_1 \leq \delta_{j_0} - \lambda_n \leq d_1$ by conditions (4.8) and estimate (4.14) for Q_{j_0} follows from estimates (4.17), (4.16).

Thus, we can assume that $m_j > 1 \ (j = 0, 1, \cdots, l)$. Arguing as above we obtain $\Delta(\eta, Q_j) = m_j \lambda_n$ and by conditions (4.8) $\delta_j - m_j \lambda_1 \leq \delta_j - m_j \lambda_n \leq d_1$ $(j = 0, 1, \cdots, l)$.

Applying Lemma 1.3 in [17] for $x_s = \rho_s^{\lambda_1^s \delta_j},\ y_s = |q(\eta^s)|,\ a = \lambda_1^s(\delta_j - \lambda_1),\ b = m_j - 1,\ c = \lambda_1^s \delta_j,\ d = m_j,\ e = \lambda_1^s d_l \ (j = 0, 1, \cdots, l;\ s \in N)$ by (4.15) - (4.18) we obtain (4.14). Note that $x_s \geq 1$ and $y_s \in [0, 1]$ for sufficiently large s.

We write

$$S(\xi) = T(\xi) - [P_0(\xi) + \sum_{j=1}^{l} Q_j(\xi) + P_1(\xi)].$$

Obvious geometric arguments show that

$$|\, S(\xi^s)\,| = o(\rho_s^{\lambda_1^s \delta_1}); \quad |\, D_1 S(\xi^s)\,| = o(\rho_s^{\lambda_1^s(\delta_1 - \lambda_1)}) \tag{4.19}$$

as $s \to \infty$. By (4.14) and (4.19) follows the existence of a constant $C_3 > 0$ such that for sufficiently large s (satisfying $\eta^s \in U(\eta)$) we get

$$|\, D_1 T(\xi^s)\,| \le C_3 \,[\,|\, P_0(\xi^s)\,| + \sum_{j=1}^{l} |\, Q_j(\xi^s)\,| + |\, P_1(\xi^s)\,|\,]\cdot \tag{4.20}$$

On the other hand, $Q_j(\xi^s) \ge 0$ $(j = 1, \cdots, l)$ by the assumption and $P_0(\xi^s) \ge 0$, $P_1(\xi^s) > 0$ for s satisfying $\eta^s \in U(\eta)$ since $P \in I_n$ (see Lemma 1.2 in [17]). This, together with (4.19) and (4.16), shows that for some constant $C_4 > 0$ for the same s

$$|\, T(\xi^s)\,| \ge C_4 \,[\, P_0(\xi^s) + \sum_{j=1}^{l} Q_j(\xi^s) + P_1(\xi^s)\,]. \tag{4.21}$$

Estimates (4.20), (4.21) contradict (4.9). □

The following example shows that condition 3) for lower order term Q in Theorem 4.1 is essential for the almost hypoellipticity of $P + Q$

Example 2. Let $n = 2$ and $P(\xi) = (\xi_1 - \xi_2)^4\,(\xi_1^2 + \xi_2^2) + \xi_2^2 + 1$. The Newton polyhedron $\Re(P)$ is the triangle in R^2 with vertices $(6,0), (0,6), (0,0) \in N_0^2\cdot$ Here $P_0(\xi) = (\xi_1 - \xi_2)^4\,(\xi_1^2 + \xi_2^2)$, $P_1(\xi) = \xi_2^2$, $\Sigma(P_0) = \{\pm\eta\} = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, $P_1(\pm\eta) = 1/2$, $\Delta(\pm\eta, P_0) = 4$, $d_0 = 6$, $d_1 = 2\cdot$ It is easily seen that all conditions of Theorem 2.1 in [15] are satisfied, and hence P is almost hypoelliptic.

Let $Q^{\pm}(\xi) = \pm(\xi_1 - \xi_2)^2\,(\xi_1^2 + 2\xi_2^2)$ be a lower order term with respect to P, where $d_Q = 4$, $\Sigma(Q) = \Sigma(P_0)$, $\Delta(\pm\eta, Q) = 2$, $d_0 - \Delta(\pm\eta, P_0) = d_Q - \Delta(\pm\eta, Q) = 2 = d_1$. Thus all the condition of Theorem 4.1 for polynomial $P + Q^+$ are satisfied, and hence $P + Q^+$ is almost hypoelliptic.

For polynomial Q^- condition 3) is not satisfied. We will show that $P + Q^-$ is not almost hypoelliptic. Indeed, it is easy to check that for the sequence $\{\xi^s = (s{+}1, s);\ s \in N\}$ $P(\xi^s) + Q^-(\xi^s) = 2$ for all $s \in N$, whereas $D_1^2[P(\xi^s) + Q^-(\xi^s)] = 18s^2(1 + o(1))$ as $s \to \infty$, i.e. $P + Q^-$ is not almost hypoelliptic.

On the other hand the following example shows that a polynomial P_1 and a lower order term Q in Theorem 4.1 may have values of any sign outside of some neighborhood of a point $\eta \in \Sigma(P_0)\cdot$

Example 3. Let $n = 2$ and $P(\xi) = (\xi_1 - \xi_2)^4(\xi_1^2 + \xi_2^2)$ $-(\xi_1^2 - 2\xi_2^2)\cdot$ Here $P_0(\xi) = (\xi_1 - \xi_2)^4(\xi_1^2 + \xi_2^2)$, $P_1(\xi) = -(\xi_1^2 - 2\xi_2^2)$, where $d_0 = 6$, $d_1 = 2$, $\Sigma(P_0) = \{\eta^{\pm}\} = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, $P_1(\eta^{\pm}) = 1$, $\Delta(\eta^{\pm}, P_0) = 4\cdot$ It is easy to verify that a) $P \in I_2$ (see [9]), b) the polynomial P satisfies all assumptions of Theorem 2.1 in [15] and P is almost hypoelliptic.

Let $Q(\xi) = -(\xi_1 - \xi_2)^2$ $(2\xi_1^2 - 3\xi_2^2)$ be a lower term with respect to $P\cdot$ Here $d_Q = 4$, $\Sigma(Q) = \Sigma(P_0)$, $\Delta(\eta^{\pm}, Q) = 2\cdot$ Since $d_Q - \Delta(\eta^{\pm}, Q) = d_1 = 2$, P_0 and

Q are polynomials with characteristics of constant multiplisity, by Theorem 4.1 the polynomial $P + Q$ is almost hypoelliptic.

It should be observed that although $P_1(\eta^{\pm}) = 1$, and there exist a neiborhood $U(\eta^{\pm})$ such that $Q(\xi) \geq 0$ for $\xi \in U(\eta^{\pm})$, it turns out that the polynomials P_1 and Q have values of any sign outside of $U(\eta^{\pm})\cdot$

Replacing in presented examples (for instance) ξ_1 by ξ_1^2 we get examples of polynomials with generalized homogeneous irregular principal parts.

References

[1] A.D. Aleksandrov, *Intrinsic geometry of convex surfaces,* OGIZ, Moscow, 1948 (in Russian). German transl., Akademie - Verlag, Berlin, 1955.

[2] V.I. Burenkov, *Investigation of spaces of differentiable functions defined on irregular domains,* Doctor of Sciences thesis. Steklov Inst. Math., 1982, 312 pp.

[3] V.I. Burenkov, *An analogue of Hörmander's theorem on hypoellipticity for functions converging to 0 at infinity,* Proc. 7th Soviet - Czechoslovak Seminar. Yerevan, 1982, 63 - 67 (in Russian)

[4] V.I. Burenkov, *Conditional hypoellipticity and Fourier multipliers in weighted L_p- spaces with an exponential weight*, Proc. of Summer School "Function spaces, differential operators, nonlinear analysis", held in Fridrichroda in 1993. B.G. Teubner, Stuttgart-Leipzig. Teubner-Texte zur Mathematik. 133 (1993), 256-265.

[5] V.I. Burenkov, *On the connection between the behaviour of a solution of a partial differential equation at infinity and its differentiability properties,* Trudy Mat. Inst. Steklov. 89 (1967), 56-68 (in Russian). English translation: Proc. Steklov Inst. Math. 89 (1967).

[6] J. Friberg, *Multi-quasi-elliptic polynomials,* Ann. Scuola Norm. Sup. Pisa. Ser 3, 21 (1967), 239-260.

[7] L. Gårding, *Linear hyperbolic partial differential equations with constant coefficients*, Acta Math. 85 (1951), 1-62.

[8] L. Gårding, B. Malgrange, *Operateurs differentiels partiellement hypoelliptiques*, Math. Scand. 9 (1961), 5-21.

[9] H.G. Ghazaryan, V.N. Margaryan, *On the behaviour of nonelliptic polynomials at infinity,* (in Russian), Izv. AN Armenii, Matematika. 39 (2004), no. 3, 1-18. English translation: Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences) 39 (2004), no. 3.

[10] H.G. Ghazaryan, V.N. Margaryan, *Behaviour at infinity of polynomials in two variables,* Topics in Analysis and its Applications, NATO Sci. Series. "Kluwer" Acad. Publ. Dortrecht - Boston - London. 147 (2004), 163-190.

[11] H.G. Ghazaryan, V.N. Margaryan, *On a class of almost hypoelliptic operators*, (in Russian), Izv. AN Armenii, Matematika. 41 (2006), no. 6, 39-56. English translation: Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences) 41 (2006), no. 6, 30-46.

[12] H.G. Ghazaryan, V.N. Margaryan, *On infinite differentiability of solutions of nonhomogeneous almost hypoelliptic equations*, Eurasian Math. J. 1 (2010), no. 1, 54-72.

[13] H.G. Ghazaryan, *On selection of infinitely differentiable solutions of a class of partially hypoelliptic equations,* Eurasian Math. J. 3 (2012), no. 1, 41-62.

[14] S. Gindikin, L.R. Volevich, *The method of Newton's holyhedron in the theory of partial differential equations,* Kluwer Academic Publishers, Doftrecht - Boston - London. 86, 1992

[15] V.V. Grushin, *On a class of hypoelliptic operators,* Mat. Sb. 83 (1970), no. 3, 456-473 (in Russian). English translation: Math USSR Sb. (1970)

[16] L. Hörmander, *The analysis of linear partial differential operators 2,* Springer - Verlag. 1983.

[17] H.G. Kazarian, *Some estimates of derivatives of polynomials of several variables,* Izv. AN Armenii, Matematika. 34 (1999), no. 3, 46-65 (in Russian). English translation: Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences). 34 (1999), no. 3, 44-63.

[18] G.G. Kazaryan, *On a family of hypoelliptic polynomials,* Izv.Akad. Nauk Arm. SSR, ser. Mat. 9 (1974), 189-211 (in Russian).

[19] G.G. Kazaryan, *On almost hypoelliptic polynomials,* Doklady Ross. Acad. Nauk. 398 (2004), no. 6, 701-703.

[20] G.G. Kazaryan, *On almost hypoelliptic polynomials increasing at infinity,* Journal of Contemporary Math. Analysis. 46 (2011), no. 6, 181-194.

[21] V.P. Mikhailov, *On the behaviour at infinity of a class of polynomials,* Trudy Mat. Inst. Steklov. 91 (1967), 59-80 (in Russian). English translation: Proc. Steklov Inst. Math. 91 (1967).

[22] S.M. Nikol'skii, *The first boundary value problem for a general linear equation,* Dokl. Akad. Nauk SSSR, 164 (1962), 767-769 (in Russian). English transl. in Soviet Math Dokl. 3, 1962.

[23] I.G. Petrovskii, *On a Cauchy problem for a system of linear differential equations in a domain of nonanalytic functions,* Bull. Mosk. Univ. (A), 1 (1938), no. 7, 1-72 (in Russian).

[24] F. Treves, *On the local solvability of linear partial differential equations,* Uspehi Mat. Nauk. 29 (1976), no. 2, 252-281. English translation: Russian Math. Surveys. 29, 1974.

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